Littlewood-Richardson, Hall-Littlewood and Kazhdan-Lusztig

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Two contributions of D.E. Littlewood

- The Littlewood-Richardson rule

$$c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}} = \langle s_\lambda, s_{\mu^{(1)}} \cdots s_{\mu^{(r)}} \rangle$$

- Hall-Littlewood functions

$$P_\mu P_\nu = \sum_\lambda f^{\lambda}_{\mu\nu}(t) P_\lambda$$

such that $g^{\lambda}_{\mu\nu}(q) = q^{n(\lambda)-n(\mu)-n(\nu)} f^{\lambda}_{\mu\nu}(q^{-1})$ (Hall algebra)

Kostka numbers and Kostka-Foulkes polynomials

- Kostka numbers are special LR coefficients

$$K_{\lambda\mu} = c^{\lambda}_{\mu_1,\mu_2,\ldots,\mu_r}$$

- Dual HL functions

$$\langle Q'_\mu, P_\nu \rangle = \delta_{\mu\nu} \quad (\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu})$$

are t-analogues of products h_μ

$$Q'_\mu = \sum_\lambda K_{\lambda\mu}(t) s_\lambda \longrightarrow h_\mu \ (t \to 1)$$

- The Kostka-Foulkes polynomials $K_{\lambda\mu}(t) \in \mathbf{N}[t]$
- The $\tilde{K}_{\lambda\mu}(q)$ are Kazhdan-Lusztig polynomials of a certain type

Roots of unity and plethysm formulae

- $t = 1$ is not the only interesting value
- For $t = \zeta$ a primitive rth root of unity

$$Q'_\lambda(X;\zeta) = Q'_\mu(X;\zeta) \prod_{i\geq 1} \left[Q'_{(i^r)}(X;\zeta)\right]^{q_i}$$

where $\lambda = (1^{m_1} 2^{m_2} \dots n^{m_n})$, $m_i = rq_i + r_i$ with $0 \leq r_i < r$, and $\mu = (1^{r_1} 2^{r_2} \dots n^{r_n})$.

- and for rectangular partitions, we obtain plethysms with power-sums

$$Q'_{(n^r)}(X;\zeta) = (-1)^{(r-1)n} p_r \circ h_n(X)$$

Interpretation

Consider the (reducible) $GL(n, \mathbf{C})$-module

$$V = \Lambda^{\nu_1}\mathbf{C}^n \otimes \Lambda^{\nu_2}\mathbf{C}^n \otimes \cdots \otimes \Lambda^{\nu_r}\mathbf{C}^n$$

and the cyclic shift operator $\gamma : \ V^{\otimes r} \mapsto V^{\otimes r}$

$$\gamma(v_1 \otimes v_2 \otimes \cdots \otimes v_r) = v_r \otimes v_1 \otimes \cdots \otimes v_{r-1}$$

Its eigenspaces $W^{(j)}$ are representations of $GL(n, \mathbf{C})$.

The previous formulae imply a combinatorial description of their characters.

Can we do the same starting with $V = V_\lambda$ irreducible ?

Generalisation

- Conjectural solution in terms of q-analogues

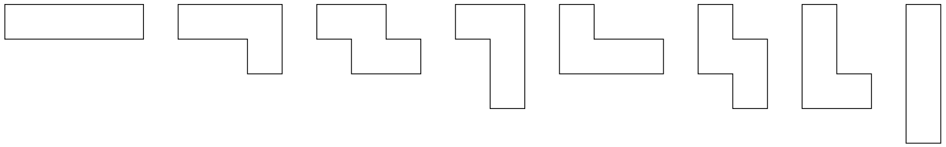
$$c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}(q)$$

  of LR coefficients

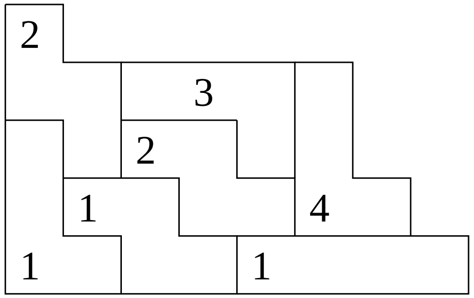
- Complete proofs only for $r = 2$ (Carré-Leclerc 1995)
- The q-LR coefficients are defined by means of *ribbon tableaux*
- They can be connected to quantum affine algebras
- and can be identified with certain Kazhdan-Lusztig polynomials

Ribbons (rim-hooks) and ribbon tableaux

Here are the ($2^3 = 8$) 4-ribbons



and a 4-ribbon tableau of shape (87661) and weight (3211)



Ribbon tableaux and products of Schur functions

A Schur function $s_\lambda(X)$ is a sum over semi-standard Young tableaux t of shape λ

$$s_\lambda(X) = \sum_{t \in \mathrm{Tab}\,(\lambda)} X^t$$

where $X^t = \prod_i x_i^{m_i}$, m_i number of occurences of i in t.

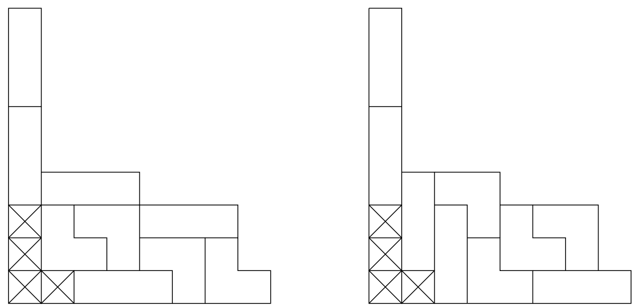
A product of r Schur functions $s_{\mu^{(i)}}$ is a sum over r-tuples of tableaux

$$s_{\mu^{(1)}} s_{\mu^{(2)}} \cdots s_{\mu^{(r)}} = \sum_{(t_1,\ldots,t_r)} X^{t_1} X^{t_2} \cdots X^{t_r}$$

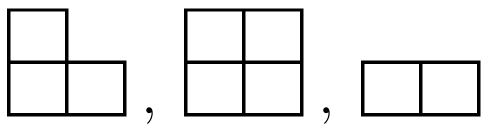
r-tuples of tableaux $\longleftrightarrow$ r-ribbon tableaux

r-core and r-quotient

The partition $\lambda = (87^2 41^5)$ has as 3-core $\nu = (211)$



and as 3-quotient the triple $((21), (22), (2))$

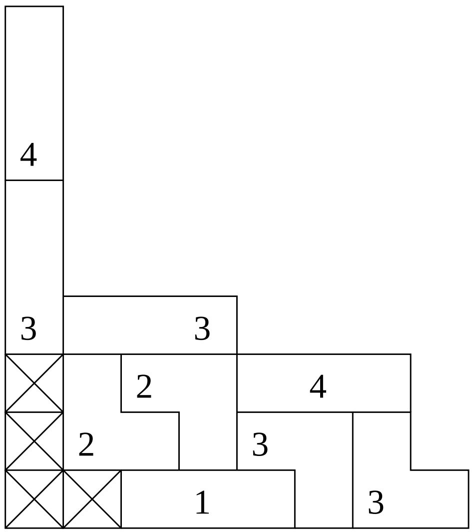


The Stanton-White bijection

Choosing as 3-core $\kappa = (211)$, the triple

$$\begin{array}{|c|c} \hline 4 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{|c|}{3} \\ \hline \end{array}\,, \quad \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array}$$

with weights $(0021), (1111), (0110)$ corresponds to the 3-ribbon tableau of shape $\lambda = (87^2 41^5)$ and weight $\mu = (1242)$.



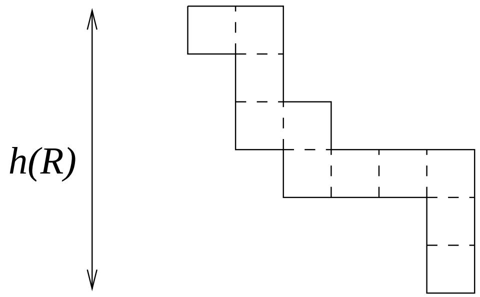
If μ is the partition with r-quotient $(\mu^{(0)}, \ldots, \mu^{(r-1)})$ and empty r-core

$$s_{\mu^{(0)}} s_{\mu^{(1)}} \cdots s_{\mu^{(r-1)}} = \sum_{T \in \mathrm{Tab}_r(\mu,\cdot)} X^T$$

where $\mathrm{Tab}_r(\mu, \cdot)$ is the set of r-ribbon tableaux of shape μ

A natural statistic on ribbon tableaux is the sum of the heights of the ribbons

Example: $r = 11$, $h(R) = 6$



Spin and cospin

The relevant statistic is rather $h(R)-1$, and for compatiblity with HL-functions, one introduces the *spin*

$$s(R)=\frac{1}{2}(h(R)-1), \quad s(T)=\sum_{R\in T} s(R)$$

(a half-integer in general) and the *cospin* (an integer)

$$\tilde{s}(T)=s_r^*(\mu)-s(T) \quad \text{for } T\in \mathrm{Tab}_r(\mu,\cdot)$$

The most general q-LR coefficients are defined by

$$\tilde{G}_\mu=\sum_{T\in \mathrm{Tab}_r(\mu,\cdot)} q^{\tilde{s}(T)}X^T=\sum_\lambda c^\lambda_{\mu^{(0)},\mu^{(1)},\ldots,\mu^{(r-1)}}(q)s_\lambda(X)$$

The 3-quotient of $\lambda = (33321)$ is $((1),(1,1),(1))$ and the q-analogue of $s_1 s_{11} s_1$ (in this order) is

$$m_{31} + (1+q)m_{22} + (2+2q+q^2)m_{211} + (3+5q+3q^2+q^3)m_{1111}$$

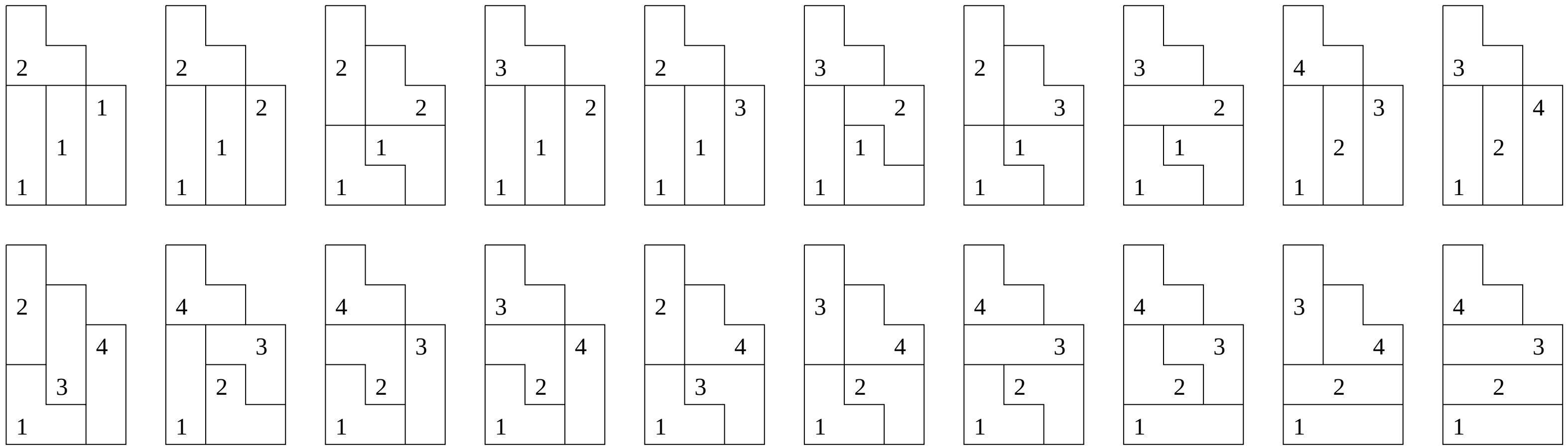
$$= (s_{31} - s_{22} - s_{211} + 2\, s_{1111}) + (1+q)(s_{22} - s_{211} + s_{1111})$$

$$+(2+2q+q^2)(s_{211} - 3\, s_{1111}) + (3+5q+3q^2+q^3)s_{1111}$$

$$= s_{31} + qs_{22} + (q+q^2)s_{211} + q^3 s_{1111}$$

The $c^{\lambda}_{\mu_1,\mu_2,\ldots,\mu_r}(q)$ are defined by an alternating sum but are in $\mathbf{N}[q]$.

The monomial expansion above is given by the 3-ribbon tableaux of shape (33321) and dominant weight



The H-functions

This is an interesting family of spin t-analogues related to HL functions.

A partition of the form $\lambda = r\mu = (r\mu_1, \ldots, r\mu_s)$ has empty r-core and its r-quotient is obtained by grouping the parts of μ according to their class modulo r

$$\lambda(i) = \{\mu_j | j \equiv -i \bmod r\}$$

For any r, the symmetric functions

$$H_\mu^{(r)}(X;t) = \sum_{T \in \mathrm{Tab}\,_r(r\mu,\cdot)} t^{s(T)} X^T$$

form a basis which is unitriangular on Schur functions

It can be proved that for $r \geq \ell(\mu)$,

$$H_\mu^{(r)}(X;t) = Q'_\mu(X;t)$$

Some conjectures for H-functions

Monotonicity $H_\mu^{(r+1)} - H_\mu^{(r)}$ is positive on the Schur basis, that is, the coefficients are in $\mathbf{N}[q]$.

Plethysm When $\mu = \nu^r$, for ζ a primitive r-th root of unity,

$$H_{\nu^r}^{(r)}(\zeta) = (-1)^{(r-1)|\nu|}\, p_r \circ s_\nu$$

and more generally, when $d|r$ and ζ is a primitive d-th root of unity,

$$H_{\nu^r}^{(r)}(\zeta) = (-1)^{(d-1)|\nu|r/d} p_d^{r/d} \circ s_\nu \ .$$

Equivalently,

$$H_{\nu^r}^{(r)}(q) \bmod 1 - q^r = \sum_{i=0}^{r-1} q^r \ell_r^{(i)} \circ s_\nu$$

Examples

The H-functions associated to the partition $\lambda = (3211)$ are

$$\begin{aligned}
H^{(2)}_{3211} &= s_{3211} + q\, s_{322} + q\, s_{331} + q\, s_{4111} \\
&\quad +(q+q^2)\, s_{421} + q^2\, s_{43} + q^2\, s_{511} + q^3\, s_{52} \\
H^{(3)}_{3211} &= s_{3211} + q\, s_{322} + (q+q^2)\, s_{331} + q\, s_{4111} \\
&\quad +(q+2q^2)\, s_{421} + (q^2+q^3)\, s_{43} + (q^2+q^3)\, s_{511} \\
&\quad +2q^3\, s_{52} + q^4\, s_{61} \\
H^{(4)}_{3211} &= s_{3211} + q\, s_{322} + (q+q^2)\, s_{331} + q\, s_{4111} + (q+2q^2+q^3)\, s_{421} \\
&\quad +(q^2+q^3+q^4)\, s_{43} + (q^2+q^3+q^4)\, s_{511} \\
&\quad +(2q^3+q^4+q^5)\, s_{52} + (q^4+q^5+q^6)\, s_{61} + q^7\, s_7 \\
&= Q'_{3211}
\end{aligned}$$

The plethysms of s_{21} with the cyclic characters $\ell_3^{(i)}$ are given by the reduction modulo $1-q^3$ of $H^{(3)}_{222111}$

$$
\begin{aligned}
&= q^9 s_{63} + (q+1)q^7 s_{621} + q^6 s_{6111} + (q+1)q^7 s_{54} \\
&+(q^3+2q^2+2q+1)q^5 s_{531} + (q^2+2q+1)q^5 s_{522} \\
&+(q^3+2q^2+2q+1)q^4 s_{5211} + (q+1)q^4 s_{51111} \\
&+(q^2+2q+1)q^5 s_{441} + (q^3+2q^2+3q+2)q^4 s_{432} \\
&+(2q^3+3q^2+3q+1)q^3 s_{4311} + (q^3+3q^2+3q+2)q^3 s_{4221} \\
&+(q^3+2q^2+2q+1)q^2 s_{42111} + q^3 s_{411111} + (q^3+1)q^3 s_{333} \\
&+(2q^3+3q^2+2q+1)q^2 s_{3321} + (q^2+2q+1)q^2 s_{33111} \\
&+(q^2+2q+1)q^2 s_{3222} + (q^3+2q^2+2q+1)q s_{32211} \\
&+(q+1)q s_{321111} + (q+1)q s_{22221} + s_{222111}
\end{aligned}
$$

More conjectures: q-fusion

- Let $\mathcal{J}^{n,l}$ be the ideal of $Sym(x_1, \ldots, x_n)$ generated by the s_λ such that $\lambda_1 - \lambda_n = l + 1$
- $\mathcal{R}^{n,l} = Sym(x_1, \ldots, x_n)/\mathcal{J}^{n,l}$
- $\mathcal{F}^{n,l} = \mathcal{R}^{n,l}/(s_{1^n} = 1)$ fusion algebra of the WZW model associated to $\widehat{\mathfrak{sl}}_n$ at level l
- $\Pi^{(n,l)} = \{\lambda | (\ell(\lambda) \leq n \text{ and } \lambda_1 - \lambda_n \leq l\}$ ((n, l)-restricted partitions)
- $\{\overline{s}_\lambda | \lambda \in \Pi^{(n,l)}\}$ basis of $\mathcal{R}^{n,l}$
- Structure constants $\overline{c}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$ are alternating sums of LR coefficients
- Replacing these by q-analogues ($\times$ some power of q) appears to yield $\overline{c}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}(q) \in \mathbf{N}[q]$ [FLOT]

Application to nonabelian theta functions ?

$\mathcal{SU}_\Sigma(n)$ moduli space of semi-stable holomorphic vector bundles with trivial determinant over a compact Riemann surface Σ of genus g.

All line bundles on $\mathcal{SU}_\Sigma(n)$ are powers of the determinant bundle $\mathcal{L}$.

The sections of $\mathcal{L}^l$ are called nonabelian theta functions of rank n and level l on Σ [Witten].

$h^0(\mathcal{L}^l) = \dim H^0(\mathcal{SU}_\Sigma(n), \mathcal{L}^l)$ is obtained by a calculation in the fusion algebra $\mathcal{F}^{(n,l)}$

$\lambda^* = -w_0(\lambda)$ highest weight of the dual representation $(V_\lambda)^*$ of $\mathfrak{sl}_n$

Let

$$\omega = \sum_{\lambda \subset (l^{n-1})} \overline{s}_\lambda \overline{s}_{\lambda^*} \quad \in \quad \mathcal{F}^{(n,l)} \, .$$

$h^0(\mathcal{L}^l)$ is equal to the constant term in ω^g.

For $n = 2$, $h^0(\mathcal{L}) = 2^g$ and $h^0(\mathcal{L}^2) = 2^{g-1}(2^g + 1)$

If we compute ω^g by using the cospin q-analogues of the fusion coefficients, we obtain q-analogues $h_q^0(\mathcal{L}^l)$

For $n = 2$ $h_q^0(\mathcal{L}) = (1 + q)^g$ and $h_q^0(\mathcal{L}^2) = \frac{1}{2}[(1 + q^2)^g + (1 + q)^{2g}]$.

Interpretation?

Ribbons and $\widehat{\mathfrak{gl}}_r$

Affine $\mathfrak{gl}_r$ is $\widehat{\mathfrak{gl}}_r = \widehat{\mathfrak{sl}}_r + \mathcal{H}_r$ where $\mathcal{H}_r$ is a Heisenberg algebra commuting with $\widehat{\mathfrak{sl}}_r'$

$\widehat{\mathfrak{sl}}_r' \cap \mathcal{H}_r = \mathbf{C}c$ (c central generator)

Bosonic Fock space $\mathcal{F} = \mathbf{C}[x_1, x_2, \ldots] \simeq Sym(t_1, t_2, \ldots)$ ($x_k = \frac{1}{k} p_k$)

Action of $\widehat{\mathfrak{gl}}_r$ on $\mathcal{F}$:

the generator B_k of $\mathcal{H}_r$ acts by $rk \frac{\partial}{\partial p_{rk}}$ for $k > 0$ and as the multiplication by p_{-rk} for $k < 0$.

Action of the generators of $\widehat{\mathfrak{sl}}_r$ particularly simple in the basis of Schur functions s_λ.

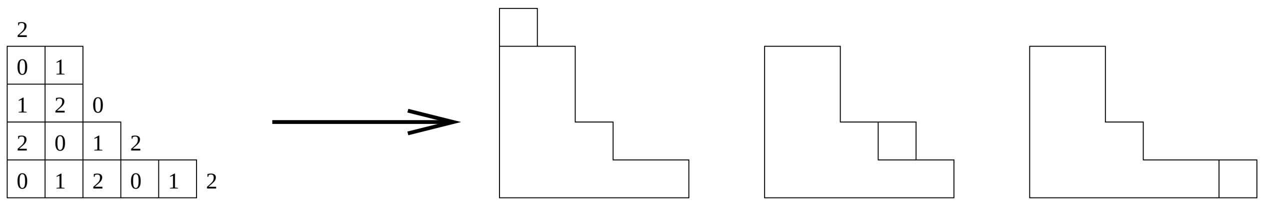
For a node γ in ith row and jth column of λ let $r(\gamma) = j - i \bmod r$.

Then,

$$e_i s_\lambda = \sum s_\nu \,, \qquad f_i s_\lambda = \sum s_\mu \,,$$

where ν (resp. μ) runs through all partitions obtained from λ by removing (resp. adding) a node of residue i.

For example, f_2 of $\widehat{\mathfrak{sl}}_3$ acts on s_{5322} by



$U(\mathcal{H}_r) = p_r \circ Sym$ is as well generated by the $V_k =$ 'multiplication by $p_r \circ h_k$' and their adjoints U_k, which have a simple graphical description

$$V_k s_\lambda = \sum (-1)^{\mathbf{h}(\mu/\lambda)} s_\mu$$

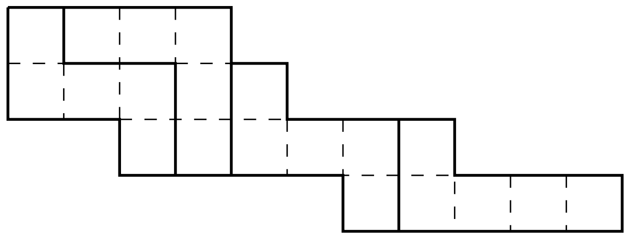
sum over all partitions μ such that μ/λ is a horizontal r-ribbon strip of weight k, and

$$\mathbf{h}(\mu/\lambda) = \sum_R (h(R) - 1)$$

sum over all the r-ribbons R tiling μ/λ. Similarly,

$$U_k s_\mu = \sum (-1)^{\mathbf{h}(\mu/\lambda)} s_\lambda$$

the sum over all partitions λ such that μ/λ is a horizontal r-ribbon strip of weight k.



A horizontal 5-ribbon strip of weight 4 and spin $\frac{7}{2}$

q-Fock space representation of $U_q(\widehat{\mathfrak{gl}}_r)$

in the $\mathbf{Q}(q)$-vector space

$$\mathcal{F} = \bigoplus_{\lambda \in \mathcal{P}} \mathbf{Q}(q)\,|\lambda\rangle$$

$\gamma = (a, b) \in \mathbf{Z}_+ \times \mathbf{Z}_+$ is an indent i-node of λ if a box of residue $i = a - b \bmod r$ can be added to λ at position (a, b)

Similarly, a node of residue i which can be removed is called a removable i-node.

$i \in \{0, 1, \ldots, r - 1\}$

λ, ν such that $\nu/\lambda = \gamma = \boxed{i}$

One defines:

$N_i(\lambda) = \sharp\{$ indent i-nodes of λ $\} - \sharp\{$ removable i-nodes of λ $\}$,

$N_i^l(\lambda, \nu) = \sharp\{$ indent i-nodes of λ on the *left* of γ (not counting γ) $\}$
$-\sharp\{$ removable i-nodes of λ on the *left* of γ $\}$,

$N_i^r(\lambda, \nu) = \sharp\{$ indent i-nodes of λ on the *right* of γ (not counting γ) $\} - \sharp\{$ removable i-nodes of λ on the *right* of γ $\}$,

$N^0(\lambda) = \sharp\{$ 0-nodes of λ $\}$.

Then [Hayashi 1990, Misra-Miwa 1990]

$$f_i|\lambda\rangle = \sum_{\mu} q^{N_i^r(\lambda,\mu)}|\mu\rangle\,, \qquad e_i|\mu\rangle = \sum_{\lambda} q^{N_i^l(\lambda,\mu)}|\lambda\rangle$$

$$q^{h_i}|\lambda\rangle = q^{N_i(\lambda)}|\lambda\rangle \quad \text{and} \quad q^D|\lambda\rangle = q^{-N^0(\lambda)}|\lambda\rangle$$

action of $U_q(\widehat{\mathfrak{sl}}_r)$

Can be extended to $U_q(\widehat{\mathfrak{gl}}_r)$ (q-wedges and q-bosons of [Kashiwara-Miwa-Stern 1996].)

Key point: 'q-bosons' B_k can be replaced by q-analogues of U_k and V_k

$$V_k|\lambda\rangle = \sum(-q)^{-\mathbf{h}(\mu/\lambda)}|\mu\rangle \qquad U_k|\mu\rangle = \sum(-q)^{-\mathbf{h}(\mu/\lambda)}|\lambda\rangle$$

This provides a basis of highest weight vectors in terms of ribbon tableaux!

Identify $\mathcal{F}_q \simeq \mathbf{Q}(q) \otimes Sym$ by $|\lambda\rangle = s_\lambda$

Define linear operator

$$\psi_q^r : \mathcal{F}_q \longrightarrow \mathcal{F}_q$$

image of the basis (h_λ)

$$\psi_q^r(h_\lambda) = V_{\lambda_1} V_{\lambda_2} \cdots V_{\lambda_r} |\emptyset\rangle$$

Then,

$$\psi_q^n(h_\mu) = \sum_{T \in \mathrm{Tab}_{\,r}(\,\cdot\,,\mu)} (-q)^{-2s(T)} s_{\mathrm{shape}(T)}$$

The image $\{\psi_q^r(g_\lambda)\}$ of any basis $\{g_\lambda\}$ of symmetric functions will be a basis of the space of $U_q(\widehat{\mathfrak{sl}}_r)$-highest weight vectors in $\mathcal{F}_q$.

Taking $g_\lambda = s_\lambda$, we have

$$\langle \psi_q^r(s_\lambda), s_\mu \rangle = (-q)^{2s_r^*(\mu)} c^\lambda_{\mu^{(0)} \ldots, \mu^{(r-1)}}(q^2)$$

$((\mu^{(0)} \ldots, \mu^{(r-1)})$ r-quotient of $\mu)$.

Canonical bases

As an $U_q(\widehat{\mathfrak{gl}}_r)$-module, $\mathcal{F}_q$ is irreducible.

But as $U_q(\widehat{\mathfrak{sl}}_r)$-module,

$$\mathcal{F}_q \simeq \bigoplus_{m \geq 0} L(\Lambda_0 - m\delta)^{\oplus p(m)}$$

Each simple $U_q(\widehat{\mathfrak{sl}}_r)$-module $L(\Lambda_0 - m\delta)$ has a canonical basis but these cannot be pieced together to form a canonical basis of the whole $\mathcal{F}_q$ under $U_q(\widehat{\mathfrak{gl}}_r)$.

Such a basis (G_λ^-) was defined in [Leclerc-T. 1996].

All the q-plethysms $\psi_q^r(s_\nu)$ are members of this basis.

The coefficients of the dual basis on Schur functions were conjectured to give the decomposition matrices of quantized Schur algebras at roots of unity.

The proof of this conjecture [Varagnolo-Vasserot 1999] allows one to identify the q-LR coefficients with parabolic KL polynomials [Leclerc-T. 2000]

Then, a result of [Kashiwara-Tanisaki 1999] shows that $c^{\lambda}_{\mu^{(0)}\ldots,\mu^{(r-1)}}(q) \in \mathbf{N}[q]$

A combinatorial proof is still wanted for $r > 2$.

An involution of $\mathcal{F}_q$

There is a unique q-semi-linear endomorphism $x \mapsto \overline{x}$ of $\mathcal{F}_q$ such that

$\overline{|\emptyset\rangle} = |\emptyset\rangle$, $\overline{f_i x} = f_i \overline{x}$ and $\overline{V_k x} = V_k \overline{x}$.

In terms of q-wedges

$$|\lambda\rangle = u_I = u_{i_1} \wedge_q u_{i_2} \wedge_q \cdots u_{i_m} \wedge_q \cdots$$

with $i_k = \lambda_k - k + 1$, $k > 0$, it is found to be given as follows.

Let $\mathcal{J}_m = \{I | \sum_k (i_k + k - 1) = m\}$ For $I \in \mathcal{J}_m$, let $\alpha_{n,k}(I)$ be the number of pairs (r, s) with $1 \leq r < s \leq k$ and $r - s \not\equiv 0 \bmod n$. Then,

$$\overline{u_I} = (-1)^{\binom{k}{2}} q^{\alpha_{n,k}(I)} u_{i_k} \wedge_q u_{i_{k-1}} \wedge_q \cdots \wedge_q u_{i_1} \wedge_q u_{i_{k+1}} \wedge_q u_{i_{k+2}} \wedge_q \cdots$$

for any $k \geq m$. This is an involution.

Upper and lower canonical bases of $\mathcal{F}_q$

Let

$$\mathcal{L}^+ = \bigoplus_\lambda \mathbf{Z}[q]|\lambda\rangle \quad \text{and} \quad \mathcal{L}^- = \bigoplus_\lambda \mathbf{Z}[q^{-1}]|\lambda\rangle$$

There exists bases G^+_λ and G^-_λ of $\mathcal{F}_q$ characterized by

(i) $\overline{G^+_\lambda} = G^+_\lambda$, $\overline{G^-_\lambda} = G^-_\lambda$

(ii) $G^+_\lambda \equiv |\lambda\rangle \bmod q\mathcal{L}^+$, $G^-_\lambda \equiv |\lambda\rangle \bmod q^{-1}\mathcal{L}^-$

We set

$$G^+_\mu = \sum_\lambda d_{\lambda\mu}(q)|\lambda\rangle$$

and

$$G^-_\lambda = \sum_\mu e_{\lambda\mu}(-q^{-1})|\mu\rangle$$

Extended affine Weyl group of type $A_{m-1}^{(1)}$

$\widehat{\mathfrak{S}}_m$ group of transformations of $\mathbf{Z}^m$ generated by permutations of the coordinates (group $\mathfrak{S}_m$) and translations in $r\mathbf{Z}^m$.

$$\mathcal{A} = \{a = (a_1, \ldots, a_m) | -r < a_1 \le \cdots \le a_m \le 0\}$$

For each $u \in \mathbf{Z}^m$ there is a unique w_u of minimal length s.t. $w_u^{-1}(u) \in \mathcal{A}$

Length function $\ell(w)$ and Bruhat order $\le$

Affine Hecke algebra $\widehat{H}_m(q)$ (convention $(T_i + q)(T_i - q^{-1}) = 0$)

Kazhdan-Lusztig basis C'_w, $w \in \widehat{\mathfrak{S}}_m$

$$C'_w = \sum_{x \le w} P_{x,w}(q) T_x$$

Let $\rho = (m-1, m-2, \ldots, 0)$.

If $\lambda, \mu \in \mathbf{Z}^m$ are two partitions with $|\lambda| = |\mu|$ and the same r-core κ, then $\lambda + \rho$ and $\mu + \rho$ are in the same $\widehat{\mathfrak{S}}_m$-orbit Ω

Let $a \in \mathcal{A} \cap \Omega$ and $\widehat{\mathfrak{S}}(a)$ be its stabilizer. It is a parabolic subgroup of $\mathfrak{S}_m$. Let w_0^a be its longest element, and $\hat{u} = w_0 u$, $\hat{v} = w_0 v$ (w_0 longest element of $\mathfrak{S}_m$), $\hat{w}_u = w_{\hat{u}} w_0^a$, $\hat{w}_v = w_{\hat{v}} w_0^a$.

Then,

$$e_{\lambda\mu}(q) = \sum_{x \in \widehat{\mathfrak{S}}(a)} (-q)^{\ell(x)} P_{w_v x, w_u}(q)$$

$$d_{\lambda\mu}(q) = \sum_{y \in \mathfrak{S}_m} (-q)^{\ell(y)} P_{y\hat{w}_u, \hat{w}_v}(q)$$

(parabolic KL polynomials of Deodhar).

Quantized Schur algebras at roots of 1

$\mathcal{S}_n(\zeta)$ with ζ a primitive r-th root of 1

$W(\lambda)$ Weyl modules. $L(\mu)$ simple modules

Conjecture

$$d_{\lambda'\mu'}(q) = \sum_{i\geq 0}[W(\lambda)^i/W(\lambda)^{(i+1)} : L(\mu)]q^i$$

where $\{W(\lambda)^i\}$ Jantzen filtration.

Extends the LLT conjecture proved by Ariki.

Proved by Varagnolo-Vasserot for $q = 1$.

Compatible with Jantzen-Schaper sum formula [Ryom-Hansen 2001]

One has $[d_{\lambda\mu}(q)] = [e_{\lambda'\mu'}(-q)]^{-1}$.

Back to HL functions

Why do we have $\tilde{K}_{\lambda\mu}(q) = c^{\lambda}_{\mu_1,\ldots,\mu_r}(q)$?

One can now deduce it from the KL interpretation, using an earlier result of Lusztig identifying $\tilde{K}_{\lambda\mu}(q)$ as a KL polynomial:

$$e_{N\lambda,N\mu}(q) = K_{\lambda\mu}(q^2) \quad (N \geq m)$$

but there is a different approach [LLT97].

More likely to explain the values at roots of unity.

Unipotent varieties

The coefficients $\tilde{g}_{\nu\mu}(q)$ of the monomial expansions

$$\tilde{Q}'_\mu(X;q) := \sum_\lambda \tilde{K}_{\lambda\mu}(q) s_\lambda = \sum_\nu \tilde{g}_{\nu\mu}(q) m_\nu$$

are known to be the Poincaré polynomials of certain algebraic varieties.

Let $u \in GL(n, \mathbf{C})$ be a unipotent element of Jordan type μ, and let $\mathcal{F}_\nu$ be the variety of ν-flags in $V = \mathbf{C}^n$

$$V_{\nu_1} \subset V_{\nu_1+\nu_2} \subset \ldots \subset V_{\nu_1+\ldots\nu_r} = V$$

where $\dim V_i = i$.

The unipotent variety $\mathcal{F}_\nu^u$ is the set of fixed points of u in $\mathcal{F}_\nu$.

Cell decompositions

[Shimomura 1980] Cell decomposition of $\mathcal{F}_\nu^u$ involving only cells of even real dimensions $\simeq \mathbf{C}^d$. Hence, the Poincaré polynomial has the form

$$\Pi_{\nu\mu}(t^2) = \sum_i t^{2i} \dim H_{2i}(\mathcal{F}_\nu^u, \mathbf{Z})$$

and $\Pi_{\nu\mu}(q) = |\mathcal{F}_\nu^u[\mathbb{F}_q]|$, which can be shown (by means of the Hall algebra) to be

$$|\mathcal{F}_\nu^u[\mathbb{F}_q]| = \tilde{g}_{\nu\mu}(q)$$

Cells are parametrized by *tabloids*.

Tabloids

For μ, ν arbitrary compositions of n, a μ-tabloid of shape ν is a filling of the diagram with row lengths $\nu_1, \nu_2, \ldots, \nu_r$ such that i occurs μ_i times, each row nondecreasing.

For example,

$$\begin{array}{|c|c|c|}
\cline{1-1}
3 \\
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 3 \\
\hline
2 & 3 \\
\cline{1-2}
\end{array}$$

is a $(5,1,3)$-tabloid of shape $(2,3,3,1)$

Inversion statistic on tabloids

Dimension $d(\mathbf{t})$ of the cell $c_{\mathbf{t}}$ explicitly given by Shimomura.

A slightly modified version $e(\mathbf{t})$ (having the same distribution) can be interpreted as a kind of 'inversion number' on r-tuple of rows (e-inversions) [Terada 1993]

Tabloid $\mathbf{t} = (w_1, \ldots, w_r) \simeq r$-tuple of row tableaux.

y the k-th letter of w_i

x the k-th letter of w_j

suppose that $x < y$ (y, x) is an e-inversion if either (a) $i < j$ or (b) $i > j$ and there is on the right of x in w_j a letter $u < y$

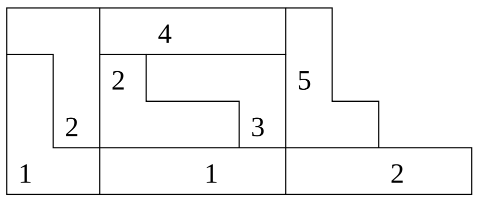
$e(\mathbf{t})$ is equal to the number of inversions (y, x) in $\mathbf{t}$.

Inversions and cospin

Stanton-White correspondence maps $\mathbf{t}$ to T such that $\tilde{s}(T) = e(\mathbf{t})$
For example,

$$\mathbf{t} \quad = \quad \begin{pmatrix} \boxed{2\,|\,3} & , & \boxed{1\,|\,1\,|\,2} & , & \boxed{4\,|\,5} & , & \boxed{2} \\ 1 \quad 1 & & 0 \quad 0 \quad 0 & & 3 \quad 1 & & 1 \end{pmatrix}$$

has $e(T) = 7$ and is mapped to



of cospin 7.

Recent progress

The generalized inversion number $e(\mathbf{t})$ mapped to cospin has been extended to arbitrary r-tuples of tableaux [Schilling-Shimozono-White 2003]

They obtain a recurrence for standard tableaux

$$W(\mu^\bullet; q) := \sum_{T \in \mathrm{STab}_r(\mu)} q^{\tilde{s}(T)} = \sum_{s \in \mathrm{Corn}(\mu^\bullet)} q^{\mathrm{inv}(s,\mu^\bullet)} W(\mu^\bullet - s; q)$$

and a formula for $D_{h_k} \tilde{G}_\mu(X; q)$

An interpretation of the cospin polynomials of ribbon tableaux of given shape and weight as Poincaré polynomials might allow a geometric proof of the properties at roots of unity (?).

A possible geometrical setting

The spin polynomials of the set of r-ribbon tableaux having *fixed diagonals* appear to be of the form

$$q^m \times (\text{product of } q\text{-binomial coefficients})$$

and when the weight is of the form $r\mu$, there is a factor $[r]_q$

(proved for $r = 2$ by Carré-Leclerc, only numerical evidence for the general case).

The cospin polynomials of diagonal spaces can probably be interpreted in terms of cell decompositions of some varieties

$$(\text{factorisations} \longleftrightarrow \text{fibrations})$$

A small example

Let us consider $\mathcal{F}_{1111}^{211}$ (complete flags of $\mathbf{C}^4$ fixed by an element of type 211).

Poincaré polynomial $\langle \tilde{Q}'_{211}, h_1^4 \rangle$

Combinatorial expression

$$\langle Q'_\mu, h_1^n \rangle = \sum_{t \in \mathrm{Stab}(\mu)} f_t(q)$$

where for each standard tableau, $f_t(q)$ is a product of q-integers

$$\begin{array}{|c|c}\hline 4 & \\ \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} = [2][1][1] = 1 + q \qquad \begin{array}{|c|c}\hline 4 & \\ \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} = [2][1][2] = 1 + 2q + q^2$$

$$\begin{array}{|c|c}\hline 3 & \\ \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline \end{array} = [1][3][2] = 1 + 2q + 2q^2 + q^3.$$

The sum is $3+5q+3q^2+q^3$ and the Poincaré polynomial is

$$\Pi_{211}(q) = 1+3q+5q^2+3q^3$$

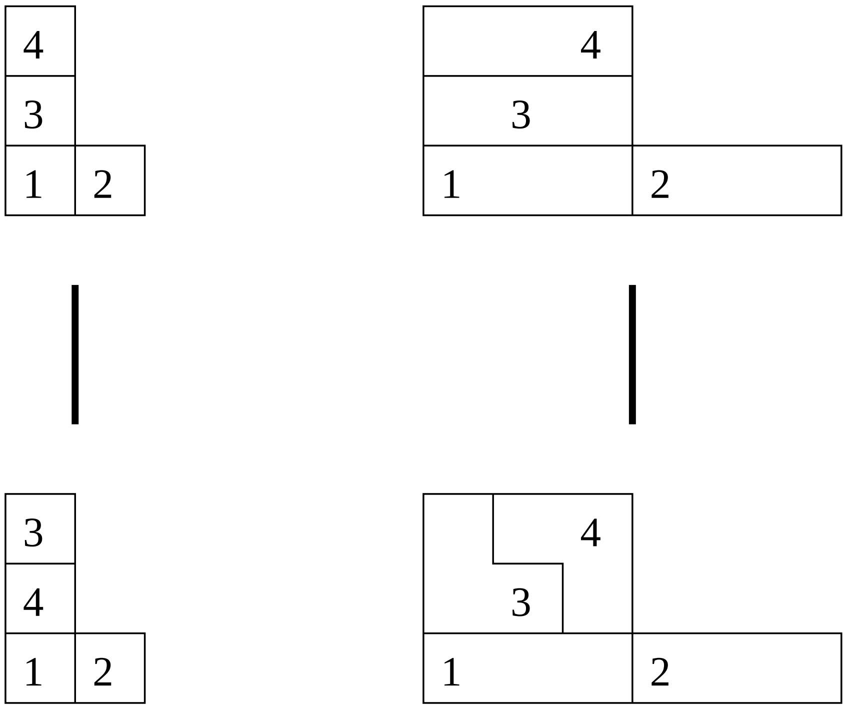
The irreducible components of $\mathcal{F}^{\mu}_{1^n}$ are parametrized by standard tableaux of shape μ.

It is probable that the partial Poincaré polynomials above correspond to cell decompositions of irreducible components and that the corresponding Bruhat order on the cells (standard tabloids) is obtained by permutations inside the columns

Under SW, this corresponds to diagonal classes

Cells of $\mathcal{F}^{211}_{1111}$

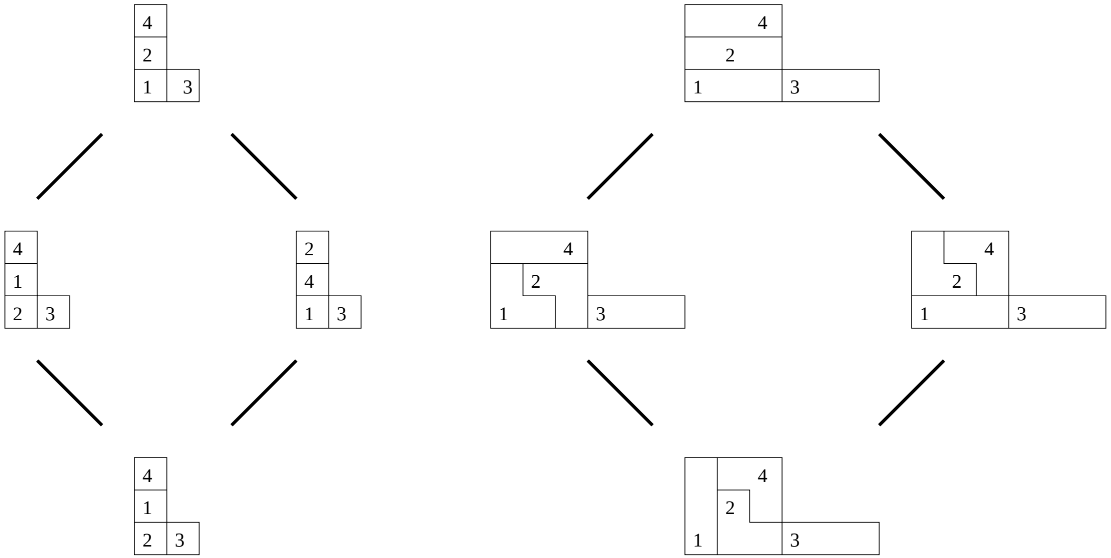
cospin



3

2

$$q^2 + q^3$$



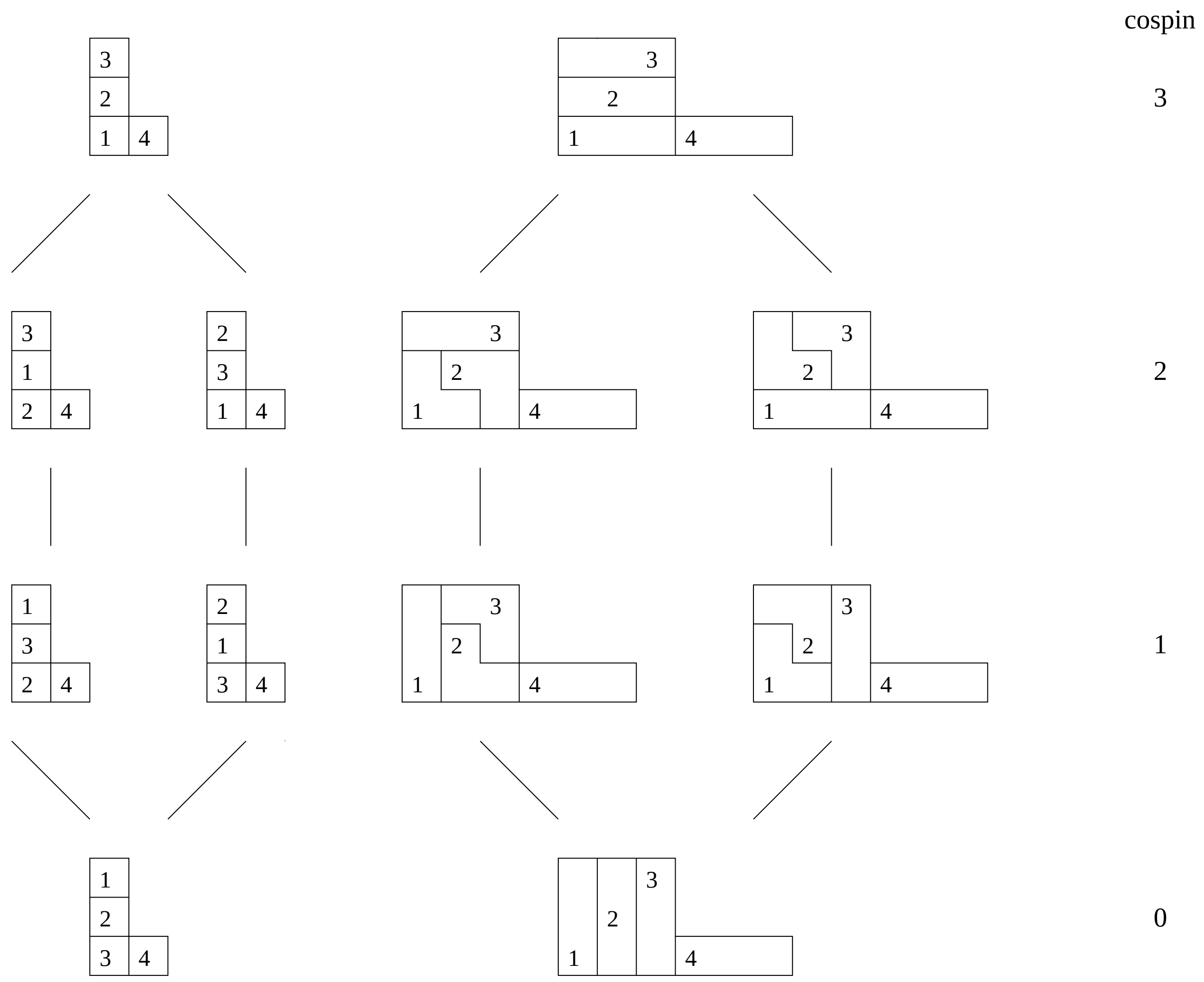
cospin

3

2

1

$$q + 2q^2 + q^3$$



$$1 + 2q + 2q^2 + q^3$$

Color and spin

The *color* of an r- ribbon tableau T of shape λ is

$$c(T) = \sum_{i=0}^{r-1} i|\lambda^{(i)}|$$

where $(\lambda^{(0)}, \ldots, \lambda^{(r)})$ is the r-quotient.

Empirically, on the set of $\mathrm{Tab}_r(\cdot, \mu)$, c and $2s$ seemed to have the same distribution

Identities in [Kirillov-Lascoux-Leclerc-T. 1994] imply that this is true for $r = 2$

Rencently, these identities have been generalized by Thomas Lam (three-parameter generating functions for domino tableaux)

Shimozono and White have constructed a bijection exchanging color and 2×spin (provides some evidence for roots of unity).

Perspectives: (q,t)-LR coefficients ?

For products of rectangular Schur functions, there are other definitions of q-LR coefficients.

Seem to be all the same, but no proof.

One definition uses the ‘Combinatorial R-matrix’

What can we do with the full R-matrix ?

Remark: Factorisation at $t=\zeta$ also for Macdonald H-polynomials

Partition λ of $n \to$ irreducible $H_n(v)$-module V_λ,

$z \in \mathbf{C} \to$ evaluation module $V_\lambda(z)$ for $\hat{H}_n(v)$.

Let $\lambda^\bullet = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(k-1)})$ non-empty partitions.

For generic $z \in \mathbf{C}$, the induced $\hat{H}_N(v)$-module

$$W = V_{\lambda^{(0)}}(1) \widehat{\otimes} V_{\lambda^{(1)}}(z) \widehat{\otimes} \cdots \widehat{\otimes} V_{\lambda^{(k-1)}}(z^{k-1})$$

is irreducible, and isomorphic to its cyclic shift

$$W' = V_{\lambda^{(k-1)}}(z^{k-1}) \widehat{\otimes} V_{\lambda^{(0)}}(1) \widehat{\otimes} V_{\lambda^{(1)}}(z) \widehat{\otimes} \cdots \widehat{\otimes} V_{\lambda^{(k-2)}}(z^{k-2})$$

Let $\Phi_{\lambda^\bullet}(z)$ be an isomorphism $W \to W'$.

It is unique up to a scalar factor, and can be interpreted as the right multiplication by an element $\phi_{\lambda^\bullet}(z) \in \hat{H}_N(v)$.

χ^λ $(\lambda \vdash N)$ irreducible characters of $H_N(v)$

e_λ be the corresponding central idempotents.

We define

$$\overline{\phi}_{\lambda^\bullet}(z) = \sum_{\nu \vdash N} \chi^\nu(\phi_{\lambda^\bullet}(z)) e_\lambda$$

and

$$G_{\lambda^\bullet}(X; v, z) = \sum_{\nu \vdash N} \chi^\nu(\phi_{\lambda^\bullet}(z)) s_\nu(X)\,.$$

Conjecture

Let $t = z^{-1}$ and $q = v^2$. It seems that for each $\lambda^\bullet$, there exist rational functions $a_{\lambda^\bullet}(q,t) \in \mathbf{Z}(q,t)$ and nonnegative integral polynomials $c^\nu_{\lambda^\bullet}(q,t) \in \mathbf{N}[q,t]$ such that

$$G_{\lambda^\bullet}\left(\frac{X}{1-q}; v^2, t^{-1}\right) = a_{\lambda^\bullet}(q,t) \sum_\nu c^\nu_{\lambda^\bullet}(q,t) s_\nu(X)$$

and such that when $\lambda^\bullet = ((m), \ldots, (m))$ contains only identical row partitions,

$$c^\nu_{\lambda^\bullet}(q,t) = K_{\nu,(m^k)}(q,t)$$

is the Macdonald (q,t)-Kostka, and in many (but not all) other cases, $c^\nu_{\lambda^\bullet}(0,t)$ reduce to the t-LR coefficients. Also, it seems that $c^\nu_{\lambda^\bullet}(1,1) = f^\nu$, the number of standard tableaux of shape ν.