A Flag Major Index for Signed Permutations (Extended Abstract)

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Abstract

A new extension of the major index, defined in terms of Coxeter elements, is introduced. For the classical Weyl groups of type B, it is equidistributed with length. For more general wreath products, as well as for dihedral groups, it appears in an explicit formula for the Hilbert series of the (diagonal action) invariant algebra. An alternative combinatorial interpretation relates this index to the decomposition of the coinvariant algebra into irreducible components.

1 Introduction

The *major index*, $major(\pi)$, of a permutation π in the symmetric group S_n is the sum (possibly zero) of all indices $1 \le i < n$ for which $\pi(i) > \pi(i+1)$. The *length* of a permutation π is the minimal number of factors in an expression of π as a product of the Coxeter generators, $(i, i+1), 1 \le i < n$. A fundamental property of the major index is its equidistribution with the length function [MM]; namely, the number of elements in S_n of a given length k is equal to the number of elements having major index k. Bijective proofs and generalizations were given in [F, FS , Ca , GG, Go, Ro3].

Candidates for a major index for the classical Weyl groups of type B have been suggested by Clarke-Foata [CF1–3], Reiner [Rei1–2], Steingrimsson[Ste], and others. Unfortunately, unlike the case of the symmetric group,

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the various alternatives are not equidistributed with the length function (defined with respect to the Coxeter generators of B_n).

In this paper we present a new definition of the major index for the groups B_n, and more generally for wreath products of the form $C_m \wr S_n$, where C_m is the cyclic group of order m. This major index is shown to be equidistributed with the length function for the groups of type B (the case $m = 2$), and to play a crucial role in the study of the actions of these groups on polynomial rings.

2 The Flag Major Index

The groups $C_m \wr S_n$ are generated by $n-1$ involutions, $s_1, \ldots, s_{n-1}$, together with an exceptional generator, s_0, of order m. The $n-1$ involutions satisfy the usual Moore-Coxeter relations (of S_n), while the exceptional generator satisfies the relations: $(s_0 s_1)^{2m} = 1$, $s_0 s_i = s_i s_0$ (for $i > 1$).

Consider now a different set of generators

$$t_k := \prod_{j=0}^{k} s_{k-j} \qquad (0 \leq k \leq n-1).$$

These are Coxeter elements [Hu, §3.16] in a distinguished flag of parabolic subgroups

$$0 < G_1 < \ldots < G_n = C_m \wr S_n$$

where $G_i \cong C_m \wr S_i$ is the subgroup of $C_m \wr S_n$ generated by $s_0, s_1, \ldots, s_{i-1}$.

An element $\pi \in C_m \wr S_n$ has a unique representation as a product

$$\pi = t_{n-1}^{k_{n-1}} t_{n-2}^{k_{n-2}} \cdots t_1^{k_1} t_0^{k_0},$$

with $0 \leq k_i < (i+1)m$ $(\forall i)$. Define the *flag major index* by

$$\textit{flag-major}(\pi) := \sum_{i=0}^{n-1} k_i.$$

For $m = 1$, this definition gives a new interpretation of a well-known parameter.

Claim. *For $m = 1$ (i.e., for the symmetric group S_n) the flag major index coincides with the major index.*

In particular, by MacMahon's classical result the flag major index is equidistributed with length, for $m=1$. This property extends to $m=2$.

Theorem 1. *For $m=2$ (i.e., for the hyperoctahedral group B_n), the flag major index is equidistributed with length.*

Here "length" is used in the usual sense, in terms of the Coxeter generators $s_0, s_1, \ldots, s_{n-1}$. Theorem 1 can be proved by an explicit bijection.

For $m \geq 3$, the flag major index is no longer equidistributed with length (with respect to $s_0, \ldots, s_{n-1}$), but nevertheless it does play a central role in the study of naturally defined algebras of polynomials, as will be shown in Sections 3 and 5.

3 Diagonal Action on Tensor Powers

Let $P_n := \boldsymbol{C}[x_1, \ldots, x_n]$ be the algebra of polynomials in n indeterminates. There is a natural action of $G := C_m \wr S_n$ on P_n, $\varphi : G \to \mathrm{Aut}(P_n)$, defined on generators by

$$\varphi(s_0)(x_j) = \begin{cases} \omega x_j, & \text{if } j=1 \\ x_j, & \text{otherwise} \end{cases} \qquad (\omega := \exp(2\pi i/m) \in \boldsymbol{C})$$

$$\varphi(s_i)(x_j) = \begin{cases} x_{i+1}, & \text{if } j=i \\ x_i, & \text{if } j=i+1 \\ x_j, & \text{if } j \notin \{i, i+1\} \end{cases} \qquad (1 \leq i \leq n-1)$$

where each $\varphi(s_i), 0 \leq i \leq n-1$, is extended to an algebra automorphism of P_n.

Consider now the tensor power $P_n^{\otimes t} := P_n \otimes \cdots \otimes P_n$ (t factors) with the natural *tensor action* φ_T of $G^t := G \times \cdots \times G$ (t factors).

The diagonal embedding

$$d : G \hookrightarrow G^t$$

defined by

$$g \longmapsto (g, \ldots, g) \in G^t \qquad (g \in G)$$

defines the *diagonal action* of G on $P_n^{\otimes t}$:

$$\varphi_D := \varphi_T \circ d.$$

The *tensor invariant algebra*:

$$\text{TIA} := \{\bar{p} \in P_n^{\otimes t} | \varphi_T(\bar{g})\bar{p} = \bar{p} \ , \forall \bar{g} \in G^t\},$$

is a subalgebra of the *diagonal invariant algebra*

$$\text{DIA} := \{\bar{p} \in P_n^{\otimes t} | \varphi_D(g)(\bar{p}) = \bar{p} \ , \forall g \in G\}.$$

Note that $\text{TIA} = (P_n^G)^{\otimes t}$, where P_n^G is the subalgebra of P_n invariant under $\varphi(G)$.

The algebra $P_n^{\otimes t}$ is $\boldsymbol{N}^t$-graded by multi-degree. Let $F_D(\bar{q})$, where $\bar{q} = (q_1, ..., q_t)$, be the multi-variate generating function (Hilbert series) for the dimensions of the homogeneous components in DIA, and let $F_T(\bar{q})$ be similarly defined for TIA. Then

Theorem 2.

$$\frac{F_D(\bar{q})}{F_T(\bar{q})} = \sum_{\pi_1 \cdots \pi_t = 1} \prod_{i=1}^{t} q_i^{flag-\mathrm{major}(\pi_i)}$$

where the sum extends over all t-tuples $(\pi_1, ..., \pi_t)$ of elements in $G = C_m \wr S_n$ such that the product $\pi_1\pi_2 \cdots \pi_t$ is equal to the identity element.

Remark. The flag major index may be defined on dihedral groups in a similar way. An exact analogue of Theorem 2 holds for dihedral groups as well.

4 A Combinatorial Interpretation of the Flag Major Index

The standard major index for permutations has a natural generalization to any finite sequence of letters from a linearly ordered alphabet (see, e.g., [F]); namely, for a finite sequence $a = (a_1, a_2, \ldots, a_n)$ of letters from a linearly ordered alphabet, define $major(a)$ to be the sum (possibly zero) of all indices $1 \leq i < n$ for which $a_i > a_{i+1}$.

Let $\omega \in \boldsymbol{C}$ be a primitive m-th root of unity. An element of the wreath product $C_m \wr S_n$ (where C_m is the cyclic group of order m) may be described as a *generalized permutation* $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$, where, for every i, $\pi(i) \in \boldsymbol{C}$, $\frac{\pi(i)}{|\pi(i)|}$ is a power of ω, and the sequence of absolute values $|\pi| :=$

$(|\pi(1)|, |\pi(2)|, \ldots, |\pi(n)|)$ is a permutation in S_n. In particular, $C_2 \wr S_n$ is the group of *signed permutations*, also known as the hyperoctahedral group, or the classical Weyl group of type B.

Consider now the linearly ordered alphabet

$$1{\cdot}\omega^{m-1} < \ldots < n{\cdot}\omega^{m-1} < \ldots\ldots < 1{\cdot}\omega^1 < \ldots < n{\cdot}\omega^1 < 1{\cdot}\omega^0 < \ldots < n{\cdot}\omega^0.$$

Theorem 3. *For any $\pi \in C_m \wr S_n$,*

$$\textit{flag-major}(\pi) = m \cdot \textit{major}(\pi) + \sum_{j=0}^{m-1} j \cdot \# \left\{ i : \frac{\pi(i)}{|\pi(i)|} = \omega^j \right\},$$

where $major(\pi)$ is defined with respect to the above order.

5 From Elements to Tableaux

A generalization of the Robinson-Schensted correspondence relates the flag-major index to a corresponding parameter for tableaux.

The major index of a (skew) standard Young tableau T, $major(T)$, is the sum of all the entries i such that $i+1$ is strictly south (and weakly west) of i.

Let $(\lambda^1, \ldots, \lambda^m)$ be an m-tuple of partitions. A *skew* $(\lambda^1, \ldots, \lambda^m)$*-diagram* is a disjoint union of m diagrams of shapes $\lambda^1, \ldots, \lambda^m$, respectively, where for each $1 < i \leq m$ the component of shape λ^i lies southwest of the component of shape λ^{i-1}. Let $n := |\lambda^1| + \ldots + |\lambda^m|$, where $|\lambda^j|$ is the size of the partition λ^j. A *skew* $(\lambda^1, \ldots, \lambda^m)$*-tableau* is obtained by replacing each cell in a skew $(\lambda^1, \ldots, \lambda^m)$-diagram by one of the integers $1, \ldots, n$. *Standard* skew tableaux are defined as usual (strictly increasing along both rows and columns).

$major_{m,n}$ for standard skew $(\lambda^1, \ldots, \lambda^m)$-tableaux is defined as follows.

$$major_{m,n}(T) := m \cdot major(T) + \sum_{j=1}^{m} (j-1) \cdot |\lambda^j|.$$

The parameter $major_{m,n}$ for tableaux appears in [Stem] as a tool in the study of coinvariant algebras and eigenvalues of representations.

Let $C_m \wr S_n$ act on the polynomial ring P_n, as in the beginning of Section 3. Let $I_{m,n}$ be the ideal of P_n generated by polynomials without a constant term which are invariant under this action. Denote by R^k the k-th homogeneous component of the *coinvariant algebra* $P_n/I_{m,n}$.

The irreducible representations of $C_m \wr S_n$ are parametrized by m-tuples of partitions $(\lambda^1, \ldots, \lambda^m)$ with $|\lambda^1| + \ldots + |\lambda^m| = n$ [Md, Part I, Appendix B].

Theorem. *[Stem] The multiplicity in R^k of the irreducible representation of $C_m \wr S_n$ corresponding to $(\lambda^1, \ldots, \lambda^m)$ is equal to the number of standard skew $(\lambda^1, \ldots, \lambda^m)$-tableaux with $major_{m,n} = k$.*

Recall that the Robinson-Schensted insertion algorithm gives a bijection between elements of S_n and pairs of standard tableaux of the same shape. This algorithm may be generalized to a bijection between elements of $C_m \wr S_n$ and pairs (P, Q) of standard tableaux of the same (skew) shape. This generalization together with Theorem 3 imply

Corollary 4. *Let (P, Q) correspond, under the generalized Robinson-Schensted correspondence, to an element $\pi \in C_m \wr S_n$. Then*

$$\textit{flag-major}(\pi) = major_{m,n}(Q).$$

Combining this corollary with Stembridge's theorem leads to an alternative algebraic proof of Theorem 1.

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