# INTERSECTIONS OF SCHUBERT CELLS AND GROUPS GENERATED BY SYMPLECTIC TRANSVECTIONS

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Abstract. We prove that the number of connected components in the intersection of two open Schubert cells in relative position $w$ in the space of complete real $n$-dimensional flags equals $3 \cdot 2^{n-1}$ for a generic permutation $w \in S_n$. Our construction combines the machinery of pseudo-line arrangements with the theory of groups generated by symplectic transvections.

The point of departure for this talk is the following result obtained in [SSV2, SSV3]. Let $N_n^0$ denote the semi-algebraic set of all unipotent upper-triangular $n \times n$ matrices $x$ with real entries such that for every $k = 1, \dots, n-1$, the minor of $x$ with rows $1, \dots, k$ and columns $n-k+1, \dots, n$ is non-zero. Then the number $\#_n$ of connected components of $N_n^0$ is as follows: $\#_2 = 2$, $\#_3 = 6$, $\#_4 = 20$, $\#_5 = 52$, and $\#_n = 3 \cdot 2^{n-1}$ for $n \geqslant 6$.

Let $\mathrm{Fl}_n$ be the space of complete flags in $\mathbb{R}^n$. Given any $f \in Fl_n$, the Schubert cell decomposition assigns to each $w \in S_n$ a Schubert cell $C_w^f$ whose dimension equals the number of inversions in $w$. For any $g \in \mathrm{Fl}_n$ we put $C_{w_1,w_2}^{f,g} = C_{w_1}^f \cap C_{w_2}^g$. Recall that $C_{w_1,w_2}^{f,g}$ depends only on the relative position of $f$ and $g$, that is, $C_{w_1,w_2}^{f,g_1}$ and $C_{w_1,w_2}^{f,g_2}$ are isomorphic, provided $g_1$, $g_2$ belong to the same $C_w^f$. For this reason, we write $C_{w_1,w_2}^w$ instead of $C_{w_1,w_2}^{f,g}$, $g \in C_w^f$. The geometry and combinatorics of pairwise intersections $C_{w_1,w_2}^w$ for arbitrary $w_1, w_2, w$ was studied in [SSV1].

Let $C^w = C_{w_0,w_0}^w$, where $w_0 = n\, n-1 \dots 1$ is the longest permutation in $S_n$. Intersections $C^w$ appeared in the literature in various contexts, and were studied (in various degrees of generality) in [BFZ,BZ,R1,R2]. It is easy to

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see that $N_n^0 = C^{w_0}$, so the result cited above gives the number of connected components in $C^{w_0}$. The main result of the present talk is the following generalization of the above result.

**Theorem 1.** *For a generic $w \in S_n$ the number of connected components in $C^w$ equals $3 \cdot 2^{n-1}$.*

Our basic construction relies on the machinery of pseudoline arrangements associated with reduced expressions in the symmetric group developed in [BFZ]. Let $w \in S_n$ be an arbitrary permutation, and let $\omega = s_{i_1} s_{i_2} \dots s_{i_m}$ be an arbitrary reduced expression for $w$. We denote by $\sigma = \sigma^\omega$ the sequence $1\,2 \dots n-1\, i_1 \dots i_m$, and by $\sigma_i$ the $i$th element of $\sigma$. A pair $(\sigma_i, \sigma_j)$, $i < j$, is called a chamber (of level $k$) if $\sigma_i = \sigma_j = k$ and $\sigma_p \neq k$ for $i < p < j$. A chamber $(\sigma_i, \sigma_j)$ is called bounded if $i > n-1$ and unbounded otherwise.

We assign to $\sigma^\omega$ an undirected graph $G^\omega$ in the following way. The vertices of $G^\omega$ are all the chambers of $\sigma^\omega$; we denote the vertex set of $G^\omega$ by $\Gamma^\omega$. Let $\gamma_1 = (\sigma_{i_1}, \sigma_{j_1})$ be a chamber of level $k_1$, $\gamma_2 = (\sigma_{i_2}, \sigma_{j_2})$ be a chamber of level $k_2$; assume without loss of generality that $i_1 < i_2$. The vertices $\gamma_1$ and $\gamma_2$ are joined by an edge in $G^\omega$ if and only if one of the following two conditions holds:

(i) $k_1 = k_2$ and $\sigma_i \neq k_1$ for $j_1 < i < i_2$;

(ii) $|k_1 - k_2| = 1$ and $i_2 < j_1 < j_2$.

Denote by $V^\omega = \mathbb{F}_2^{\Gamma^\omega}$ the vector space over $\mathbb{F}_2$ with the basis $e_\gamma$, $\gamma \in \Gamma^\omega$. The graph $G^\omega$ induces an alternating bilinear form $\Omega^\omega$ on $V^\omega$, namely,

$$\Omega^\omega = \sum_{(\alpha,\beta) \in G^\omega} e_\alpha^* \wedge e_\beta^*.$$

For any $\gamma \in \Gamma^\omega$ we define a linear transformation $\tau_\gamma \colon V^\omega \to V^\omega$ by the following rule:

$$\tau_\gamma v = v - \Omega^\omega(v, e_\gamma) e_\gamma \qquad \text{for any } v \in V^\omega;$$

$\tau_\gamma$ is called the symplectic transvection at $\gamma$ with respect to $\Omega^\omega$. Let $\Gamma_0^\omega \subset \Gamma^\omega$ be the set of bounded chambers of $\sigma^\omega$. We denote by $\mathfrak{G}^\omega$ the group of linear transformations $V^\omega \to V^\omega$ generated by $\{\tau_\gamma, \gamma \in \Gamma_0^\omega\}$.

The following result is a generalization of the main theorem of [SSV2].

**Theorem 2.** *For any $w \in S_n$ and any reduced expression $\omega$ for $w$, the number of connected components of $C^w$ equals the number of orbits of $\mathfrak{G}^\omega$ in $V^\omega$.*

Our construction of $\mathfrak{G}^\omega$ is a special case of a more general construction, which uses an arbitrary finite set $\Gamma$, an arbitrary undirected graph $G$ without loops and multiple edges on the vertex set $\Gamma$, and an arbitrary subset $\Gamma_0 \subseteq \Gamma$. The orbits of $\mathfrak{G}$ in $V = \mathbb{F}_2^\Gamma$ in case $\Gamma_0 = \Gamma$ were studied in [Ja] under certain assumptions of general position; the number of orbits in this case is equal to 3. Here we prove that under similar assumptions the number of $\mathfrak{G}$-orbits in $V$ equals $3 \cdot 2^r$, where $r = |\Gamma| - |\Gamma_0|$.

Let us define the linear operator $L\colon V \to V^*$ by $\Omega(x,y) = (x, Ly)$, where $(\cdot,\cdot)$ is the standard pairing between $V$ and $V^*$; evidently, $L^* = L$. Let $V_0 = \mathbb{F}_2^{\Gamma_0}$ be the space spanned by $e_\gamma$, $\gamma \in \Gamma_0$; by $\iota$ we denote the injection $V_0 \hookrightarrow V$, and by $\iota^*$ its dual $V^* \to V_0^*$. Combining $L$ with $\iota$ and $\iota^*$ we get the maps $\Lambda = L \circ \iota\colon V_0 \to V^*$ and $L_0 = \iota^* \circ \Lambda\colon V_0 \to V_0^*$; evidently, $L_0^* = i^* \circ L^* \circ i = i^* \circ L \circ i = L_0$.

The following observation turns out to be crucial for our construction. Let $\Omega_0$ denote the restriction of the form $\Omega$ to $V_0$.

**Theorem 3.** *Let $\Gamma$, $\Gamma_0$ and $G$ satisfy condition*

$$(* +) \qquad \Lambda^*\colon V \to V_0^* \text{ is a surjection,}$$

*then the $\mathfrak{G}$-action on $V_0$ by symplectic transvections with respect to $\Omega_0$ is dual to the $\mathfrak{G}$-action on $V_0^*$ induced by $\Lambda^*$.*

We say that a subset $\Delta \subseteq \Gamma_0$ is a transversal if $|\Gamma_0| - |\Delta| = \dim\ker L_0$ and the restriction of $\Omega$ to $\mathbb{F}_2^{\Delta}$ is nondegenerate. A transversal is called nonspecial if the subgraph of $G$ induced by $\Delta$ is connected and contains six vertices that span a subgraph isomorphic to the Dynkin diagram $E_6$.

**Theorem 4.** *Let $\Gamma$, $\Gamma_0$ and $G$ satisfy $(*)$ and let there exist two nonspecial transversals $\Delta_1, \Delta_2 \subseteq \Gamma_0$ such that*

$$(**) \qquad \Delta_1 \cup \Delta_2 = \Gamma_0,$$

$$(***) \qquad (\gamma_1, \gamma_2) \in G \text{ for some pair } \gamma_1, \gamma_2 \in \Delta_1 \cap \Delta_2.$$

*Then the number of orbits of $\mathfrak{G}$ on $V$ equals $3 \cdot 2^r$, where $r = |\Gamma| - |\Gamma_0|$.*

To apply Theorem 4 we have to verify its assumptions for the triple $(\Gamma^\omega, \Gamma_0^\omega, G^\omega)$. First of all, we have the following proposition.

**Proposition 5.** *For any $w \in S_n$ and any reduced expression $\omega$ for $w$, the linear operator $(\Lambda^\omega)^*$ satisfies $(*)$.*

Let $\omega$ be a reduced expression for some $w \in S_n$. If the subgraph $G_0^\omega$ of $G^\omega$ induced by $\Gamma_0^\omega$ is not connected, then $w$ can be decomposed into the product of two permutations on the index sets $[1, \ldots, i]$ and $[i, \ldots, n]$ with $1 < i < n$; moreover, $G_0^{\omega'}$ remains not connected for any other reduced expression $\omega'$ of $w$. Therefore, the fraction of permutations for which $G_0^\omega$ is connected tends to 1 with $n \to \infty$, and hence this property is generic. On the other hand, the existence of a reduced expression $\omega$ such that $\sigma^\omega$ contains three consequent indices $i-1$, $i$, $i+1$ at least $k$ times is also generic for any fixed $k$.

**Proposition 6.** *Let $w \in S_n$ possess a reduced expression $\omega$ such that $G_0^\omega$ is connected and $\sigma^\omega$ contains each of the indices $i-1$, $i$, $i+1$ at least 5 times for some $i$, $1 < i < n-2$. Then there exist two nonspecial transversals $\Delta_1, \Delta_2 \subseteq \Gamma_0^\omega$ satisfying $(**)$ and $(***)$.*

To get Theorem 1 from Theorem 4 it remains to observe that $r^\omega = |\Gamma^\omega| - |\Gamma_0^\omega|$ is just the number of unbounded chambers, $r^\omega = n - 1$.

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