STACKED LATTICE BOXES AND THEOREMS OF LIOUVILLE, BELL AND MACMAHON

BY

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This is a brief summary of the work presented in [1] together with some comments about more recent work.

Arithmetically we are considering $L_k(n)$, the number of solutions of

$$n = x_1x_2 + x_2x_3 + x_3x_4 + \cdots + x_kx_{k+1} \tag{1}$$

in positive integers x_i. In the mid-nineteenth century, Liouville developed a mysterious method for studying such problems. I say "mysterious" because he stated all his central theorems without proof. However numerous mathematicians (most notably Humbert, Pepin and Bell) managed to find proofs of the Liouville Theorems. The theorems apparently have their genesis in the theory of elliptic functions (as Humbert [6] observed); however, Pepin [9] was able to develop elementary methods to prove many of Liouville's results.

According to Bell [4], the last result of Liouville [7] to be proved was equivalent to the assertion that

$$L_4(n) = D_2(n) - n\,D_0(n) - D_{0,0}(n)\,, \tag{2}$$

where

$$D_{i_1 i_2 \dots i_r}(n) = \sum_{\substack{j_1+j_2+\cdots+j_r=n \\ j_1,\dots,j_r \geqq 1}} D_{i_1}(j_1)D_{i_2}(j_2)\cdots D_{i_r}(j_r)\,, \tag{3}$$

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and

$$D_i(n) = \sum_{j|n} j^i \, . \tag{4}$$

The phrase "stacked lattice boxes" derives from the fact that a solution of (1) may be viewed as yielding n as the area of the following figure

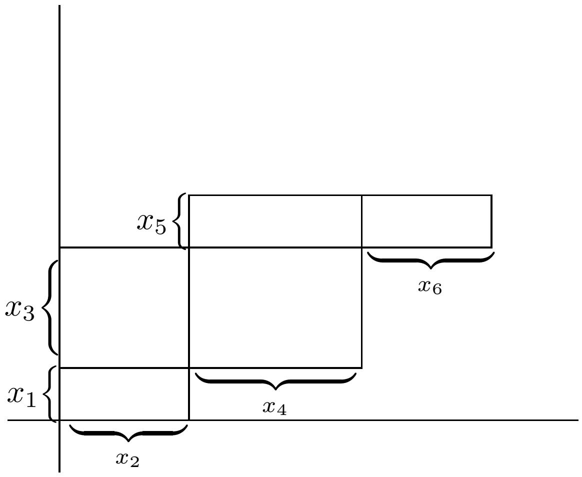

Figure 1

In addition to (2), Liouville's "last" theorem, one can derive fairly easily (with a little help from MacMahon [8]), that

$$L_1(n) = D_0(n) \tag{5}$$

$$L_2(n) = D_1(n) - D_0(n) \tag{6}$$

$$L_3(n) = \frac{1}{2}D_{0,0}(n) + \frac{1}{2}D_0(n) - \frac{1}{2}D_1(n)\, . \tag{7}$$

Our approach to these questions is to use q-series extensively following the ideas of W. N. Bailey [3]. This allows us to prove

$$\begin{aligned} L_5(n) = &\frac{1}{6}D_{0,0,0}(n) + D_{0,0}(n) + \frac{1}{2}D_{0,1}(n) \\ &\left(2n - \frac{1}{6}\right) D_0(n) - \frac{11}{6}D_2(n)\, . \end{aligned} \tag{8}$$

These discoveries dovetail interestingly with the study (begin in [2]) of the q-series:

$$M_k(q) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\binom{n+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^{n-1})(1-q^n)^{k+1}} . \tag{9}$$

Let us denote the generating function for $L_k(n)$ by $\mathcal{L}_k(q)$.

Using the methods of [2], we can show that

$$M_1(q) = \mathcal{L}_1(q) \tag{10}$$

$$M_2(q) = \mathcal{L}_1(q) + \mathcal{L}_2(q) + \mathcal{L}_3(q)\,, \tag{11}$$

and

$$M_3(q) = \mathcal{L}_1(q) + 2\mathcal{L}_2(q) + 3\mathcal{L}_3(q) + 2\mathcal{L}_4(q) + \mathcal{L}_5(q)\,. \tag{12}$$

The table of coefficients in these identities, namely

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & 1 & \\ 1 & 2 & 3 & 2 & 1 \end{array}$$

suggests the trinomial coefficients [5; p. 78]. However if we examine the next suggested identity we find instead

$$\begin{aligned} M_4(q) =& \mathcal{L}_1(q) + 3\mathcal{L}_2(q) + 6\mathcal{L}_3(q) + 7\mathcal{L}_4(q) \\ & + 6\mathcal{L}_5(q) + 3\mathcal{L}_6(q) + \mathcal{L}_7(q) \\ & - (q^7 + 2q^8 + 6q^9 + 11q^{10} + 22q^{11} + 33q^{12} + 57q^{13} + \dots) \end{aligned} \tag{13}$$

So having surveyed the world of the mysterious Liouville identities, we are led to a new “Liouville mystery.”

References

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