Tableaux, hyperplanes, and representations

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Classically, there are two, very different, kinds of tableaux which are important in representation theory:

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 3 & 7 & 10 & 17 \\ \hline
2 & 4 & 8 & 14 & 18 \\ \hline
5 & 11 & 12 \\ \cline{1-3}
6 & 15 & 16 \\ \cline{1-3}
9 \\ \cline{1-1}
13 \\ \cline{1-1}
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 1 & 2 & 2 \\ \hline
2 & 2 & 3 & 3 & 6 \\ \hline
3 & 4 & 4 \\ \cline{1-3}
5 & 5 & 5 \\ \cline{1-3}
6 \\ \cline{1-1}
7 \\ \cline{1-1}
\end{array}$$

(a) standard tableaux and (b) column strict tableaux.

There are connections between them but here we would like to stress that standard tableaux and column strict tableaux come from different sources and serve different purposes in representation theory.

Standard tableaux model the representations of the symmetric group S_n. The main results are:

(S1) The irreducible representations of the symmetric group S_n are indexed by partitions with n boxes.

(S2) The dimension of the irreducible representation indexed by λ is the number of standard tableaux of shape λ.

(S3) The character χ^λ of the irreducible representation of S_n indexed by λ is given by

$$\chi^\lambda(\sigma) = \sum_T \mathrm{wt}^\mu(T),$$

where the sum is over all standard tableaux of shape λ, μ is the cycle type of the permutation σ and $\mathrm{wt}^\mu(T)$ is given by

$$\mathrm{wt}^\mu(T) = \prod_{i=1}^{n} f(i,T),$$

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where

$$f(i,T) = \begin{cases} -1, & \text{if } i \notin B(\mu) \text{ and } i+1 \text{ is sw of } i, \\ 0, & \text{if } i, i+1 \notin B(\mu),\ i+1 \text{ is ne of } i, \text{ and } i+2 \text{ is sw of } i+1, \\ 1, & \text{otherwise,} \end{cases}$$

and $B(\mu) = \{\mu_1+\mu_2+\cdots+\mu_k \mid 1 \le k \le \ell\}$ if $\mu = (\mu_1, \ldots, \mu_\ell)$. In the formula for $f(i,T)$, sw means strictly south and weakly west and ne means strictly north and weakly east. This fundamental formula for the characters of the symmetric group is due to Roichman [Ro] and Fomin and Greene [FG]. See also [Ra1].

Column strict tableaux model the Schur functions and the representations of the general linear group $GL_n(\mathbb{C})$. The main results are

(GL1) The irreducible polynomial representations of the general linear group $GL_n(\mathbb{C})$ are indexed by partitions λ with $\le n$ rows.

(GL2) The dimension of the irreducible representation indexed by λ is the number of column strict tableaux of shape λ filled with elements from the set $\{1, 2, \ldots, n\}$.

(GL3) The character of the irreducible representation of $GL_n(\mathbb{C})$ indexed by λ is the Schur function

$$s_\lambda(x_1, \ldots, x_n) = \sum_T x^T,$$

where the sum is over all column strict tableaux T of shape λ filled with elements from the set $\{1, 2, \ldots, n\}$ and x^T is given by

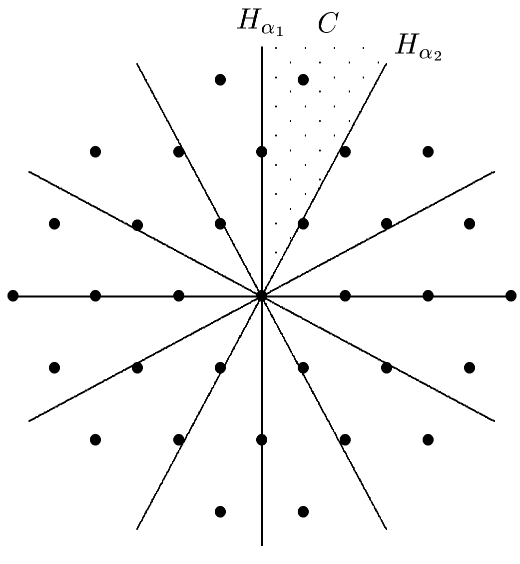
$$x^T = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}, \qquad \text{where } \mu_i = \#\text{ of } i\text{'s in } T.$$

Recently both types of tableaux, standard and column strict, have been generalized to the Weyl group/root system setting. These generalizations both use the same basic combinatorial structure as a starting point. The necessary combinatorial data consists of three things:

(a) A finite group W generated by reflections (in $\mathbb{R}^n$),

(b) A W-invariant lattice P contained in $\mathbb{R}^n$,

(c) The choice of a fundamental chamber C for W.



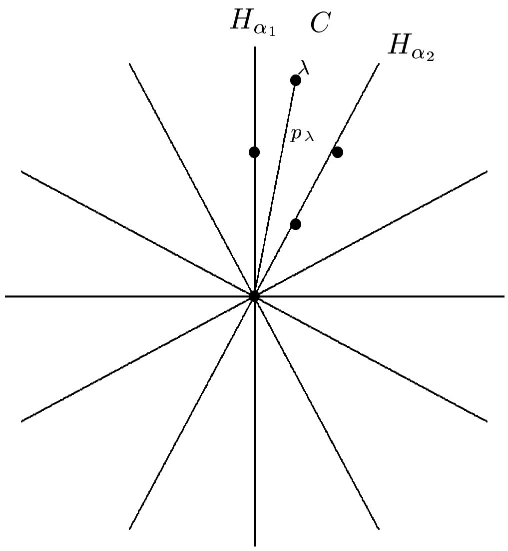
(1)

In this picture W is the group generated by the reflections in the hyperplanes and P is the lattice given by the $\bullet$ dots. In the "Type A" situation, when W is the symmetric group, the generalized standard tableaux defined below can be identified with the classical standard tableaux and the generalized column strict tableaux can be identified with the classical column strict tableaux.

Generalized column strict tableaux

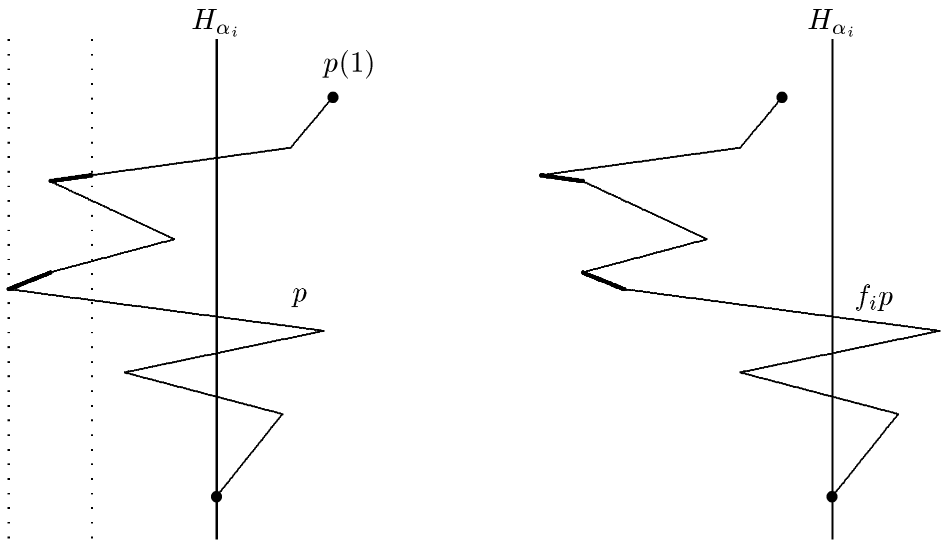
The generalization of column strict tableaux is due to P. Littelmann [Li1-3] and is called the *path model.* There is a path model associated to each symmetrizable Kac-Moody group G and the resulting column strict tableaux model the representations of G. Let us describe these generalized column strict tableaux.

The elements of P are called integral weights and the elements of $P^+ = P \cap \bar{C}$ (where $\bar{C}$ is the closure of the chamber C) are *dominant integral weights.* The *highest weight path* to λ is the straight line path from 0 to λ given by



$$\begin{array}{rccc} p_\lambda\colon & [0,1] & \longrightarrow & \mathbb{R}^n \\ & t & \longmapsto & t\lambda. \end{array}$$

Label the walls of C by $H_{\alpha_1}, \ldots, H_{\alpha_n}$. There is one *root operator* f_i for each wall H_{α_i} of the chamber C and its action on paths is given by



so that the dark parts of the path p are reflected (in a mirror parallel to H_{α_i}) to form the path $f_i p$. The left dotted line is the affine hyperplane parallel to H_{α_i} which intersects the path p at its leftmost (most negative) point (relative to H_{α_i}) and the distance between the dotted lines is exactly the "lattice unit" (distance between lines of lattice points in P parallel to H_{α_i}). The definition of the action of the f_i can also be given by a formula, see [Li3].

The set of *generalized column strict tableaux of shape* λ is the set

$$\mathcal{T}^\lambda = \{f_{i_k} \cdots f_{i_2} f_{i_1} p_\lambda\}$$

consisting of all possible applications of a sequence of root operators to the highest weight path p_λ. Amazingly, the set $\mathcal{T}^\lambda$ is finite! In the literature, the set $\mathcal{T}^\lambda$ is the set of *LS paths*.

The main results are

(G1) The irreducible representations of the group G are indexed by dominant integral weights λ.

(G2) The dimension of the irreducible representation of G indexed by λ is the number of generalized column strict tableaux of shape λ.

(G3) The character of the irreducible representation of G indexed by λ is the Weyl character

$$\chi^\lambda = \sum_p e^{p(1)},$$

where the sum is over all generalized column strict tableaux p of shape λ and $p(1)$ is the endpoint of the path p.

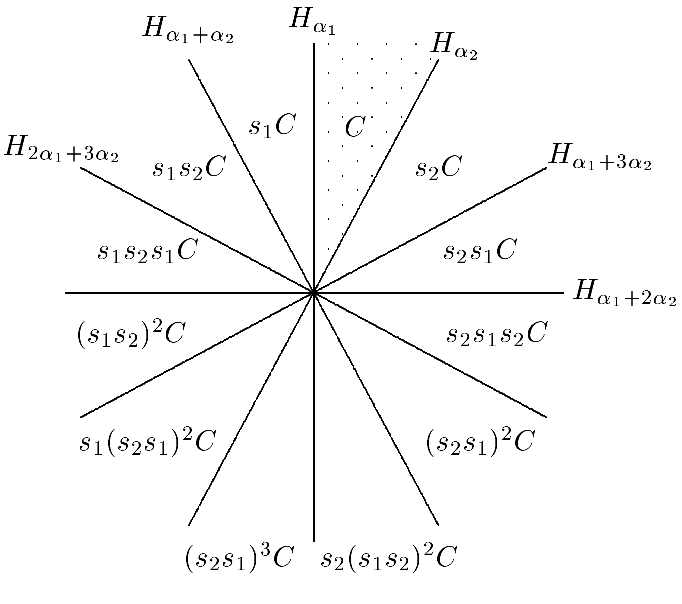
Generalized standard tableaux

The generalized of standard tableaux given in [Ra1-3] model representations of affine Hecke algebras. Let us describe these generalized standard tableaux.

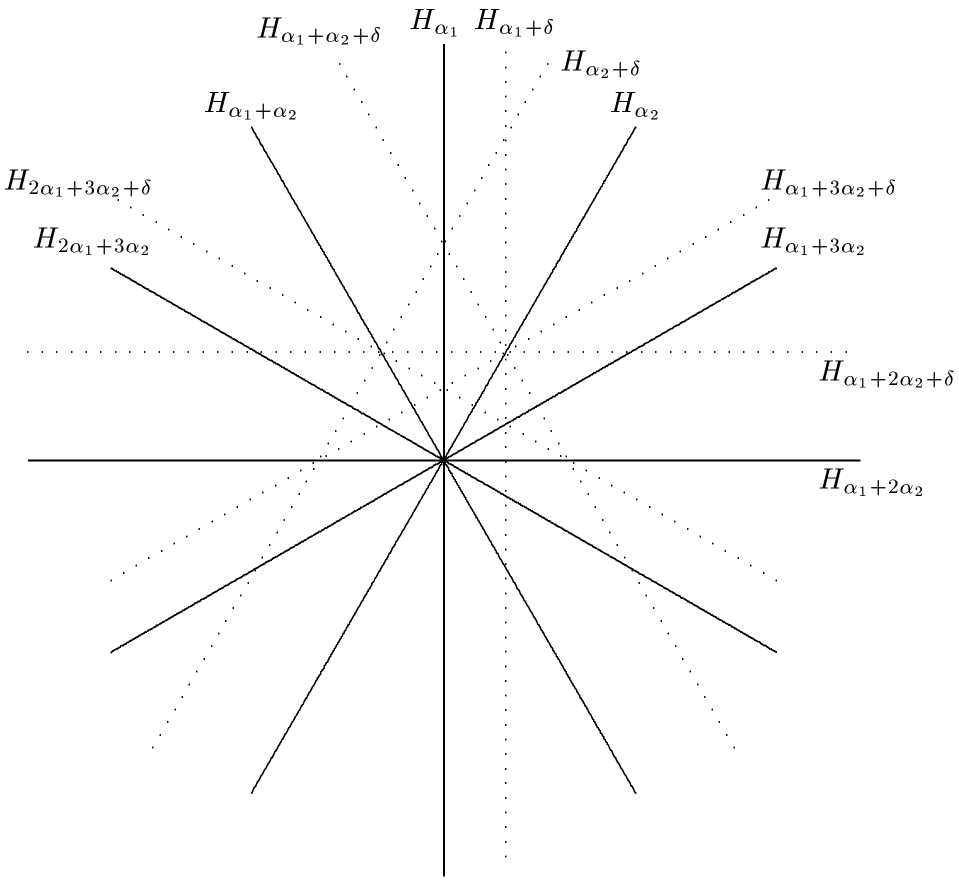
The chambers are in bijection with the elements of W ($w \leftrightarrow wC$). The *inversion set* of an element $w \in W$ is

$$R(w) = \{\alpha \mid wC \text{ is on the negative side of } H_\alpha \},$$

where the positive side of a hyperplane H_α is the side containing the chamber C.



Label the hyperplanes by H_α and label their shifts (by one "lattice unit" in the direction of C) by $H_{\alpha+\delta}$.



Suppose that γ is an element of $\bar{C}$ (the closure of the chamber C) and define

$$Z(\gamma) = \{\alpha \mid \gamma \text{ is on } H_\alpha \} \qquad \text{and} \qquad P(\gamma) = \{\alpha \mid \gamma \text{ is on } H_{\alpha+\delta} \}.$$

A *placed shape* is a pair (γ, J) where $\gamma \in \bar{C}$ and $J \subseteq P(\gamma)$. A *placed skew shape* is a placed shape which satisfies a simple but rather technical combinatorial condition which is not important to make precise here (see [Ra2]). The set of *generalized standard tableaux of shape* (γ, J) is

$$\mathcal{F}^{(\gamma,J)} = \{w \in W \mid R(w) \cap Z(\gamma) = \emptyset,\ R(w) \cap P(\gamma) = J\},$$

where $R(w)$ is the inversion set of w.

The main results are

(H1) The irreducible calibrated representations of the affine Hecke algebra $\tilde{H}$ are indexed by placed skew shapes (γ, J).

(H2) The dimension of the irreducible representation of the affine Hecke algebra $\tilde{H}$ indexed by the placed skew shape (γ, J) is the number generalized standard tableaux of shape (γ, J).

(H3) The character of the irreducible representation of the affine Hecke algebra $\tilde{H}$ indexed by the placed skew shape (γ, J) is

$$\chi^{(\gamma,J)} = \sum_w e^{w\gamma},$$

where the sum is over all generalized standard tableaux w of shape (γ, J).

Is there a relation between them?

One of the main open questions is whether there is a connection between the generalized column strict tableaux and the generalized standard tableaux. A priori there seems to be none whatsoever. On the other hand, recent work with H. Pittie [PR] has shown that the LS paths *are* connected to the affine Hecke algebra, so maybe there is a possibility of making this connection.

The LS paths are connected to the affine Hecke algebra as follows. The ∞-affine Hecke algebra $\tilde{H}(\infty)$ has two natural bases

$$\{X^\lambda T_w \mid \lambda \in P, w \in W\} \qquad \text{and} \qquad \{T_v X^\mu \mid \mu \in P, v \in W\}.$$

The formula in [PR] is, if $\lambda \in P^+$, $w \in W$,

$$X^\lambda T_w = \sum_{p \in \mathcal{T}^\lambda_{\leq w}} T_{v(p,w)^{-1}} X^{p(1)},$$

where $\mathcal{T}^\lambda_{\leq w}$ is the set of paths in $\mathcal{T}^\lambda$ with initial direction $\leq w$,
$p(1)$ is the endpoint of the path p, and
$v(p,w)^{-1}$ is the final direction of the path p (with respect to w).

This shows that multiplication in the affine Hecke algebra can be controlled with the combinatorics of LS paths!

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