Tiling groups and algorithms for tiling

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We present algorithms for tiling a simply connected region of the plane (*i.e.* polygonal region without holes) with various kinds of tiles. Many of the algorithms use the concept of tiling group of J. H. Conway and Lagarias [1], and ideas of W. P. Thurston on using the geometry of the group [12]. The concept of tiling groups is best illustrated by examples.

The first problem studied using tiling groups is the domino tiling problem. Given a simply connected polyomino P (polygon in $\mathbf{R}^2$, with vertices in $\mathbf{Z}^2$, horizontal or vertical edges, and no "holes"), find a tiling of P using dominoes, *i.e.* 1×2 and 2×1 rectangles, if it exists.

A moment's thought shows that the problem can be reduced *via* duality to perfect matching. However in [1] Conway and Lagarias used a radically different approach by introducing the *domino tiling group*. Let a represent a horizontal step to the right, a^{-1} a horizontal step to the left, b a vertical step going up, and b^{-1} a vertical step going down. Then the perimeters of the dominoes, read in a counterclockwise order from the lower left corner, correspond to the words $a^2ba^{-2}b^{-1}$ and $ab^2a^{-1}b^{-2}$. The perimeter of any simply connected polyomino P corresponds to a word $w(P)$.

Let $F = \langle a, b | \rangle$ denote the free group on two generators and in general let $\langle A|R\rangle$ denote the group generated by the elements of A with relations $r = e$ for $r \in R$. (Thus the discrete plane $\mathbf{Z}^2$ has the group presentation $\mathbf{Z}^2 = \langle a, b|aba^{-1}b^{-1}\rangle$). The domino tiling group is defined as

$$G = \langle a, b|a^2ba^{-2}b^{-1}, ab^2a^{-1}b^{-2}\rangle.$$

It is easy to see that if P is tileable by dominos, then $w(P)$ is equal to the identity in G [1]. Moreover, by studying the geometry of the Cayley graph of G, Thurston obtained an efficient and elegant algorithm for deciding if P is tileable and constructing a tiling if one exists [12].

The same approach can be used for other tiling problems.

- Tiling with "bars", $1 \times m$ and $m \times 1$ rectangles with m fixed [2].
- Tiling with $m \times n$ and $n \times m$ rectangles with n, m fixed [2].
- When the tiling group is equal to $\mathbf{Z}^2$, it suggests that the corresponding tiling problem may be "easy". This is confirmed in the case of tiling with "bibones" [3].
- When the underlying planar lattice is the triangular lattice, one can also use tiling groups to get tiling algorithms for lozenges, "leaning dominoes" and other families of tiles [12, 10].
- One can also use the tiling group to obtain some information on tiling with "tetrominos" [9, 8], on tiling with "tribones" [12], and on the tileability of a planar polygon with squares [7].
- The tiling group approach can be used to give a new proof of the result that if a rectangle R is tiled with rectangles, each having at least one side of integral length, then R has a side of integral length [5].

- A generalization of the notion of tiling group can also be used to construct an example of a tiling problem which is not "Escher-equivalent" to any problem in which all the tiles are polygonal [4].

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