Connected heaps are nice animals

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Abstract

The general quest of this paper (and a few others...) is the exact enumeration of large classes of square lattice animals. More than 10 years ago, it was understood that *directed* animals can be conveniently described in terms of *heaps of dimers*. The number of n-celled directed animals grows like 3^n. We define in this paper a set of heaps — called *connected* heaps — that are in one-to-one correspondence with a large class of animals. We obtain a functional equation for their generating function. An analysis of this equation shows that their number grows like $(3.58...)^n$. We find exact solutions for several subclasses, containing directed animals. One of them has an algebraic generating function and growth constant 3.5: this is the largest class of animals ever counted exactly. We obtain similar results for triangular lattice animals.

Résumé

L'objet général de cet article (et de quelques autres...) est l'énumération exacte de grandes classes d'animaux sur réseau carré. Il y a plus de 10 ans, l'introduction des empilements de dominos est venu éclairer considérablement les résultats connus auparavant sur les animaux *dirigés*. Rappelons que le nombre d'animaux dirigés à n cellules croît comme 3^n. Nous définissons ici une famille d'empilements — dits *connexes* — qui sont en bijection avec une "grande" famille d'animaux : leur nombre croît comme $(3.58...)^n$. Nous obtenons une équation fonctionnelle qui régit leur série génératrice. Nous énumérons exactement plusieurs sous-classes, contenant toutes les animaux dirigés. L'une d'elles donne une série génératrice algébrique. Les nombres correspondants croissent comme 3.5^n: c'est la plus vaste classe d'animaux jamais énumérée exactement. Nous obtenons des résultats analogues sur le réseau triangulaire.

1 Introduction

Many mathematical models of physical phenomena have a combinatorial nature. Perhaps two of the most famous of these are the self-avoiding walk and polygon models of polymers. A related problem – which also has a physical interpretation of its own [9] – is that of counting *polyominoes*. A *polyomino* is a finite connected union of cells on a lattice. In this paper we consider polyominoes with hexagonal and square cells. If we replace each cell of a polyomino by a vertex at its centre, we obtain the corresponding *animal*, which lives on the dual lattice (Fig. 1).

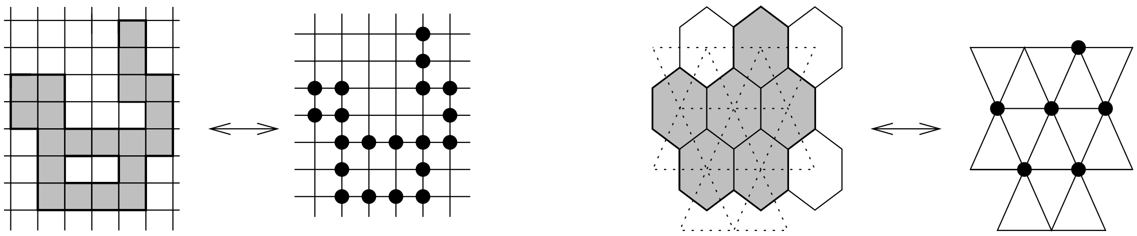

Figure 1: Polyominoes with square and hexagonal cells, and the corresponding animals.

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Although polyominoes have been intensively studied for more than 40 years [10, 11, 13], exact results on general polyominoes have remained elusive. However, some asymptotic results are known. Let c_n denote the number of polyominoes of n cells on the square lattice. A concatenation argument [14] shows that there exists a constant μ, called the *connective constant*, such that

$$\lim_{n\to\infty}(c_n)^{1/n} = \mu.$$

The exact value of μ is unknown, though numerical studies [6] have shown that $\mu \simeq 4.06$. The best published[1] bounds on μ are [16]

$$3.72 < \mu < 4.65.$$

It is a measure of the complexity of the problem that not even the first digit of μ is known rigorously.

Given the difficulty in solving this problem, what rigorous work can be done towards better understanding polyominoes? Perhaps the most fruitful work has been in the investigation of large subclasses of polyominoes.

All the subclasses of polyominoes that have been solved so far have had at least one of the following two properties: *convexity* or *directedness*. Convexity is now well understood (*e.g.* [2]), and the largest subclass of polyominoes having a convexity property (*column convex* polyominoes) has been enumerated; it is understood that convexity will limit the connective constant to a maximal value of 3.20... (on the square lattice). The largest class of directed polyominoes has also been solved and it is understood that this property limits the connective constant to be at most 3 (see Table 1 for details).

Model	μ	Nature of the series	Who solved it (first...)
Rectangles	1	q-series	obvious...
Ferrers Diagrams (Partitions)	1	q-series	Euler [8]
Stacks	1	q-series	Auluck [1]
Staircase (Parallelogram)	2.30...	q-series	Klarner & Rivest [17]
Directed Convex	2.30...	q-series	Bousquet & Viennot [4]
Convex	2.30...	q-series	Bousquet & Fédou [3]
Bargraph (Compositions)	2	rational	obvious...
Directed Column Convex	2.62...	rational	Moser, Klarner [13]
Column Convex	3.20...	rational	Temperley [18]
Directed	3	algebraic	Dhar [7]

Table 1: Some of the solved subclasses of square lattice polyominoes, and their connective constants.

So to be able to enumerate larger classes of polyominoes, classes without the properties of convexity and directedness need to be investigated.

In this paper we investigate some new larger classes of polyominoes with square and hexagonal cells. These classes are conveniently described in terms of *heaps of dimers*. Heaps have already proved extremely useful in the enumeration of directed animals: we recall in Section 2 the main definitions and results. We define in Section 3 *connected* heaps, and show they are in one-to-one correspondence with a class of animals. We also give a functional equation defining their generating function. The associated connective constant is about 3.58, but we have not been able to evaluate it exactly yet. However, we find an algebraic generating function for a subclass of connected heaps having a growth constant of 3.5. This is the largest class of animals ever enumerated (Section 4). We give in the last section partial asymptotic information about the generating function of connected heaps. We solve two functional equations related to that of connected heaps.

All our square lattice results have triangular analogues.

2 Heaps of dimers and directed animals

A square lattice animal is said to be *directed* if all its vertices can be reached from a specific vertex, called the *root*, by a path formed of North and East steps that only visits vertices of the animal. A similar notion

[1] Klarner [15] announced to have obtained a better lower bound of 3.9.

exists for triangular lattice animals (Fig. 2). These two models were first solved by Dhar [7] and are now well understood. Perhaps the simplest solution is based on the fact that these animals are *heaps of dimers*.

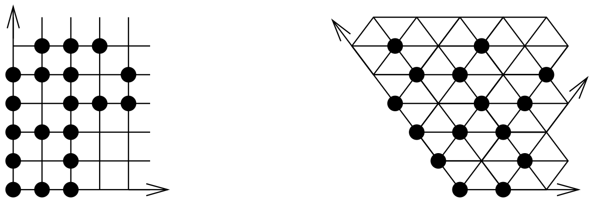

Figure 2: Directed animals on the square and triangular lattices.

The notion of heaps is a nice geometric version of partially commutative monoids, due to Viennot [19]. Intuitively, a heap of dimers is obtained by dropping a finite number of dimers until each of them falls either on the axis, or on another dimer (Fig. 3.*a*). The heap is *strict* if no dimer has another dimer exactly above it (Fig. 3.*b*). The dimers that touch the axis are *minimal*. If there is only one minimal dimer, the heap is a *pyramid*. If, moreover, this minimal dimer is the rightmost one, the heap is a *demi-pyramid*.

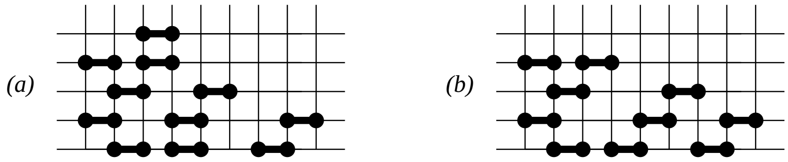

Figure 3: Two heaps of dimers; each has three minimal dimers.

It was observed by Viennot that directed animals on the square (resp. triangular) lattice are in one-to-one correspondence with *strict* pyramids of dimers (resp. pyramids of dimers). This correspondence is simply obtained by rotating the animal so that the preferred direction becomes North, and replacing each vertex by a dimer (Figs. 4 and 5).

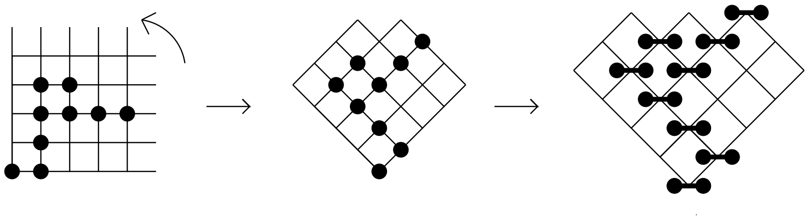

Figure 4: A directed animal on the square lattice and the associated strict pyramid.

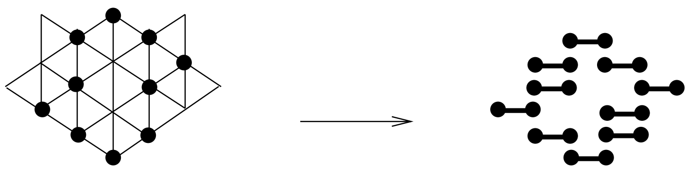

Figure 5: A directed animal on the triangular lattice and the associated pyramid.

The reason why using the notion of heap is interesting here is that there exists a *monoid structure* on the set of heaps (the product of two heaps is obtained by putting one heap above the other and dropping its pieces). This yields unambiguous factorizations for heaps of dimers.

Let us begin with the factorization of strict pyramids. A strict pyramid is either a strict demi-pyramid or the product of a strict demi-pyramid and a strict pyramid (Fig. 6). Let $P_s(x)$ denote the generating function for strict pyramids, counted according to the number of dimers, and let $D_s(x)$ denote the generating function for strict demi-pyramids. Then $P_s(x) = D_s(x)(1 + P_s(x))$. Now, a demi-pyramid having several dimers is the product of a single dimer and either one or two demi-pyramids (Fig. 7), which implies $D_s(x) = x + xD_s(x) + xD_s(x)^2$.

These equations are readily solved yielding:

$$D_s(x) = \frac{1 - x - \sqrt{(1+x)(1-3x)}}{2x} \quad \text{and} \quad P_s(x) = \frac{1}{2}\left(\sqrt{\frac{1+x}{1-3x}} - 1\right).$$

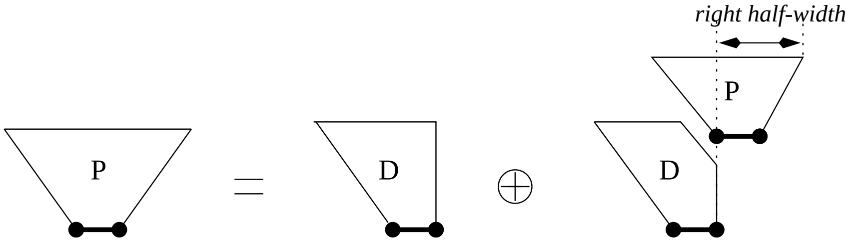

Figure 6: The factorization of pyramids.

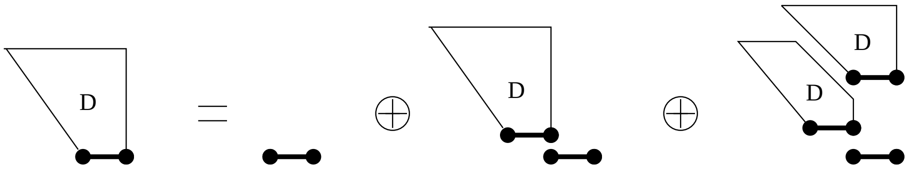

Figure 7: The factorization of demi-pyramids.

We can obtain analogous results for non-strict pyramids of dimers (*i.e.*, directed animals on the triangular lattice) by a similar factorisation. Alternatively, we can use a nice mapping between strict and ordinary heaps: To obtain an ordinary heap from a strict heap, we have to allow dimers to be placed directly on top of each other. We can do this by replacing each dimer with a column of dimers (Fig. 8).

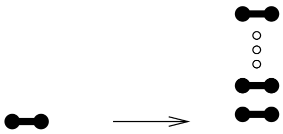

Figure 8: From strict heaps to ordinary heaps.

This corresponds to replacing the variable x in our generating functions by $\frac{x}{1-x}$. Using this mapping (and its inverse) we can take any strict heap (square lattice) result and easily find the corresponding ordinary heap (triangular lattice) result, and vice-versa. For instance, the generating function for directed animals on the triangular lattice is simply given by $P_t(x) = P_s(x/(1-x))$.

The above factorization of (strict) pyramids allows us to take into account an additional parameter, which will play an important role in Section 4. This parameter is the *right half-width* of the pyramid, which we define as the number of columns of the heap to the right of the minimal dimer (Fig. 6). For instance, a single column of dimers has right half-width 0. We thus obtain the following refined result (the first part of it was implicit in [12]).

Proposition 1 *Let $P_s(x,w)$ (resp. $P_t(x,w)$) be the generating function for strict pyramids (resp. pyramids), counted according to the number of dimers (x) and the right half-width (w). Then*

$$P_s(x,w) = \frac{1}{2}\,\frac{\sqrt{(1+x)(1-3x)} - 1 + x(1+2w)}{w - x(1+w+w^2)}$$

and

$$P_t(x,w) = P_s\left(\frac{x}{1-x}, w\right) = \frac{1}{2}\,\frac{\sqrt{1-4x} - 1 + 2x(1+w)}{w - x(1+2w+w^2)}.$$

3 Connected heaps of dimers

A heap of dimers is *connected* if its projection onto the horizontal axis is connected. For instance, the heap of Fig. 3.b is connected, while that of Fig. 3.a is not. We define a bijection between (strict) connected heaps of dimers and a class of polyominoes on the (square) triangular lattice as follows: we first identify the polyomino components of the heap (Fig. 9.2). To connect the polyominoes, one then starts from the rightmost and works towards the left, lifting each polyomino that has another above it until all the polyominoes become connected (Figs. 9.2 and 9.3). The reverse bijection is obtained by transforming the polyomino back into a collection of dimers and then allowing the dimers to fall back down towards the axis.

The class of hexagonally celled polyominoes defined by connected heaps of dimers is in fact equivalent to a family of polyominoes studied by Klarner [14]. His numerical study yielded the lower bound 4 for the corresponding connective constant. We have been able to improve upon the accuracy of Klarner's estimate both by analysis of a functional equation that describes connected heaps, and also by the exact enumeration of large subclasses of connected heaps.

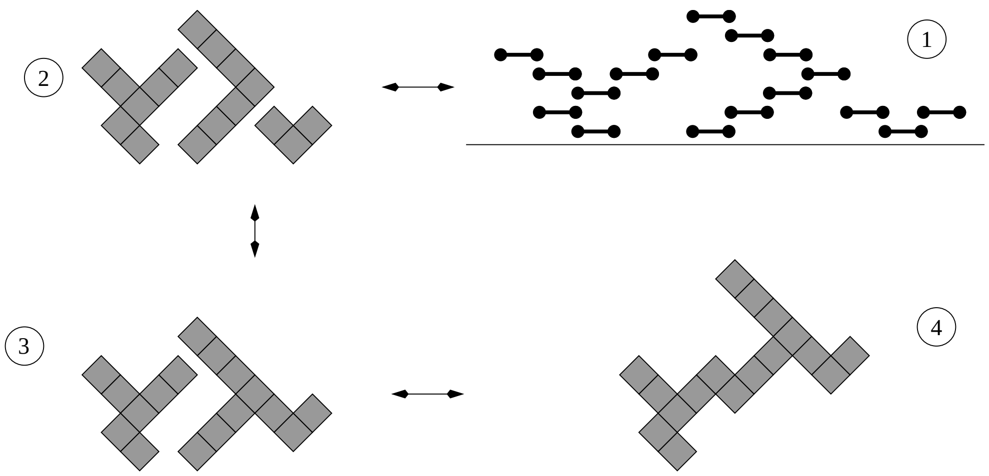

Figure 9: (1) A strict connected heap of dimers and its corresponding polyomino (4). Steps (2) and (3) show the intermediate steps of the bijection.

Proposition 2 *Let $S(x,y)$ (resp. $T(x,y)$) be the generating function for strict (resp. general) connected heaps, where the variable y counts the number of dimers in the rightmost column, and x counts the remaining dimers. Then $S(x,y)$ and $T(x,y)$ satisfy the following functional equations:*

$$\begin{aligned} S(x,y) &= y + (1+y)\,S\left(x, \frac{x(1+y)}{1-xy}\right) - S(x,x), \\ T(x,y) &= \frac{y}{1-y} + \frac{1}{1-y}\,T\left(x, \frac{x}{1-y}\right) - T(x,x). \end{aligned}$$

As the mapping between heaps and strict heaps implies, we observe that $T(x,y) = S(\frac{x}{1-x}, \frac{y}{1-y})$.

Proof. One can interpret these equations as the description of the growth of a connected heap column by column, and explain them graphically. We do so here for general connected heaps (series $T(x, y)$).

A connected heap H is either a single column of dimers ($= \frac{y}{1-y}$) or has width $m > 1$. In this case, it can be grown from a heap H' of width $m - 1$ by adding cells in the mth column. More precisely (see Fig. 10):

1. Insert a column of dimers to the above-right of each dimer in the rightmost column of H'. This is represented by the substitution $y \longmapsto \frac{x}{1-y}$.
2. Insert a column of dimers to the below right of the lowest rightmost dimer of H'. The bottom dimer of this column will form a new minimal piece in the heap. This operation is represented by multiplying by $\frac{1}{1-y}$.

We must avoid counting the cases where we have not added any new dimers to H', which is done by subtracting $T(x, x)$. ∎

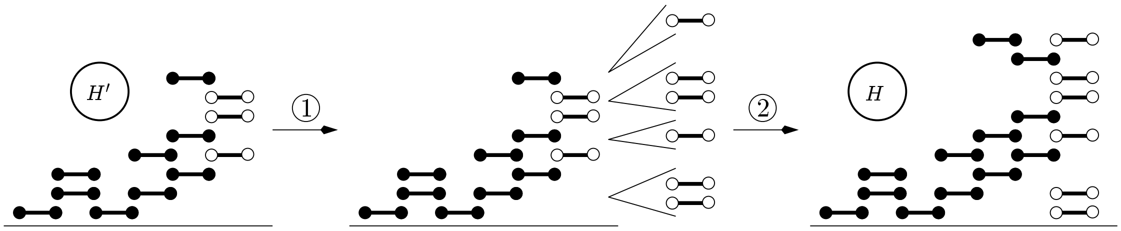

Figure 10: A recursive construction for connected heaps of dimers.

We have so far been unable to solve either functional equation of Proposition 2. Nor have we been able to find exactly the radius of convergence of the solution at $y = x$, which is identical to the reciprocal of the connective constant (about 3.58 for the square lattice and 4.58 for the triangular lattice). However, we solve exactly, in the next section, a subclass of connected heaps whose connective constant is exactly 3.5 (and 4.5 for the triangular lattice).

4 Multi-pyramids

Any heap of dimers can be factored into pyramids by successively pushing upwards the leftmost minimal piece. Ideally, we would like to be able to use this factorisation to obtain the generating function for connected heaps. However, there exist connected heaps for which the removal of a pyramid factor disconnects the remaining heap (Fig. 11). Because of this we cannot define connected heaps recursively in terms of pyramids.

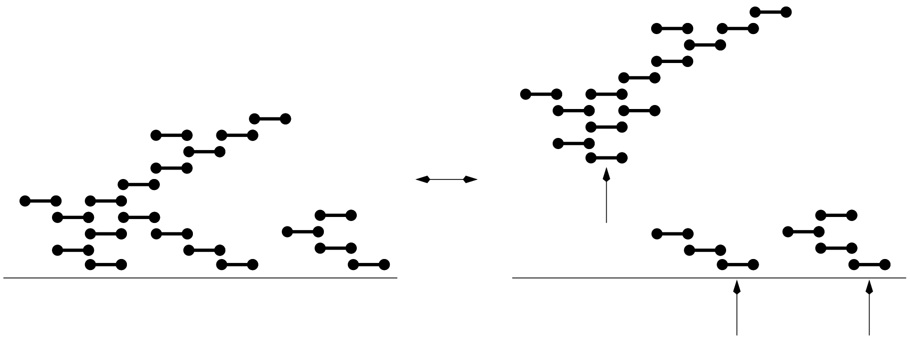

Figure 11: A connected heap and its pyramid factors. Note that after the leftmost pyramid factor is removed, the remaining heap is no longer connected.

However, we can define an exactly enumerable subclass of connected heaps by requiring that the heap remains connected upon the removal of successive pyramid factors (Fig. 12). We call this subclass *multi-pyramids*.

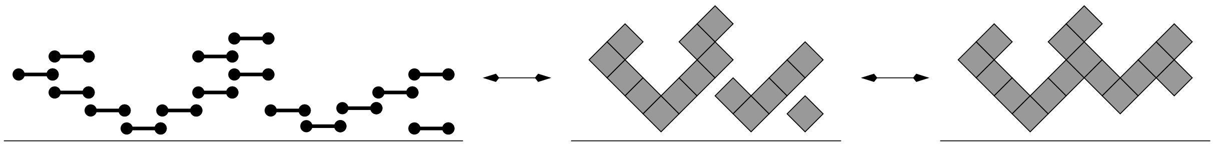

Figure 12: A multi-pyramid formed from three pyramids of dimers.

Each multi-pyramid is (see Fig. 13):

- either a single pyramid (if it has only a single minimal piece).
- or the product of a pyramid and a multi-pyramid. Note that conversely, the pyramid factor can be placed in a number of ways equal to its right half-width.

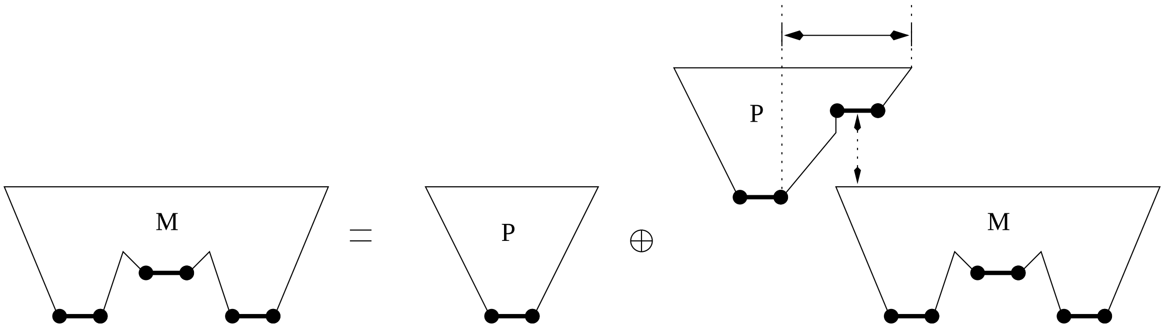

Figure 13: The recursive factorisation of multi-pyramids. Each new pyramid factor can be added to a multi-pyramid in a number of ways equal to its right half-width.

Proposition 1 gives the generating functions $P_s(x,w)$ and $P_t(x,w)$ for strict pyramids and general pyramids respectively, counted by their size and right half-width. The above factorisation of multi-pyramids yields, for the square lattice,

$$M_s(x) = P_s(x,1) + M_s(x)\frac{\partial P_s}{\partial w}(x,1),$$

where $M_s(x)$ denotes the generating function for strict multi-pyramids. A similar equation holds for the triangular lattice.

Proposition 3 *The generating functions for strict multi-pyramids and general multi-pyramids are respectively:*

$$\begin{aligned} M_s(x) &= \frac{1}{2x}\frac{(1-x)(1-2x)-(1-4x)\sqrt{(1-3x)(1+x)}}{2-7x}, \\ M_t(x) &= \frac{1}{2x}\frac{(1-2x)(1-3x)-(1-5x)\sqrt{1-4x}}{2-9x} = M_s\left(\frac{x}{1-x}\right). \end{aligned}$$

Hence the connective constants for multi-pyramids on the square and triangular lattices are 3.5 *and* 4.5 *respectively.*

The class of multi-pyramids is the largest class of lattice animals for which an exact generating function is known. We have to qualify what is meant by this last statement; the current lower bound of 3.9 for the connective constant of square lattice polyominoes is calculated by enumerating polyominoes of bounded column height by a transfer matrix method [15]. In such studies, only an estimate of the largest eigenvalue of the transfer matrix is usually published, and not the corresponding rational generating function, whose degree grows exponentially with the bounding height.

5 Asymptotic results and saw-tooth connected heaps

In this remaining section we discuss exclusively classes of animals on the triangular lattice and their corresponding heaps. Recall the functional equation for connected heaps:

$$T(x,y) = \frac{y}{1-y} + \frac{1}{1-y}\, T\left(x, \frac{x}{1-y}\right) - T(x,x). \tag{1}$$

Let us consider the series $T(x,xy)$: it is a series in x with polynomial coefficients in y. We are mainly interested in the case $y=1$. From (1), we have been able to determine the shape of the singularity diagram for the function $T(x,xy)$. A nearly identical result exists for the generating function $S(x,xy)$ counting strict connected heaps.

Proposition 4 *Let $\rho(y)$ denote the radius of convergence of $T(x,xy)$ for y fixed. Then there exists $\rho_c \le 1/4$ such that*

$$\rho(y) = \begin{cases} \rho_c & \text{for } 0 < y \le y_c, \\ 1/y - 1/y^2 & \text{for } y_c < y, \end{cases}$$

where

$$y_c = \frac{1+\sqrt{1-4\rho_c}}{2\rho_c} \ge 2 .$$

In particular $\rho(1) = \rho_c$.

Fig. 14 is a schematic plot of the singularity diagram of $T(x,xy)$.

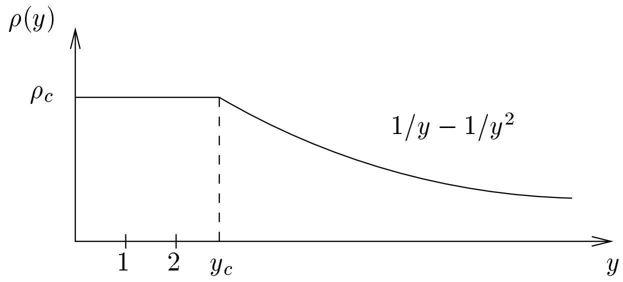

Figure 14: Schematic plot of the singularity diagram of $T(x,xy)$.

We have made numeric estimates of $\rho(y)$, using the ratio of successive coefficients in the expansion of $T(x,xy)$ in powers of x. From such analysis we have found that $\rho_c \simeq 0.218$, and hence $\mu \simeq 4.587$ and $y_c \simeq 3.114$. See Fig. 15.

Let us now discuss two variations of the functional equation (1). By removing the term that corresponds to the creation of new minimal pieces from this equation, we obtain a functional equation for demi-pyramids of dimers (pyramids whose minimal dimer is the leftmost one):

$$D_t(x,y) = \frac{y}{1-y} + D_t\left(x, \frac{x}{1-y}\right) - D_t(x,x). \tag{2}$$

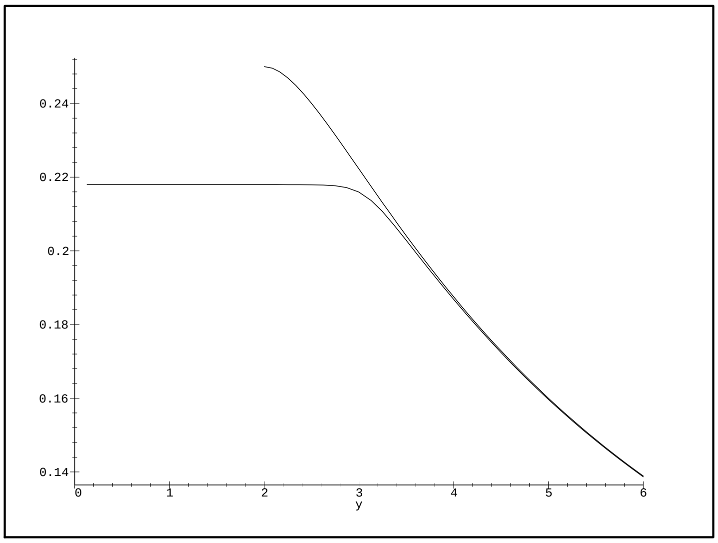

Figure 15: Plot of $\rho(y)$ for connected heaps. Plotted are numerical data (the ratio the 79^{th} and 80^{th} coefficients in the expansion of $T(x, xy)$ in powers of x), and the function $1/y - 1/y^2$, for $y \geq 2$, which is the predicted value of the radius for $y \geq y_c = 3.114...$

Again, we are able to determine that the shape of the singularity diagram of $D_t(x, xy)$ is identical to that given by Proposition 4. Moreover, because we know $\rho(1) = \rho_c = 1/4$ from the solution of demi-pyramids (see Section 2), the singularity diagram is completely determined.

We can iterate Eq. (2), which yields the following closed form expression of $D_t(x, y)$:

$$D_t(x,y) = \sum_{n \geq 1} \frac{x^{n-1} y}{P_n[P_n - yP_{n-1}]} \tag{3}$$

where $P_n = P_n(x)$ is the matching polynomial of a line with n vertices. Equivalently, the generating function for these polynomials is

$$\sum_{n \geq 0} P_n(x) t^n = \frac{1}{1 - t + t^2 x}. \tag{4}$$

Expression (3) is a simple extension of the known enumeration of Dyck paths of fixed height [5]. The singularity diagram of $D_t(x, xy)$ can be directly derived from (3).

The functional equations for demi-pyramids and connected heaps are very similar, excepting the factor that corresponds to the generation of new minimal pieces, $\frac{1}{1-y} = 1 + y + y^2 + \cdots$. One way of extending our solution for demi-pyramids towards a solution for connected heaps would be to replace this factor with only the first few terms of its expansion, that is, only allowing short columns to be inserted. By doing so we would obtain the functional equation for a class of heaps that interpolates between the demi-pyramid and connected heap models.

Exploring this idea, we investigated the following functional equation:

$$N(x,y) = \frac{y}{1-y} + (1+y) N\left(x, \frac{x}{1-y}\right) - N(x,x). \tag{5}$$

The lower edge of the animals described by the solution of this equation is constrained so that (when drawn from left to right) it may only grow downwards diagonally, but is able to grow straight up, something like

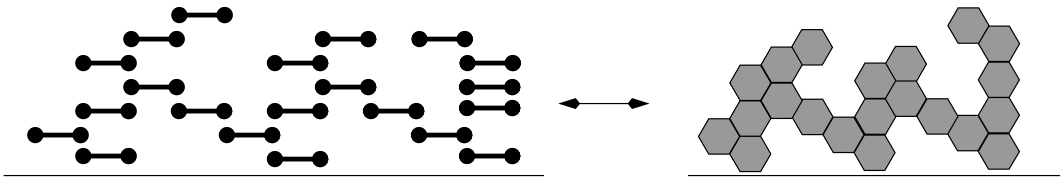

Figure 16: An example of a sawtooth connected heap.

the edge of a saw-tooth. Because of this similarity, we call these animals *saw-tooth animals*, and denote their generating function by N (Fig. 16).

Once again, the functional equation satisfied by $N(x, y)$ implies that the singularity diagram of $N(x, xy)$ is identical to that given in Proposition 4. By iterating (5), we obtain the following expression for $N(x, y)$:

$$N(x,y) = \sum_{n \geq 1} \frac{x^{n-1}y}{P_n - yP_{n-1}} \prod_{j=1}^{n-1} \left(1 + \frac{P_j}{P_{j+1}}\right)$$

where the polynomials P_n are given by (4). From this expression, we can completely determine the shape of the singularity diagram for $N(x, xy)$. We find that $\rho_c = \rho(1) = \sqrt{5} - 2$, so that the connective constant for saw-tooth animals is $\sqrt{5} + 2 \simeq 4.236$.

The method that allowed us to obtain this result looks interesting. We plan to apply it to other functional equations that interpolate between directed animals and connected heaps. In the limit, it would yield the exact value of the connective constant for connected heaps.

Another plan is the investigation of the average width of saw-tooth animals and connected heaps: for n-celled directed animals on the triangular lattice, it is known to grow like $\sqrt{n}$, and it would be interesting to know whether saw-tooth animals and connected heaps have similar behaviour.

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