Noncommutative Ribbons and Quasi-differential Operators

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Abstract

This paper is devoted to a noncommutative generalisation of a classical result occuring in the context of the modular representation theory of the symmetric group. We prove that a noncommutative Schur ribbon function R_I is annihilated by the quasi-differential operator D_{P_k} if and only if the composition I is the external border of a k-core.

1 Introduction

Ordinary Schur functions can be interpreted as the Frobenius characteristics of the irreducible representations of the symmetric group in characteristic 0. When one wants to work with modular representations of the symmetric group, things become much more complicated since the algebra of the symmetric group does not remain semisimple in this new context. One can however show that the two Grothendieck rings naturally associated with irreducible and projective indecomposable modules of the symmetric group in characteristic k are respectively isomorphic to

$$Sym/\mathcal{I}_k \quad \text{and} \quad Sym_{-k}$$

where $\mathcal{I}_k$ denotes the ideal of the algebra of symmetric functions Sym which is generated by the power sums indexed by a multiple of k and where Sym_{-k} denotes the subalgebra of Sym generated by the power sums which are indexed by a non-multiple of k. The Schur functions that belong to Sym_{-k} are of special interest since they are exactly the Frobenius characteristics of the Specht modules. One can show that these Schur functions are characterized by the fact that they are indexed by k-cores.

This paper is intented as a first step towards the generalization to noncommutative symmetric functions of the previous framework. We indeed show that the good noncommutative analogues of the Schur functions, i.e. the so called noncommutative ribbon Schur functions, belong to a noncommutative analogue of Sym_{-k} (coming from Lazard's elimination theorem) iff they are indexed by a composition which is the border of a k-core.

The main problem which remains clearly open would be to find a good representation theoretic interpretation of such a result. Since the representation theoretic interpretation of noncommutative symmetric functions is given by the 0-Hecke algebra, there is certainly some two parameter (q and t) deformation of the Hecke algebra where one should both consider degeneracies at $q = 0$ and at t a k-root of unity that would give us the required representation theoretic interpretation of our work. This question is unfortunately still open !

2 Preliminaries

2.1 Noncommutative symmetric functions

The algebra of *noncommutative symmetric functions* defined in [G-T] is the free associative algebra

$$\mathbf{Sym} = \mathbb{C}\langle S_1, S_2, \ldots \rangle$$

generated by an infinite sequence of noncommutative indeterminates S_k, called the *complete* symmetric functions. It is convenient to set $S_0 = 1$.

Let t be another indeterminate commuting with the S_k. If one introduces the generating series

$$\sigma(t) := \sum_{k \geq 0} S_k t^k,$$

it is possible to define other families of noncommutative symmetric functions by the following relations :

$$\begin{cases} \lambda(t) = \sigma(-t)^{-1}, \\ \frac{d}{dt}\sigma(t) = \sigma(t)\psi(t)\ , \quad \sigma(t) = \exp(\phi(t)), \end{cases}$$

where $\lambda(t), \psi(t), \phi(t)$ are the generating series given by

$$\lambda(t) := \sum_{k\geq 0} \Lambda_k t^k,$$

$$\psi(t) := \sum_{k=1}^{\infty} \Psi_k t^{k-1}\ , \qquad \phi(t) := \sum_{k=1}^{\infty} \frac{\Phi_k}{k} t^k.$$

The noncommutative symmetric functions Λ_k are called *elementary* symmetric functions and Ψ_k and Φ_k are respectively called *power sums* of *first* and *second kind.*

The algebra **Sym** can also be endowed with a Hopf algebra structure. Its coproduct Δ is defined by any of the following equivalent formulas:

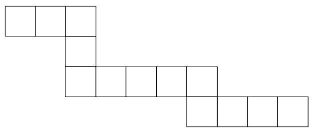
$$\Delta(S_n) = \sum_{k=0}^{n} S_k \otimes S_{n-k}, \qquad \Delta(\Lambda_n) = \sum_{k=0}^{n} \Lambda_k \otimes \Lambda_{n-k},$$

$$\Delta(\Psi_n) = 1 \otimes \Psi_n + \Psi_n \otimes 1, \qquad \Delta(\Phi_n) = 1 \otimes \Phi_n + \Phi_n \otimes 1.$$

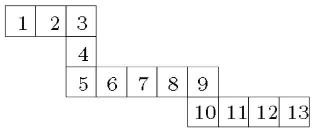
The free Lie algebra generated by the family $(\Phi_n)_{n\geq 1}$ or equivalently by $(\Psi_n)_{n\geq 1}$ is then exactly the Lie algebra of the primitive elements for Δ.

A *composition* is just a sequence $I = (i_1, i_2, \dots, i_r)$ of strictly positive integers. If m denotes the sum $\sum_{j=1}^{r} i_j$ of the parts of such a composition I, we will say that I is a composition of m. If I is a composition of m, we will write $I \models m$ and call m the *weight* of I. The integer r will be called the *length* of I and denoted by $\ell(I)$. Compositions can be represented by ribbon diagrams, i. e. by connected skew Ferrers diagrams that do not contain 2 by 2 squares. One associates with the composition $I = (i_1, i_2, \dots, i_r)$ the ribbon diagram whose j-th row has exactly i_j boxes.

Example 2.1 The composition $I = (3, 1, 5, 4)$ is a composition of 13. It can be represented by the ribbon diagram given below.



It is often useful to number the boxes of a ribbon diagram by starting from the top left and finishing at the bottom right. The previous ribbon diagram can then be numbered as follows :



Definition 2.2 *If $I = (i_1, i_2, \dots, i_r)$ is a composition, we shall denote by $\overline{I}$ the mirror image of I defined by $\overline{I} = (i_r, i_{r-1}, \dots, i_1)$.*

We can also equip the set of all compositions of a given integer m with the *reverse refinement order*, denoted $\preceq$. For instance, the compositions J of 4 such that $J \preceq (1, 2, 1)$ are exactly $(1, 2, 1)$, $(3, 1)$, $(1, 3)$ and (4).

The basis of **Sym** are naturally indexed by compositions. Let us indeed define for every family $(F_n)_{n\geq 1}$ of homogeneous noncommutative symmetric functions and for every composition $I = (i_1, i_2, \dots, i_r)$, the noncommutative symmetric function $F^I = F_{i_1} F_{i_2} \dots F_{i_r}$. The families $(S^I)_I$, $(\Lambda^I)_I$, $(\Phi^I)_I$ and $(\Psi^I)_I$ are homogeneous linear basis of **Sym**.

There is also another very important basis of **Sym** that is indexed by compositions. If I is a composition we can then define the ribbon Schur function R_I by setting:

$$R_I = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)} S^J ,$$

one can then show that the family $(R_I)_I$ is a basis of **Sym**.

We will use extensively the following rule for multiplying two noncommutative ribbon Schur functions.

Proposition 2.3 *Let $I = (i_1, i_2, \ldots, i_r)$ and $J = (j_1, j_2, \ldots, j_s)$ be two compositions. Then we have :*

$$R_I \, R_J = R_{I \cdot J} + R_{I \triangleright J}$$

where we set $I \cdot J = (i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_s)$ and $I \triangleright J = (i_1, i_2, \ldots, i_r + j_1, j_2, \ldots, j_s)$

2.2 Quasi symmetric functions

Malvenuto and Reutenauer (cf. [MvR]) showed that the dual bialgebra **Sym*** of the noncommutative Hopf algebra **Sym** can be identified with the algebra $Qsym$ of quasi-symmetric functions, introduced by Gessel (see [Ge]). This last algebra is defined as follows. Let X be an infinite alphabet totally ordered by some total order $<$. A commutative polynomial $P \in \mathbb{C}[X]$ is then said to be quasi symmetric if one has

$$(P, x_1^{i_1} \ldots x_n^{i_n}) = (P, y_1^{i_1} \ldots y_n^{i_n})$$

for every sequence $(i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$ and every strictly increasing sequences $(x_1 < x_2 < \ldots < x_n)$ and $(y_1 < y_2 < \ldots < y_n)$ of letters of X. The set of all quasi symmetric polynomials of $\mathbb{C}[X]$ form an algebra, denoted by $Qsym$, and called the algebra of quasi symmetric functions.

A natural basis of $Qsym$ is formed by the *quasi-monomial functions*, defined by setting

$$M_I = \sum_{y_1 < y_2 < \ldots < y_p} y_1^{i_1} y_2^{i_2} \ldots y_p^{i_p} ,$$

for every $I = (i_1, \ldots, i_p)$. Another convenient basis is constituted by the quasi ribbon functions defined by setting

$$F_I = \sum_{J \succeq I} M_J$$

for every I. One can then introduce a pairing between $Qsym$ and **Sym** by setting equivalently

$$\langle R_I, F_J \rangle = \delta_{IJ}, \text{ or } \langle S^I, M_J \rangle = \delta_{IJ} .$$

With this pairing, $Qsym$ becomes exactly the Hopf dual of **Sym** (cf [G-T] for more details). The graded dual basis of (Ψ^I) will then be denoted by (P_I).

2.3 Commutative symmetric functions

The usual algebra of commutative symmetric functions will be denoted here by Sym. We refer the reader to [Macd] or to [LS] for any details concerning the classical theory of symmetric functions. The Schur functions $(s_\lambda)_\lambda$ form in particular an important basis of Sym indexed by partitions.

One can define a morphism c from the algebra of noncommutative symmetric functions into the algera of commutative symmetric functions by asking that

$$c(S_n) = h_n$$

(using here Macdonald's notations) for every $n \geq 1$.

The image of a noncommutative symmetric function F under this morphism will be called the *commutative image* of F. This terminology is justified by the fact that $c(\Lambda_n) = e_n$ and $c(\Psi_n) = c(\Phi_n) = p_n$ for every $n \geq 1$, using again Macdonald notations.

One can also show that the commutative image of a noncommutative ribbon function R_I is the so called ribbon Schur function r_I. This last commutative symmetric function is defined in the following way. A composition I can be represented as a skew Ferrers diagram λ/μ and the ribbon Schur function r_I is then just the skew Schur Function $s_{\lambda/\mu}$ (cf [Macd] for details). The following figure gives the example of the composition $I = (3, 1, 5, 4)$ interpreted as the skew Ferrers diagram $(10\,7\,3\,3/6\,2\,2)$.

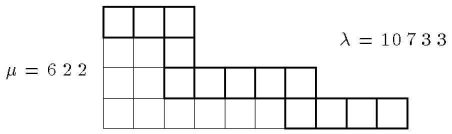


Figure 1: The ribbon Schur function $r_{3\,1\,5\,4}$

Remark 2.4 The two symmetric functions r_I and $r_{\overline{I}}$ are always equal.

Several properties of noncommutative Schur ribbons are inherited by their commutative images. In particular, the multiplication rule stated in Proposition 2.3 holds true for commutative ribbon Schur functions.

2.4 Some combinatorial results for ribbon diagrams

Definition 2.5 *Let I be a composition interpreted as a skew Ferrers diagram λ/μ. We say that a composition J of weight k is removable from I at the position i if and only if:*

1. *the boxes numbered from i to $i+k-1$ in the ribbon diagram of I form a ribbon diagram of shape J ;*
2. *one still gets a skew Ferrers diagram by removing from the Ferrers diagram λ/μ the boxes numbered from i to $1+k-1$.*

Example 2.6 The next figure shows that a composition of weight 4 is removable from the composition $(3,1,5,4)$ at position 6, but not at position 4. By removing a ribbon from another, one obtains two disconnected ribbons (possibly empty).

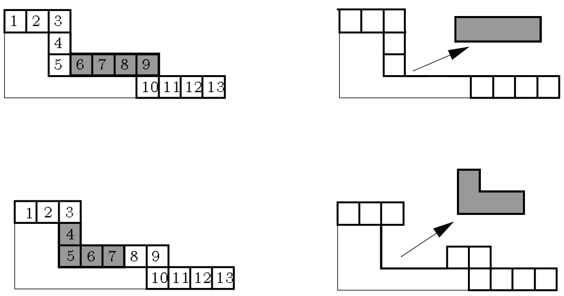


Figure 2: Removable (top) and non removable (bottom) compositions

Remark 2.7 *Notice that saying that a composition of weight k is removable from a composition I, is equivalent to saying that a ribbon diagram of* **length** *k is removable from the ribbon diagram associated with I.*

Well known objects in the theory of representations are the so called k-cores.

Definition 2.8 *Let λ be a partition. We say that λ is a k-core if it is not possible to remove any composition of weight k at the border of the Ferrers diagram λ and obtain another Ferrers diagram.*

Definition 2.9 *Let n be a positive integer, let $I = (i_1, i_2, \ldots, i_r)$ be a composition of n and let i be an integer in $\{1, 2, \ldots, n\}$. We say that the composition I passes through i if there exists $k \in \{1, 2, \ldots, r\}$ such that $\sum_{j=1}^{k} i_j = i$.*

For instance the composition $I = (3,1,5,4)$ of 13 passes exactly through 3, 4, 9, 13.

Proposition 2.10 *Let I be a composition. Then a composition of weight k is removable from I at position i if and only if I passes through $i+k-1$ and I does not pass through $i-1$.*

Sketch of the proof. The above condition guarantees that what is left after the removal of a composition of weight k is still a Ferrers diagram. $\diamondsuit$

Definition 2.11 *We will say that a composition I is k-solid if no composition of weight k is removable from I.*

Remark 2.12 *It is straightforward from the definitions that a composition I is k-solid if and only if it is the border of a k-core.*

Lemma 2.13 *Let I be a composition of n. Then I is k-solid iff either $n < k$, or when $n > k$ if I satisfies:*

i) I passes through $n-k$,

ii) I does not pass through k,

iii) if I does not pass through i then I does not pass through $i+k$.

Proof. A ribbon of length k is certainly not k-solid as it is possible to remove it entirely. Of course, ribbons of length smaller than k are k-solid. Let us then suppose that n is greater than k. Conditions *i*), *ii*) and *iii*) respectively insure that no composition of length k is removable at the end of I, at the beginning of I and at any other position i of I (according to Proposition 2.10). $\diamondsuit$

2.5 Differential and quasi differential operators

The algebra of commutative symmetric functions Sym is equipped with a canonical scalar product $(\ ,\ )$ which is defined by requiring that the Schur functions form an orthonormal basis for it, i.e.

$$(s_\lambda, s_\mu) = \delta_{\lambda\mu}$$

for all partitions λ and μ. It is worth noticing that the algebra of $Qsym$ of quasi symmetric functions contains the algebra Sym of symmetric functions.

The scalar product defined in Sym is related to the pairing between **Sym** and $Qsym$ since one can prove that

$$\langle F, f\rangle = (c(F), f) \tag{1}$$

for every noncommutative symmetric function F and every quasi symmetric function f which is in fact a symmetric function of Sym (see [Ge] or [G-T]).

Let f be a symmetric function. The differential operator D_f is defined as the adjoint of the multiplication operator $M_f : g \longrightarrow fg$, i.e. by setting :

$$(D_f g, h) = (g, M_f h) = (g, fh)$$

for every $g, h \in Sym$. One can prove in particular that $D_{p_k} = k\frac{\partial}{\partial p_k}$. (cf [Macd], or [LS]). The Murnaghan-Nakayama rule explicitly describes the actions of these last differential operators on commutative ribbon Schur functions.

Proposition 2.14 (Murnaghan-Nakayama rule) *Let I be a composition. Then one has :*

$$D_{p_k}(r_I) = \sum_{\substack{J \text{ removable from } I \\ J \models k}} (-1)^{\ell(J)-1} r_{I_1} r_{I_2}$$

where I_1 and I_2 are the ribbons obtained by removing J from I in all possible ways.

The following corollary of the previous proposition can be obtained by simple computations. We use it extensively in the proofs of following theorems and lemmas.

Corollary 2.15 *Let I be a composition of the integer k. In the expansion of the commutative ribbon Schur function r_I in the basis of power sums, the term p_k appears with coefficient $\frac{(-1)^{\ell(I)-1}}{k}$. Moreover no other monomial of this expansion contains the term p_k.*

It is also interesting to introduce a notion of quasi differential operator in the context of noncommutative symmetric functions. Let f be a quasi symmetric function. The quasi differential operator D_f is then defined by setting :

$$\langle D_f(F), g\rangle = \langle F, gf\rangle$$

for every noncommutative symmetic function F and every quasi symmetric function g. The previous property of the pairing given at the beginning of this section shows that

$$c(D_f(F)) = D_f(c(F))$$

when f is a quasi symmetric function which is indeed a symmetric function. This justifies the fact that we used the same notation for differential operators and quasi differential operators.

3 Commutative ribbons and differential operators

3.1 A tensor decomposition of Sym

One can clearly decompose Sym as follows :

$$Sym = Sym_k \otimes Sym_{-k},$$

where $Sym_k = \mathbb{C}[p_k, p_{2k}, p_{3k}, \ldots]$ is the algebra generated by the power sums indexed by multiples of k and where Sym_{-k} is the algebra generated by the remaining power sums.

This decomposition is of interest in the theory of modular representations of the symmetric group. The Schur functions that are in Sym_{-k} are indeed exactly the Frobenius characteristics of the Specht k-modular representations of the symmetric group (cf. [CR] and [R]). These Schur functions can be characterized exactly as follows.

Theorem 3.1 *A Schur function s_λ is in Sym_{-k} if and only if λ is a k-core.*

We want to generalize this result to the noncommutative case where the ribbons Schur functions R_I's play the role of the usual Schur functions. Our first step in this direction will be to characterize the commutative ribbon Schur functions that belongs to Sym_{-k}. When one expands a ribbon Schur function r_I of Sym_{-k} into the basis of the power sums, no p_{jk} appears in the expansion. Therefore a ribbon Schur functions r_I of Sym_{-k} satisfies :

$$D_{p_{jk}}(r_I) = jk\frac{\partial(r_I)}{\partial p_{jk}} = 0$$

for every $j \geq 1$. In the main theorem of this section (Theorem 3.2), we characterize the compositions I such that $D_{p_k}(r_I) = 0$, in a further corollary (3.10) we show that this condition is equivalent to the fact that :

$$D_{p_{jk}}(r_I) = 0$$

for every $j \geq 1$.

3.2 Ribbon Schur functions in Sym_{-k}

The following theorem is the main result of this section. It basically states that a ribbon Schur function is annihilated by the operator D_{p_k} iff it is indexed by a k-solid composition or by the mirror image of such a composition.

Theorem 3.2 *Let I be a composition. Then the two following assertions are equivalent :*

1. $D_{p_k}(r_I) = 0$
2. *I or $\overline{I}$ is k-solid.*

Remark 3.3 Before entering into the proof, it is worth noticing that the conditions "I is k-solid" and "$\overline{I}$ is k-solid" are not equivalent. For instance, the composition $I = 1\,3\,3$ is 3-solid. It is easy to check that no composition of weight 3 (in other words, no ribbon of length 3) can be removed from it. On the other hand, its mirror image, the composition $\overline{I} = 3\,3\,1$, is not 3-solid. Indeed, it is possible to remove two ribbons of length 3 from it, one at the beginning (position 1) and one at the end (position 5). However, $D_{p_3}(r_{3\,3\,1})$ is equal to zero. Indeed, using Murnaghan-Nakayama rule, one gets

$$D_{p_3}(r_{3\,3\,1}) = r_{3\,1} - r_{3\,1} = 0.$$

Proof. Let us now show that condition 2 implies condition 1. If I is k-solid, then no ribbon of length k is removable from it and hence, by Proposition 2.14, $D_{p_k}(r_I) = 0$. Of course, if $\overline{I}$ is k-solid, one has $D_{p_k}(r_I) = D_{p_k}(r_{\overline{I}}) = 0$.
We have split the proof of the converse of our theorem in the several following lemmas which we give without demonstration. The proofs of these lemmas and of every result of this paper can be found in its journal version. $\diamondsuit$

Lemma 3.4 *Let I be a composition of $n \geq k$ such that $D_{p_k}(r_I) = 0$. Then either I or $\overline{I}$ satisfies the conditions i) and ii) of Lemma 2.13.*

We are only left with proving that if I is a composition of $n \geq k$ that satisfies the conditions *i*) and *ii*) of 2.13, and such that $D_{p_k}(r_I) = 0$ then I also satisfies the condition and *iii*) of Lemma 2.13. This property is proven in the following lemma.

Lemma 3.5 *Let I be a composition of $n \geq k$ that satisfies conditions i) and ii) of 2.13 and such that $D_{p_k}(r_I) = 0$. Then, if I does not pass through some integer i, with $n \geq i + k$, I does not pass through $i + k$ either.*

This also ends the proof of Theorem 3.2. We can now give some consequences of our result. Let us start with the following remark.

Remark 3.6 Let I be a k-solid composition. Then if I passes through i, I passes also through $i - k$. This condition is indeed clearly equivalent to the condition *iii*) of Lemma 2.13.

We deduce from this remark the following corollaries.

Corollary 3.7 *Let I be a k-solid composition of $n > k$. Let us consider the euclidean division decomposition $n = qk + r$ (with $0 \leq r < k$) of n. Then I passes through $r, r+k, r+2k, \ldots, r+qk = n$ and I does not pass through $k, 2k, 3k, \ldots, qk$.*

Corollary 3.8 *Let I be a composition of n such that $D_{p_k}(r_I) = 0$. Then $n \bmod k \neq 0$.*

Corollary 3.9 *Let I be a composition of n and let $n = qk + r$ be the euclidean division decomposition of n. Then the compositions I is k-solid iff one can decompose it as $I = I_0 \cdot I_1 \cdot \ldots \cdot I_q$ where I_j are compositions with the following properties :*

- *I_0 is a composition of r,*
- *I_j is a composition of k for all $j \in \{1, \ldots, q\}$,*
- *for all $j \in \{1, \ldots, q-1\}$, I_{j+1} is a composition less fine than or equal to I_j,*
- *none of the compositions $I_1, \ldots, I_q$ passes through $k - r$*
- *if I_0 does not pass through an integer i, then none of the compositions $I_1, \ldots, I_q$ passes through $k - r + i$.*

Hence, the compositions I such that $D_{p_k}(r_I) = 0$ are the compositions described in the previous corollary or their mirror images.

Corollary 3.10 *Let I be a composition. Then one has $D_{p_{jk}}(r_I) = 0$ for every $j \geq 1$, iff I or $\overline{I}$ is k-solid.*

Example 3.11 The composition $\underbrace{1\,1\,1\,1}_{I_0}\,\underbrace{2\,1\,2}_{I_1}\,\underbrace{2\,1\,2}_{I_2}\,\underbrace{2\,3}_{I_3}\,\underbrace{5}_{I_4}$ is 5-solid.

The composition 6 6 4 2 3 1 2 3 1 1 1 2 is annihilated by D_{p_6} because its mirror image, the composition $\underbrace{2}_{I_0}\,\underbrace{1\,1\,1\,3}_{I_1}\underbrace{2\,1\,3}_{I_2}\,\underbrace{2\,4}_{I_3}\,\underbrace{6}_{I_4}\,\underbrace{6}_{I_5}$, is 6-solid.

4 Noncommutative ribbons and quasi differential operators

4.1 A tensor decomposition of Sym

As in the commutative case, the algebra **Sym** has an analogous interesting decomposition. Let $\Pi = (\Pi_n)_{n\geq 1}$ a sequence of Lie idempotents with $deg(\Pi_n) = n$. For any positive integer k, we can then define the subalgebra $\mathbf{Sym}_k(\Pi)$ of **Sym** generated by the elements of the sequence Π which are indexed by some multiple of k, i. e. :

$$\mathbf{Sym}_k(\Pi) = \mathbb{C}\langle \Pi_k\,, \Pi_{2k}, \Pi_{3k}, \ldots \rangle.$$

We must now introduce another important subalgebra of **Sym**. Let us denote by T_k the set of all compositions of the form :

$$t = (i_1 k, \ldots , i_r k, i_{r+1})$$

with $i_{r+1} \not\equiv 0 \pmod k$. We associate with every element $t \in T_k$ the noncommutative symmetric function $\Pi[t]$ defined by

$$\Pi[t] = [\Pi_{i_1 k}\,, [\Pi_{i_2 k}, [\ldots , [\Pi_{i_r k}, \Pi_{i_{r+1}}] \ldots]]].$$

The subalgebra $\mathbf{Sym}_{-k}(\Pi)$ is then defined by setting

$$\mathbf{Sym}_{-k}(\Pi) = \mathbb{C}\langle \Pi[t] \mid t \in T_k \rangle.$$

According to Lazard's elimination theorem (see [B]), the subalgebra $\mathbf{Sym}_{-k}(\Pi)$ is freely generated by the family $\{\Pi[t] \mid t \in T\}$ and one has moreover the following tensor decomposition of **Sym** :

$$\mathbf{Sym} = \mathbf{Sym}_k(\Pi) \otimes \mathbf{Sym}_{-k}(\Pi).$$

An important fact is that the algebra $\mathbf{Sym}_{-k}(\Pi)$ does not depend on the sequence Π of Lie idempotents that was used to define it.

Proposition 4.1 *Let $\Pi = (\Pi_n)_{n\geq 1}$ and $\Pi' = (\Pi'_n)_{n\geq 1}$ be two sequences of homogeneous Lie idempotents. Then one has*

$$\mathbf{Sym}_{-k}(\Pi) = \mathbf{Sym}_{-k}(\Pi').$$

Sketch of the proof. The result is a consequence of the free Lie algebra version of the Lazard's Elimination Theorem and uses simple arguments of degree homogeneity modulo k. A complete proof can be found in the journal version of this paper. $\diamondsuit$

Therefore, the algebra $\mathbf{Sym}_{-k}(\Pi)$ can be denoted simply by $\mathbf{Sym}_{-k}$. In particular one has $\mathbf{Sym}_{-k} = \mathbf{Sym}_{-k}(\Psi)$, where $\Psi = (\Psi_n)_{n\geq 1}$, since (Ψ_n) is a sequence of homogeneous Lie idempotents.

Lemma 4.2 *Let $\Pi = (\Pi_n)_{n\geq 1}$ be a sequence of homogeneous Lie idempotents , such that $deg(\Pi_n) = n$ and let (Π^*_I) be the graded dual basis of (Π_I). Then $\Pi^*_n = nP_n$.*

Proposition 4.3 *Let $\Pi = (\Pi_n)_{n\geq 1}$ a sequence of Lie homogeneous idempotents. Then one has :*

$$\mathbf{Sym}_{-k}(\Pi) = \bigcap_{\substack{r\geq 1 \\ m_1, m_2, \ldots , m_r \in \mathbb{N}}} ker D_{P_{m_1 k, m_2 k, \ldots , m_r k}}$$

Sketch of the proof. It is possible to show that a noncommutative symmetric function F is in $\mathbf{Sym}_{-k}(\Psi)$ if and only if it satisfies

$$\langle \Delta(F), P_{m_1 k, m_2 k, \ldots, m_r k} \otimes P_I \rangle = 0$$

for all $m_1, m_2, \ldots, m_r \in \mathbb{N}$ and every composition I, and the latter condition is clearly equivalent to

$$D_{P_{m_1 k, m_2 k, \ldots, m_r k}}(F) = 0$$

for all $m_1, m_2, \ldots, m_r \in \mathbb{N}$. ◊

4.2 The ribbon functions in $\mathbf{Sym}_{-k}$

The set of all noncommutative ribbon Schur functions form a basis of the algebra $\mathbf{Sym}$. We want to characterize the noncommutative ribbons that belong to $\mathbf{Sym}_{-k}$.

Theorem 4.4 *Let I be a composition. Then the following conditions are equivalent :*

i) $R_I \in \mathbf{Sym}_{-k}$,

ii) $D_{P_k}(R_I) = 0$,

iii) Either I or $\overline{I}$ is k-solid.

iv) $D_{P_{m_1 k, m_2 k, \ldots, m_r k}}(R_I) = 0$ *for all integers* $m_1, m_2, \ldots, m_r \geq 1$.

Note first that conditions $i)$ and $iv)$ are equivalent according to Proposition 4.3 and that $iv)$ obviously implies $ii)$. In each of the following subsections, we show the remaining implications.

4.2.1 Condition $ii)$ implies condition $iii)$

Lemma 4.5 *Let I be a composition such that $D_{P_k}(R_I) = 0$, then either I or $\overline{I}$ is k-solid.*

Proof. Suppose that $D_{P_k}(R_I) = 0$. This means that one has :

$$\langle R_I, P_k F \rangle = 0$$

for all quasi-symmetric functions F. Since every symmetric function is a quasi-symmetric function, we get in particular that

$$\langle R_I, P_k f \rangle = 0$$

for every symmetric function f. Using now property (1) of Section 2.5, we get

$$(r_I, p_k f) = 0$$

for every symmetric function f (notice that indeed $P_k = p_k$). Hence we have $D_{p_k}(r_I) = 0$ and we can conclude to our Lemma by using the results of the previous section. ◊

4.2.2 Condition $iii)$ implies condition $iv)$

Since the P_K's form a linear basis of the algebra of the quasi-symmetric functions, condition $iv)$ is equivalent to the fact that

$$\langle R_I, P_{m_1 k, m_2 k, \ldots, m_r k} P_K \rangle = 0,$$

for all integers $m_1, m_2, \ldots, m_r \geq 1$ and for every composition K. This last condition is itself equivalent to

$$\langle \Delta(R_I), P_{m_1 k, m_2 k, \ldots, m_r k} \otimes P_K \rangle = 0$$

for all integers $m_1, m_2, \ldots, m_r \geq 1$ and for every composition K. Hence, to prove that condition $iii)$ implies condition $iv)$ it suffices to prove the following lemma.

Lemma 4.6 *Let I be a composition such that either I or $\overline{I}$ is k-solid. Then one has*

$$\langle \Delta(R_I), P_{m_1k,m_2k,\dots,m_rk} \otimes P_K \rangle = 0$$

for all integers $m_1, m_2, \dots, m_r \geq 1$ and for every composition K.

Proof. Let I be a composition such that either I or $\overline{I}$ is k-solid. By definition of a noncommutative ribbon Schur function, note first that one has,

$$\Delta(R_I) = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)} \Delta(S^J).$$

Hence it suffices to prove that one has

$$\sum_{J \preceq I} (-1)^{\ell(J)} \langle \Delta(S^J), P_{m_1k,m_2k,\dots,m_rk} \otimes P_K \rangle = 0 \qquad (2)$$

for every composition K. Recall now that for all composition $J = (j_1, j_2, \dots, j_s)$, one has

$$\Delta(S^J) = \Delta(S_{j_1})\Delta(S_{j_2})\dots\Delta(S_{j_s})$$

By definition of Δ, one gets

$$\Delta(S^J) = \sum_{l_1=0}^{j_1}\sum_{l_2=0}^{j_2}\dots\sum_{l_s=0}^{j_s} S_{l_1}S_{l_2}\dots S_{l_s} \otimes S_{j_1-l_1}S_{j_2-l_2}\dots S_{j_s-l_s}.$$

By substituting this last expression in (2), we see that we have to show that :

$$\sum_{J \preceq I} (-1)^{\ell(J)} \langle \sum_{l_1=0}^{j_1}\sum_{l_2=0}^{j_2}\dots\sum_{l_s=0}^{j_s} S_{l_1}S_{l_2}\dots S_{l_s} \otimes S_{j_1-l_1}S_{j_2-l_2}\dots S_{j_s-l_s}, P_{m_1k,m_2k,\dots,m_rk} \otimes P_K \rangle$$

$$= \sum_{J \preceq I} (-1)^{\ell(J)} \sum_{l_1=0}^{j_1}\sum_{l_2=0}^{j_2}\dots\sum_{l_s=0}^{j_s} \langle S_{l_1}S_{l_2}\dots S_{l_s}, P_{m_1k,m_2k,\dots,m_rk} \rangle \langle S_{j_1-l_1}S_{j_2-l_2}\dots S_{j_s-l_s}, P_K \rangle = 0.$$

for every composition K. Using the expansion of S_l on the basis of noncommutative power sums of the first kind (cf [G-T]), one gets

$$S^L = \sum_{C \succeq L} \alpha_{C,L} \Psi^C$$

for every composition L, where $\alpha_{C,L}$ is a rational number that depends on the composition L and its refinement C. Therefore the only terms of the internal sum that are different than zero are those in which the l_j's that are not equal to zero form a composition which is an anti-refinement of the composition $M = m_1k, m_2k, \dots, m_rk$. Hence we have to show that :

$$\sum_{J \preceq I} (-1)^{\ell(J)} \left(\sum_{L \preceq M} \alpha_{M,L} \left(\sum_{J'} \langle S_{J'}, P_K \rangle \right) \right) = 0$$

for every composition K, where the most internal sum in the above expression is taken over all compositions J' of the integer $n - (m_1 + m_2 + \dots + m_r)k$, which can be obtained from J in the following way :

1. take a sequence of non negative integers $l_1, l_2, \dots, l_s$, such that the l_j's that are not equal to zero form a composition L which is an anti-refinement of the composition $M = m_1k, m_2k, \dots, m_rk$, and such that $l_t \leq j_t$ for all $t = 1, 2, \dots, s$

2. form then J' by subtracting in J the integer l_t from j_t for all $t = 1, 2, \dots, s$ (if a part j_t of J is equal to l_t, then this part "disappears" in J' when l_t is subtracted).

We can now rewrite the above identity as follows :

$$\sum_{L\leq M} \alpha_{M,L} \left(\sum_{J\preceq I} (-1)^{\ell(J)} \left(\sum_{J'} \langle S_{J'}, P_K\rangle \right) \right) = 0 \ .$$

Of course it suffices to prove that all the terms

$$\sum_{J\preceq I} (-1)^{\ell(J)} \left(\sum_{J'} \langle S_{J'}, P_K\rangle \right)$$

are zero for every composition K. Now, any composition L which is an anti-refinement of $M = m_1k, m_2k, \ldots, m_rk$ is still a composition of the form $n_1k, n_2k, \ldots, n_qk$. Therefore it suffices to prove that if I is a k-solid composition or the mirror image of a k-solid composition and if $m_1k, m_2k, \ldots, m_rk$ is a sequence of multiples of k, then the identity

$$\sum_{J\preceq I} (-1)^{\ell(J)} \left(\sum_{J'} \langle S_{J'}, P_K\rangle \right) = 0 \qquad (3)$$

holds for every composition K.

Let us remind that J' is any composition obtained by taking any subsequence $j_{l_1}, j_{l_2}, \ldots, j_{l_r}$ of r parts of J such that $j_{l_t} \geq m_tk$ for all $t = 1, 2, \ldots, r$, and by subtracting the integer m_tk from j_{l_t} for all $t = 1, 2, \ldots, r$. If a part j_{l_t} of J is equal to m_tk, then this part "disappears" in J' after the subtraction.

The fact that equation (3) holds for every composition K is clearly equivalent to the following identity

$$\sum_{J\preceq I} (-1)^{\ell(J)} \left(\sum_{J'} S_{J'} \right) = 0 \qquad (4)$$

where the internal sum is taken over all compositions J' of $n-(m_1+m_2+\ldots+m_r)k$ obtained using the process described above.
This last property is then an immediate consequence of the next lemma 4.11, which says that

$$\sum_{J\preceq I} (-1)^{\ell(J)} \sum_{J'} J'$$

is identically equal to zero when I is either a k-solid or the mirror image a of k-solid composition. $\diamondsuit$

This result might be clearer with the following example.

Example 4.7 Let I be the 3-solid composition : $1\,1\,2\,3\,3$. The anti-refinements of I are :

$$\{1\,1\,2\,3\,3, 2\,2\,3\,3, 1\,3\,3\,3, 1\,1\,5\,3, 1\,1\,2\,6,\ 4\,3\,3, 2\,5\,3, 2\,2\,6, 1\,6\,3, 1\,3\,6, 1\,1\,8, 7\,3, 4\,6, 2\,8, 1\,9, 10\}.$$

Consider the sequence $M = \{3, 3\}$. By subtracting the integers of this sequence from the anti-refinements of I (where this is possible), we obtain the followed signed compositions :

$$\begin{array}{lll}
1\,1\,2\,3\,3 & \longrightarrow & -1\,1\,2 \\
2\,2\,3\,3 & \longrightarrow & 2\,2 \\
1\,3\,3\,3 & \longrightarrow & 1\,3 + 1\,3 + 1\,3 \text{ (3 times !)} \\
1\,1\,5\,3 & \longrightarrow & 1\,1\,2 \\
4\,3\,3 & \longrightarrow & -1\,3 - 1\,3 - 4 \\
2\,5\,3 & \longrightarrow & -2\,2 \\
1\,6\,3 & \longrightarrow & -1\,3 \\
1\,3\,6 & \longrightarrow & -1\,3 \\
7\,3 & \longrightarrow & 4 \\
4\,6 & \longrightarrow & 1\,3
\end{array}$$

It is straightforward to check that the sum of these signed compositions is identically equal to zero.

Before stating the next lemma, we need to introduce some notations.

Definition 4.8 *Let $J = (j_1, j_2, \dots, j_s)$ be a composition, k and r be two positive integers with $r \leq s$. Let $m_1, m_2, \dots, m_r$ be another sequence of positive integers and let $l_1 < l_2 < \cdots < l_r$ be elements of $\{1, 2, \dots, s\}$ such that $j_{l_t} \geq m_t k$ for all $t = 1, 2, \dots, r$.*
We note by $(J; l_1, l_2, \dots, l_r)^{\frown m_1 k, m_2 k, \dots, m_r k}$ the composition obtained from J by subtracting the integer $m_t k$ from its l_t-th part for all $t = 1, 2, \dots, r$. If $j_{l_t} = m_t k$, then the l_t-th part is erased from J.

Example 4.9

$$((1,3,8,6,7); 2,4,5)^{\frown 2,4,4} = (1,1,8,2,3)$$
$$((1,3,8,6,7); 2,4,5)^{\frown 2,6,4} = (1,1,8,3)$$

Definition 4.10 *Let I be a composition and let $m_1, m_2, \dots, m_r$ be another fixed sequence of positive integers. We note by $\mathcal{E}_I$ the set defined by :*

$$\mathcal{E}_I = \{(J; l_1, \dots, l_r) \mid J = (j_1, \dots, j_s) \preceq I;\ 1 \leq l_1 < \cdots < l_r \leq s;\ j_{l_t} \geq m_t k,\ \textit{for all } t = 1, \dots, r\}$$

The elements of this set are pairs made of a composition and a sequence of indeces corresponding to parts of the composition from which it is possible to subtract the integers $m_t k$. Notice that the above set also depends on the sequence $m_1, m_2, \dots, m_r$, we decided to omit this dependence in the notation $\mathcal{E}_I$ for clearness.

Lemma 4.11 *Let I be a composition of n such that either I or $\overline{I}$ is k-solid, and let $m_1, m_2, \dots, m_r$ be another fixed sequence of positive integers. Then the formal sum :*

$$\sum_{(J; l_1, \dots, l_r) \in \mathcal{E}_I} (-1)^{\ell(J)} (J; l_1, l_2, \dots, l_r)^{\frown m_1 k, m_2 k, \dots, m_r k} \tag{5}$$

is identically equal to 0.

Sketch of the proof. The proof is bijective. The argument is based on an involution which couples up pairs of elements of the set $\mathcal{E}_I$ whose contribution to the above sum is equal in absolute value but opposite in sign. $\diamondsuit$

Using the last lemma, we immediately obtain equation (4). This ends the proof of Lemma 4.6.

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