Rooted maps and hypermaps on surfaces

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Résumé

Nous présentons une étude des cartes pointées indépendamment de leur genre. Nous prouvons tout d'abord l'existence d'un nouveau type d'équation (différentielle de Riccati) pour la série génératrice des cartes pointées orientables indépendamment du genre, énumérées en fonction des nombres de sommets et d'arêtes. La résolution de cette équation conduit à une fraction continue très simple, qui se traduit par une nouvelle relation généralisant l'équation très connue de Dyck pour les arbres planaires pointés. Nous effectuons également un travail similaire pour les arbres, pour les cartes sur les surfaces localement orientables, et enfin pour les hypercartes.

Abstract

We present a study of rooted maps without regard to genus. First we prove the existence of a new kind of equation for the generating series of orientable rooted maps regardless to genus enumerated with respect to edges and vertices. This is Riccati's equation. Solving this equation leads to a very simple continued fraction, and then to a new equation generalizing the well known Dyck equation for rooted planar trees. Then we give similar results for trees, maps on locally orientable surfaces, and finally for hypermaps.

Keywords
map, hypermap, Dyck's equation, Riccati's equation, continued fraction

1. Introduction

The recent history of maps began with W. Tutte in 1962. Since this date, maps have almost always been studied for a fixed genus associated surface. Three principal approaches characterize this combinatorial domain:

1. The bijective approach was the first used (see [18]). Its principle is to construct a one-to-one correspondence between the family of the studied maps and another family of objects for which are known efficient enumeration methods. This approach was developed by R. Cori, B. Vauquelin and D. Arquès [6, 7, 15].
2. The algebraic approach is more recent and was developed by D. Jackson and T. Visentin in [17]. A rooted map can be seen as a pair of permutations acting transitively on the set of half-edges. This approach applies algebraic combinatorics methods of the symmetric group to these permutations to derive an expression of the series of rooted maps with characters.
3. The topological approach consists in applying to rooted maps a topological operation interpreted in terms of a functional equation for the generating series of the studied maps. Many articles based on this kind of approach can be found in the literature [1-5, 8-12, 19, 21].

We know only three papers that quickly deal with maps without regard to genus. The older one [21] simply gives a recurrence relation for the number of an equivalent of orientable rooted maps regardless to genus, with respect to the number of edges. The second paper [17] used the algebraic approach to give a closed form for the generating function of maps regardless to genus Me(y,z) with respect to vertices (y) and edges (z)

$$\mathrm{Me}(y,z) = 2z\frac{\partial}{\partial z}\ln\sum_{n\geq 0}\frac{y(y+1)\ldots(y+2n-1)}{2^n n!}z^n .$$

In the newest paper [13], the authors describe a bijection between orientable rooted maps regardless to genus and a family of description trees.

The aim of this paper is to give a "complete" topological study of rooted maps regardless to genus. In the second section of this paper we quickly recall the basic notions of map theory. In Section 3 we derive a new differential functional equation for the generating series of orientable rooted maps without regard to genus, with respect to the numbers of edges and vertices. This equation is obtained by a new interpretation of the classical operation consisting in removing the root edge of the map. In Section 4 we first express the series of orientable maps as a continued fraction, leading to a new relation that generalizes the well known Dyck equation for rooted planar trees. Thus we study orientable rooted trees with respect to edges, and we enumerate orientable rooted maps and trees. Then we obtain similar results for locally orientable rooted maps and trees. In Section 5 we give similar results for rooted hypermaps. Finally in Section 6 we give some enumerating tables.

Remark 1. Continued fractions never appeared in the literature to express the generating series of maps. However, in [6, 7], using the bijective approach, D. Arquès introduced multi-continued fractions to express the generating series of rooted planar maps and hypermaps.

2. Definitions

For the convenience of the reader we recall quickly some definitions used in the following (for more details about combinatorial maps refer for example to [14]).

2.1 Topological map

A *topological orientable map* C on an orientable surface Σ of $\mathbb{R}^3$ is a partition of Σ in three finite sets of *cells*:

i. The set of the vertices of C, that is a finite set of points ;
ii. The set of the edges of C, that is a finite set of simple opened Jordan arcs, disjoint in pairs, whose extremities are vertices ;
iii. The set of the faces of C. Each face is homeomorphic to an open disc, and its border is a union of vertices and edges.

The *genus of the map C* is the genus of the surface Σ.

A cell is called *incident* to another cell if one of them is in the border of the other. An *isthmus* is an edge incident on both sides to the same face.

2.2 Combinatorial map

We call *half-edge* an oriented edge of the map. We denote by B the set of all half-edges of the map. To each half-edge, are associated in an evident way its initial vertex, its final vertex, and the underlying edge.

α (respectively σ) is the permutation on B associating each half-edge h with its opposite half-edge (resp. the first half-edge met by turning around the initial vertex of h in the positive way chosen on the orientable surface). α is a fixed point free involution. The cycles of α (resp. σ) represent the edges (resp. vertices) of the map.

$\overline{\sigma}$ is the permutation $\sigma o \alpha$ on B. The cycles of $\overline{\sigma}$ represent the oriented borders of the faces of the map.

In the following, a vertex (resp. edge, face) will be, depending on the context, either the topological object defined at 2.1, or the cycle of σ (resp. α, $\overline{\sigma}$) according to the previous definitions.

The triplet (B, σ, α) is called the combinatorial definition of the associated topological orientable map C.

2.3 Rooted map

A map is called a *rooted map* if a half-edge $\tilde{h}$ is chosen. The half-edge $\tilde{h}$ is called the *root half-edge* of the map, and its initial vertex is called the *root vertex* of the map.

We call *external face* (or *root face*), the face $\overline{\sigma}^*(\tilde{h})$ generated by the root half-edge $\tilde{h}$. The planar map with only a vertex and no edge is also regarded as rooted, even though it contains no half-edge.

Two orientable rooted maps with the same genus are isomorphic if there exists an homeomorphism of the associated surface, preserving its orientation, mapping the vertices, edges, faces and the root half-edge of the first map respectively on those of the second one.

An isomorphic class of orientable rooted maps of genus g will simply be called an *orientable rooted map*. Our goal is to enumerate these equivalence classes of orientable rooted maps independently of their genus.

3. A differential equation for the series of orientable rooted maps

We present here the first topological equation for the generating series of orientable rooted maps regardless to genus, with respect to vertices and edges. We denote by M(y,z) the generating series of orientable rooted maps of any genus (we simply call them orientable rooted maps in the following), where the exponent of y (resp. z) refers to the number of vertices (resp. edges) in the map.

Theorem 1. *The generating series M(y,z) of orientable rooted maps is the solution of the Riccati equation:*

$$M(y,z) = y + zM(y,z)^2 + zM(y,z) + 2z^2 \frac{\partial}{\partial z}\left[M(y,z)\right] \tag{1}$$

Proof. Let C denote an orientable rooted map with root half-edge $\tilde{h}$. The proof will be based on the topological operation of deleting the root half-edge $\tilde{h}$ as introduced by W. Tutte [19] in the study of planar maps. Four cases are possible. The first two terms (in the right part of the equation) look like the first two terms of Tutte's equation (see for example [19]), and are obtained in the same way (but generalized to maps enumerated without regard to genus). The last two terms are of a new type.

First case. If C is the rooted planar map reduced to a vertex, the contribution in eq. (1) is $y^1z^0=y$.

Second case. If the edge supporting $\tilde{h}$ is an isthmus whose deletion disconnects the map into two maps C_1 and C_2, then we choose $\sigma(\tilde{h})$ to be the root half-edge on the first map C_1 and $\bar{\sigma}(\tilde{h})$ on the second one C_2 in order to be able to reconstruct the initial rooted map C (Fig. 1). The contribution of this case in eq. (1) is $zM(y,z)^2$.

The contribution of the first map C_1 is $M(y_1,z_1)$. The contribution of C_2 is $M(y_2,z_2)$. During the reconstruction of the map C, all the edges add up ($z_1=z_2=z$), and we do not forget the added root isthmus (multiplication by z). The vertices add up too ($y_1=y_2=y$).

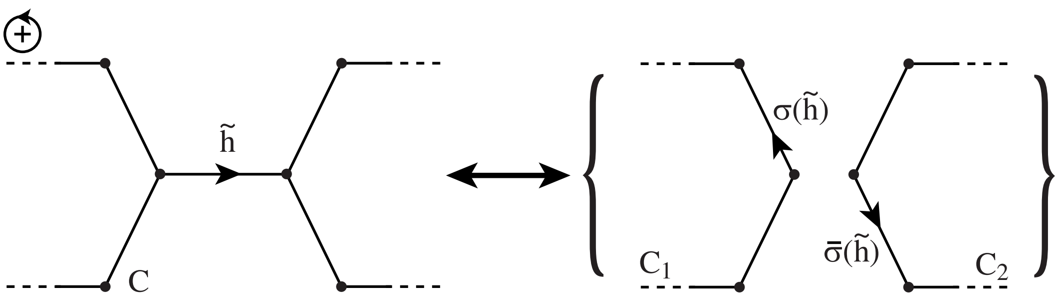

Fig. 1: The root half-edge $\tilde{h}$ is an isthmus whose removal disconnects the orientable map C into two orientable rooted maps C_1 and C_2.

Third case. If $\tilde{h}$ is a planar loop and is the oriented border of the root face, we remove the loop and root the resulting map C_1 with $\sigma(\tilde{h})$ (Fig. 2). C_1 is a general orientable rooted map to which we add a loop during the inverse step of reconstruction. So the contribution of this case in (1) is zM(y,z).

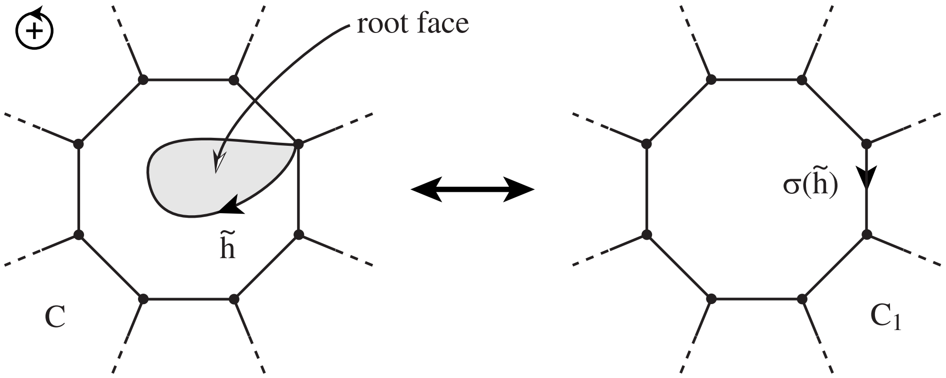

Fig. 2: The root half-edge $\tilde{h}$ is a planar loop, and the root face is "inside" the loop.

Fourth case. Here we group together all the cases which have not been studied above. Two subcases may be encountered:

Subcase 1. The root half-edge $\tilde{h}$ is not an isthmus and is not a planar loop border of the root face (Fig. 3 and 4) ;

Subcase 2. The root half-edge $\tilde{h}$ is an isthmus whose removal does not disconnect the map C (Fig. 5).

a) Removal of the root half-edge $\tilde{h}$.
We remove the root half-edge $\tilde{h}$. It gives the map C_1. We root C_1 with both root half-edges $h_1 = \sigma_{C_1}(\tilde{h})$ (where $\sigma_{C_1}(\tilde{h})$ is the first half-edge in C_1 among $\sigma(\tilde{h})$ and $\sigma^2(\tilde{h})$) and $h_2 = \overline{\sigma}(\tilde{h})$ to be able to reconstruct C from C_1. We obtain a map C_1 with two root half-edges, possibly equal.

b) Addition of a root half-edge in a double rooted map: the inverse operation.
Let C_1 be a map with two root half-edges h_1 and h_2 (with initial vertices v_1 and v_2 respectively). We reconstruct the rooted map C from C_1 by adding an edge from v_1 to v_2, in the angular sectors a_1 and a_2 defined by h_1 and h_2 (more precisely for k=1,2 we consider the sector between $\sigma^{-1}(h_k)$ and h_k) and we root the half-edge oriented from v_1 to v_2. If the angular sectors a_1 and a_2 are in the same face of C_1, the root half-edge $\tilde{h}$ of C is embedded in this face, splitting it in two new faces of C: $\tilde{h}$ is not an isthmus (Fig. 3). It could be a planar loop, if a_1=a_2, but in this case the root face is "outside" the loop (Fig. 4). If a_1 and a_2 are not in the same face of C_1, the root half-edge $\tilde{h}$ of C is embedded on an added handle that collects both faces generated by h_1 and h_2 into one face: $\tilde{h}$ is an isthmus of C whose removal does not disconnect the map (Fig. 5).

To summarize we must choose a half-edge anywhere (eventually equal to the first root half-edge) in any map. To convey this operation on the generating series, we first choose an edge by applying the operator z.∂/∂z to the series M(y,z). There are two possible half-edges associated to the chosen underlying edge, so we multiply by 2, and the contribution of the added edge is z. Thus the contribution of this case in (1) is $2z^2 \frac{\partial}{\partial z}\left[M(y,z)\right]$.

This concludes the proof of Theorem 1.

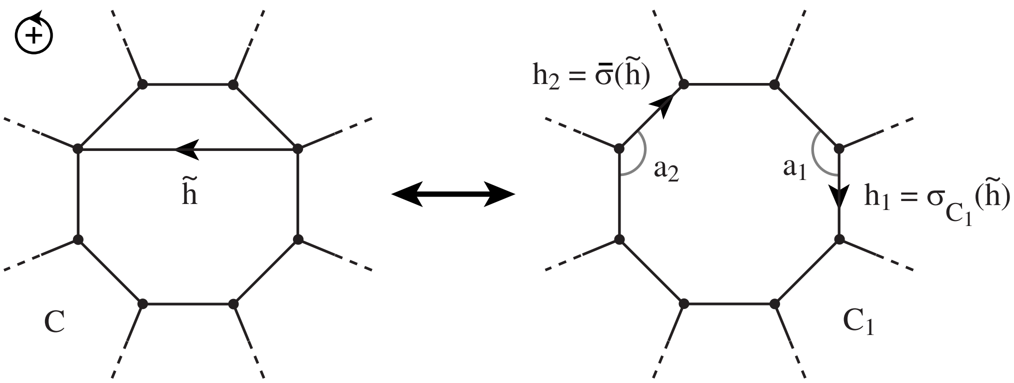

Fig. 3: (Subcase 1) The second root half-edge h_2 of C_1 is in the face generated by h_1: the root half-edge $\tilde{h}$ of the map C is not an isthmus.

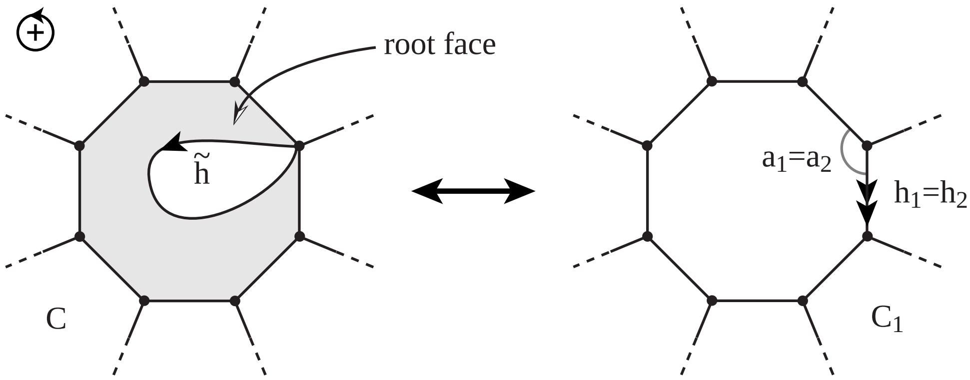

Fig. 4: (Particular situation of subcase 1 and Fig. 3) The same half-edge of C_1 is rooted twice (h_1=h_2): the root half-edge $\tilde{h}$ of the map C is a planar loop, and the root face is “outside” the loop.

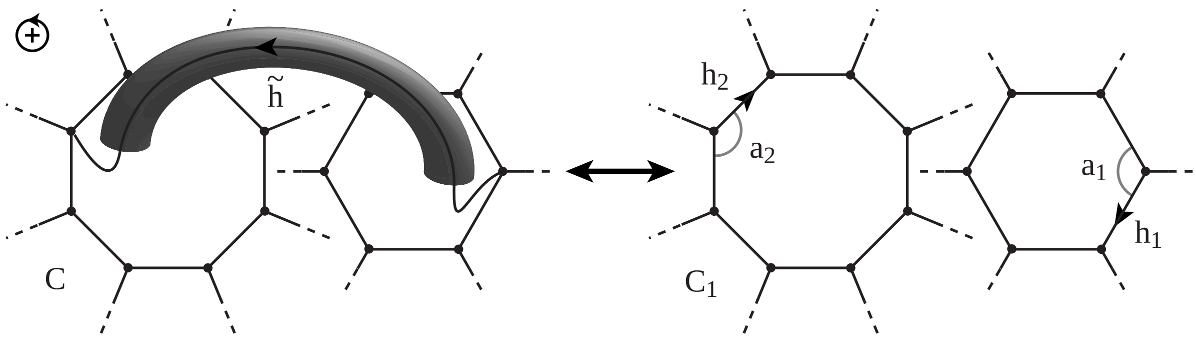

Fig. 5: (Subcase 2) The second root half-edge h_2 of C_1 is not in the face generated by h_1: the root half-edge $\tilde{h}$ of the map C is an isthmus whose removal does not disconnect the map.

4. Enumeration of orientable rooted maps and trees

In this section we solve eq. (1) to obtain a continued fraction form of the generating series of orientable rooted maps. Then we study orientable rooted trees, and finally we give explicit formulae for the enumeration of the number of rooted maps and trees with n edges.

4.1 A continued fraction for orientable rooted maps

Equation (1) is a Riccati differential equation. We present in Theorem 2 an iterative solution of (1), which leads to a very nice continued fraction form of the generating series of orientable rooted maps.

Theorem 2. *The generating series M(y,z) of orientable rooted maps with respect to the number of vertices and edges is:*

$$M(y,z) = \cfrac{y}{1-\cfrac{(y+1)z}{1-\cfrac{(y+2)z}{1-\cfrac{(y+3)z}{1-\dots}}}} \tag{2}$$

Proof. The proof is by recurrence.
Let us first define the sequence $(M_k(y,z))_{k\geq 0}$ of series by the set of equations:

1) $M_0(y,z)$ is the desired generating function M(y,z) of rooted maps ;
2) For any integer k in $\mathbb{N}$, $M_{k+1}(y,z)$ is obtained from $M_k(y,z)$ by the relation

$$M_k(y,z) = \frac{y+k}{1-zM_{k+1}(y,z)} \tag{α_k}$$

Then for every k, $M_k(y,z)$ is a solution of the equation

$$M_k(y,z) = (y+k) + zM_k(y,z)^2 + zM_k(y,z) + 2z^2\frac{\partial}{\partial z}\left[M_k(y,z)\right] \tag{β_k}$$

This result is proved by recurrence:

- For k=0, this is Theorem 1 and Equation (1).
- Let k be a positive integer and let us suppose that $M_k(y,z)$ is a solution of (β_k):

$$M_k(y,z) = (y+k) + zM_k(y,z)^2 + zM_k(y,z) + 2z^2\frac{\partial}{\partial z}\left[M_k(y,z)\right]$$

  Now we substitute $M_k(y,z)$ by its expression with respect to $M_{k+1}(y,z)$ from (α_k). After little algebra one obtains the equation (β_{k+1}).

The result is then proved by recurrence and the interpretation of this set of equations $(\alpha_k)_{k\geq 0}$ gives the continued fraction form of M(y,z). In fact,

$$M(y,z) = M_0(y,z) = \frac{y}{1-zM_1(y,z)} = \cfrac{y}{1-z\cfrac{y+1}{1-zM_2(y,z)}} = \dots$$

When iterating the process, one obtains:

$$M(y,z) = \cfrac{y}{1-\cfrac{(y+1)z}{1-\cfrac{(y+2)z}{1-\cfrac{(y+3)z}{1-\dots}}}}$$

This concludes the proof of Theorem 2.

In Theorem 2 a new relation on maps appears:
Corollary 1. *The generating series M(y,z) of orientable rooted maps is the solution of the following generalized Dyck equation:*

$$M(y,z) = y + zM(y,z)M(y+1,z) \tag{3}$$

Proof. Straightforward if we remark that the continued fraction can be rewritten as

$$M(y,z) = \frac{y}{1-zM(y+1,z)}.$$

Remark 2. As reminded in the introduction, the generating function for rooted orientable maps can be written

$$M(y,z) = y + 2z\frac{\partial}{\partial z}\ln\sum_{n\geq 0}\frac{y(y+1)\ldots(y+2n-1)}{2^n\, n!}z^n$$

(we add the vertex map) with an algebraic combinatorics point of view (see [17]). It can be verified that this formula is the solution of Eq. (1) and (2).

4.2 Orientable rooted trees

Let us recall that a *tree* (of any genus) is a map with only one face. We denote by T(z) the generating series of orientable rooted trees, where the exponent of z refers to the number of edges in the tree.

By duality it exists a one-to-one correspondence between rooted trees and rooted monopoles (rooted maps with only one vertex). Thus the series of trees is the coefficient of y^1 in the series of maps and:

$$T(z) = \left[\frac{M(y,z)}{y}\right]_{|y=0} \tag{4}$$

Then we can write the following results:

Corollary 2.

1) *The generating series T(z) of orientable rooted trees is the solution of the following differential equation:*

$$T(z) = 1 + zT(z) + 2z^2\frac{\partial}{\partial z}\left[T(z)\right] \tag{5}$$

2) *The generating series T(z) of orientable rooted trees is:*

$$T(z) = \cfrac{1}{1-\cfrac{z}{1-\cfrac{2z}{1-\cfrac{3z}{1-\cfrac{4z}{1-\ldots}}}}} \tag{6}$$

3) *Both generating series of orientable rooted trees and orientable rooted maps are linked by the relation:*

$$T(z) = \frac{1}{1-zM(1,z)} \tag{7}$$

Proof. 1) and 2) are direct consequences of eq. (1), (2) and (4). 3) is a direct consequence of (3) and (4).

4.3 Explicit enumeration formulae

From previous expressions, we deduce explicit formulae enumerating orientable rooted maps and trees with a given number of edges.

Corollary 3.

1) *The number of orientable rooted trees with n edges is:*

$$\frac{(2n)!}{2^n n!} \tag{8}$$

2) *The number of orientable rooted maps with n edges is:*

$$\frac{1}{2^{n+1}}\sum_{i=0}^{n}(-1)^i\sum_{\substack{k_1+\ldots+k_{i+1}=n+1\\ k_1,\ldots,k_{i+1}>0}}\prod_{j=1}^{i+1}\frac{(2k_j)!}{k_j!} \tag{9}$$

Proof.

1) Assume that T is the series $\sum_{k\geq 0} t_k z^k$. By searching the coefficient of z^n in (5), we obtain the recurrence relation $t_n = (2n-1)t_{n-1}$. This enumeration has been previously obtained through a very different way by T. Walsh and A. Lehman in [21].
2) From (7), we express the generating series M(1,z) with respect to T(z), and we obtain (9) with little algebra.

Remark 3. The number of rooted trees with n edges is equivalent to the number of permutations σ with one cycle over 2n elements (with two chosen ones for the root edge) divided by $2^{n-1}(n-1)!$. We can also note that this is the very classical enumeration formula for fixed-point-free involutions, namely the odd factorial.

5. Locally orientable maps

Locally orientable maps can be obtained from oriented ones by reversing a subset of edges outside a given spanning tree (see [16 p. 111]). In particular the series X of an orientable family of maps and the series $\tilde{X}$ of the associated locally orientable family are linked by

$$\tilde{X}(y,z) = 2X\left(\frac{y}{2}, 2z\right). \tag{10}$$

This allows us to obtained easily some formulae on locally orientable rooted maps and trees. Let $\tilde{M}(y,z)$ (resp. $\tilde{T}(z)$) denote the generating series of locally orientable rooted maps (resp. trees).

Theorem 3.

1) *The generating series $\tilde{M}(y,z)$ of locally orientable rooted maps is the solution of the following Riccati differential equation:*

$$\tilde{M}(y,z) = y + z\tilde{M}(y,z)^2 + 2z\tilde{M}(y,z) + 4z^2 \frac{\partial}{\partial z}\left[\tilde{M}(y,z)\right] \tag{11}$$

2) *The generating series $\tilde{M}(y,z)$ of locally orientable rooted maps is:*

$$\tilde{M}(y,z) = \cfrac{y}{1 - \cfrac{(y+2)z}{1 - \cfrac{(y+4)z}{1 - \cfrac{(y+6)z}{1 - \dots}}}} \tag{12}$$

3) *The generating series $\tilde{M}(y,z)$ of locally orientable rooted maps is the solution of the following generalized Dyck equation:*

$$\tilde{M}(y,z) = y + z\tilde{M}(y,z)\tilde{M}(y+2,z) \tag{13}$$

Corollary 4.

1) *Both generating series of locally orientable rooted trees and maps are linked by the relation:*

$$\tilde{T}(z) = \frac{1}{1 - z\tilde{M}(2,z)} \tag{14}$$

2) *The generating series $\tilde{T}(z)$ of locally orientable rooted trees is:*

$$\tilde{T}(z) = \cfrac{1}{1 - \cfrac{2z}{1 - \cfrac{4z}{1 - \cfrac{6z}{1 - \dots}}}} \tag{15}$$

Remark 4. These formulae (11-15) can also be obtained directly with the topological method used to obtain eq. (1-7).

6. Rooted hypermaps

We call a map *two-colorable* if its vertices can be colored with exactly two colors, any edge being incident to two vertices with different colors.

The two-colorable property is compatible with the equivalence relation whose classes are rooted maps. Then we can call a two-colorable rooted map, a *rooted hypermap*. Usually the smallest hypermap considered is the edge with two vertices. Here, to simplify our purpose we assume that the smallest hypermap is the vertex.

This definition is equivalent [see 20] to the combinatorial definition of a hypermap [14]. Our goal is to enumerate these equivalence classes of rooted hypermaps independently of their associated surface.

We denote by H(y,z) (resp. $\tilde{H}(y,z)$) the generating series of orientable (resp. locally orientable) rooted hypermaps without respect to genus.

We can prove the following results in a similar way as for rooted maps:

Theorem 4.

1) *The generating series H(y,z) of orientable rooted hypermaps is the solution of the following Riccati differential equation:*

$$H(y,z) = y + zH(y,z)^2 + z^2 \frac{\partial}{\partial z}\left[H(y,z)\right] \tag{16}$$

2) *The generating series $\tilde{H}(y,z)$ of locally orientable rooted hypermaps is the solution of the following Riccati differential equation:*

$$\tilde{H}(y,z) = y + z\tilde{H}(y,z)^2 + 2z^2 \frac{\partial}{\partial z}\left[\tilde{H}(y,z)\right] \tag{17}$$

Theorem 5.

1) *The generating series H(y,z) of orientable rooted hypermaps is:*

$$H(y,z) = \cfrac{y}{1 - \cfrac{yz}{1 - \cfrac{(y+1)z}{1 - \cfrac{(y+1)z}{1 - \cfrac{(y+2)z}{1 - \dots}}}}} \tag{18}$$

2) *The generating series $\tilde{H}(y,z)$ of locally orientable rooted hypermaps is:*

$$\tilde{H}(y,z) = \cfrac{y}{1 - \cfrac{yz}{1 - \cfrac{(y+2)z}{1 - \cfrac{(y+2)z}{1 - \cfrac{(y+4)z}{1 - \dots}}}}} \tag{19}$$

Corollary 5.

1) *The generating series H(y,z) of orientable rooted hypermaps is the solution of the following generalized Dyck equation:*

$$H(y,z) = y + zH(y,z)H(y+1,z) + yz\left\{H(y,z) - H(y+1,z)\right\} \tag{20}$$

2) *The generating series $\tilde{H}(y,z)$ of locally orientable rooted hypermaps is the solution of the following generalized Dyck equation:*

$$\tilde{H}(y,z) = y + z\tilde{H}(y,z)\tilde{H}(y+2,z) + yz\left\{\tilde{H}(y,z) - \tilde{H}(y+2,z)\right\} \tag{21}$$

7. Tables

Here we present enumerating tables for the first terms of M(y,z) (table 1), $\tilde{M}(y,z)$ (table 2), H(y,z) (table 3), $\tilde{H}(y,z)$ (table 4), and finally T(z) and $\tilde{T}(z)$ (table 5).

z\y	1	2	3	4	5	6	7	8	9
0	1								
1	1	1							
2	3	5	2						
3	15	32	22	5					
4	105	260	234	93	14				
5	945	2589	2750	1450	386	42			
6	10395	30669	36500	22950	8178	1586	132		
7	135135	422232	546476	388136	166110	43400	6476	429	
8	2027025	6633360	9163236	7123780	3463634	1092560	220708	26333	1430

Table 1: The number of orientable rooted maps regardless to genus, with respect to edges (z) and vertices (y).

z\y	1	2	3	4	5	6	7	8	9
0	1								
1	2	1							
2	12	10	2						
3	120	128	44	5					
4	1680	2080	936	186	14				
5	30240	41424	22000	5800	772	42			
6	665280	981408	584000	183600	32712	3172	132		
7	17297280	27022848	17487232	6210176	1328880	173600	12952	429	
8	518918400	849070080	586447104	227960960	55418144	8740480	882832	52666	1430

Table 2: The number of locally orientable rooted maps regardless to genus, with respect to edges (z) and vertices (y).

z\y	2	3	4	5	6	7	8	9	10
1	1								
2	1	2							
3	2	6	5						
4	6	22	29	14					
5	24	100	165	130	42				
6	120	548	1041	1044	562	132			
7	720	3528	7406	8638	5992	2380	429		
8	5040	26136	59210	76830	62472	32276	9949	1430	
9	40320	219168	527764	742644	675454	411624	166263	41226	4862

Table 3: The number of orientable rooted hypermaps regardless to genus, with respect to edges (z) and vertices (y).

z\y	2	3	4	5	6	7	8	9	10
1	1								
2	2	2							
3	8	12	5						
4	48	88	58	14					
5	384	800	660	260	42				
6	3840	8768	8328	4176	1124	132			
7	46080	112896	118496	69104	23968	4760	429		
8	645120	1672704	1894720	1229280	499776	129104	19898	1430	
9	10321920	28053504	33776896	23764608	10807264	3292992	665052	82452	4862

Table 4: The number of locally orientable rooted hypermaps regardless to genus, with respect to edges (z) and vertices (y).

z	T(z)	$\tilde{T}(z)$
0	1	1
1	1	2
2	3	12
3	15	120
4	105	1680
5	945	30240
6	10395	665280
7	135135	17297280
8	2027025	518918400
9	34459425	17643225600
10	654729075	670442572800
11	13749310575	28158588057600
12	316234143225	1295295050649600
13	7905853580625	64764752532480000
14	213458046676875	3497296636753920000
15	6190283353629375	202843204931727360000

Table 5: The number of orientable rooted trees T(z) and locally orientable ones $\tilde{T}(z)$, regardless to genus, with respect to edges.

8. Conclusion

We first studied orientable rooted maps regardless to genus with regard to vertices and edges. We obtained a new differential functional equation (1) for the corresponding series. It is the first time such a differential equation is used in map enumeration. Then we expressed this series with a simple continued fraction (2), leading to a generalization (3) of Dyck's equation for rooted planar trees. The importance of Dyck's equation in the study of rooted planar trees induces us to look at its generalization (3) with a great interest.

Then we studied orientable rooted trees. We gave a differential equation (5), a continued fraction (6), and a astonishing relation between maps and trees (7). Then we presented explicit enumerations formulae (8) and (9). Finally we gave similar results for locally orientable maps and trees (Theorem 3, Corollary 4), and for rooted hypermaps (Theorems 4 and 5, Corollary 5).

These results open some new problems. We are now working on the topological interpretations of the generalized Dyck equations (3, 13, 20-21), and the relations between trees and maps (7) and (14). Moreover, the continued fractions found here appear in other papers (see for example [22]). The links to maps area are to study.

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