Equivalence of the Bethe Ansatz and Gessel-Viennot Theorem for Non-intersecting Paths

R. Brak†, J. W. Essam‡ and A. L. Owczarek†*
†Department of Mathematics and Statistics,
The University of Melbourne,
Parkville, Victoria 3052, Australia
‡Department of Mathematics,
Royal Holloway College, University of London,
Egham, Surrey TW20 0EX, England.

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Abstract

We show how the problem of non-intersecting lattice paths on the directed square lattice can be formulated as difference equations. The difference equations are encoded by the action of various "transfer matrices". We state several theorems that demonstrate how the coordinate Bethe Ansatz for the eigenvectors of the transfer matrices, given certain conditions hold, is equivalent to the Gessel-Viennot determinant for the number of configurations of N non-intersecting lattice paths on the directed square lattice. Another way of viewing this result is that it is a linear algebra proof of the Gessel-Viennot theorem for the particular case considered in this paper. This is significant as the Bethe Ansatz is potentially capable of solving various lattice paths problems, such as osculating lattice paths, which are beyond the scope of the Gessel-Viennot theorem.

Résumé

*email: brak@maths.mu.oz.au, j.essam@vms.rhbnc.ac.uk, aleks@ms.unimelb.edu.au

1 Introduction

Non-intersecting paths have been extensively studied in various contexts [1, 2, 3, 4, 5, 6], culminating in the Gessel-Viennot theorem [7, 8]. All these studies express the number of configurations as the value of a determinant. Non-intersecting paths arise in another context, that of vertex models in statistical mechanics [9], where it was noticed [10, 6], that if the vertices of the "six-vertex" model are drawn in a particular way they could be interpreted as lattice paths. If one of the vertices had weight zero, giving a five-vertex model, the resulting paths were non-intersecting. The vertex models are traditionally solved by expressing the partition function (a generating function) in terms of transfer matrices. The partition function is then evaluated by either of two very powerful techniques, that of commuting transfer matrices [11] or by direct diagonalisation of the transfer matrices using the coordinate Bethe Ansatz [12, 13].

In this paper we will show that the Bethe Ansatz and the Gessel-Viennot Theorem are equivalent so long as the eigenvectors of the $N = 1$ path transfer matrix span its row (or column) space.

The connection between the two methods is potentially significant as preliminary work on osculating lattice paths and their relation to alternating sign matrices [14] has shown that the Bethe Ansatz has the potential to solve lattice paths problems which are beyond the scope of the Gessel-Viennot theorem.

Definitions and Notation The $\mathcal{S}$ be a square lattice rotated 45° directed in the North-East and South-East directions. Label the vertices of $\mathcal{S}$ with orthogonal coordinates (x, y). A N-vertex is an N-tuple of distinct vertices of $\mathcal{S}$ all of which have the same x coordinate. If $\mathbf{y}^i = (y_1^i, \dots, y_N^i)$ and $\mathbf{y}^f = (y_1^f, \dots, y_N^f)$ are N-vertices of $\mathcal{S}$, a N-path from $\mathbf{y}^i$ to $\mathbf{y}^f$ is a N-tuple $\omega = (\omega_1, \dots, \omega_N)$ such that ω_i is a path from y_α^i to y_α^f. The N-path is non-intersecting if the paths ω_α are vertex disjoint. Assign a weight to every edge of $\mathcal{S}$ and a weight to each of the vertices of the N-vertex $\mathbf{y}^i$. We define the weight, $w(\omega_\alpha)$ of a path ω_α as the product of the weighs of its edges and the weight of the initial vertex. The weight $W(\omega)$ of the N-path is the product of the weights of its components.

We shall only consider the following special case of edge weights: the weights of all edges of $\mathcal{S}$ above the line $y = L > 0$, and below the line $y = 0$ are set to zero. This restricts the paths to a strip of width L – see figure 1. By controlling the width of the strip we can still obtain paths in the half plane and full plane. Let Ω_L^N be the set of all non-intersecting N-paths with non-zero weights from $\mathbf{y}^i$ to $\mathbf{y}^f$. Note, all the paths are necessarily the same length, say t. We are interested in computing the the strip generating function

$$\overset{=\mathcal{N}}{Z_t}(\mathbf{y}^i \to \mathbf{y}^f) = \sum_{\omega \in \Omega_L^N} W(\omega) \tag{1.1}$$

Note, the double bar above the Z denotes the strip.

We require the following sub-domains of $\mathbb{Z}^N$

$$\overset{o}{\mathcal{S}}_L = \{y \,|\, 1 \le y \le L, y \in \mathbb{Z} \text{ and } y \text{ odd}\}, \tag{1.2a}$$

$$\overset{e}{\mathcal{S}}_L = \{y \,|\, 0 \le y \le L, y \in \mathbb{Z} \text{ and } y \text{ even}\}, \tag{1.2b}$$

$$\mathcal{S}_L = \{y \,|\, 0 \le y \le L, y \in \mathbb{Z}\}, \tag{1.2c}$$

$$\overset{o}{\mathcal{U}}_L = \{(y_1, \ldots, y_N) \,|\, 1 \le y_1 < \ldots < y_N \le L,\ y_i \in \overset{o}{\mathcal{S}}_L\} \tag{1.2d}$$

$$\overset{e}{\mathcal{U}}_L = \{(y_1, \ldots, y_N) \,|\, 0 \le y_1 < \ldots < y_N \le L,\ y_i \in \overset{e}{\mathcal{S}}_L\} \tag{1.2e}$$

$$\mathcal{U}_L = \{(y_1, \ldots, y_N) \,|\, 0 \le y_1 < \ldots < y_N \le L,\ y_i \in \mathcal{S}_L\} \tag{1.2f}$$

We will use $\overset{p}{\mathcal{U}}_L$ to denote $\overset{o}{\mathcal{U}}_L$ or $\overset{e}{\mathcal{U}}_L$. We will only consider the case that L is odd so that $|\ \overset{e}{\mathcal{U}}_L\ | = |\ \overset{o}{\mathcal{U}}_L\ | = \binom{\frac{1}{2}(L+1)}{N}$. (If L is even a null space enters the subsequent analysis of the transfer matrices leading to a distracting complication.)

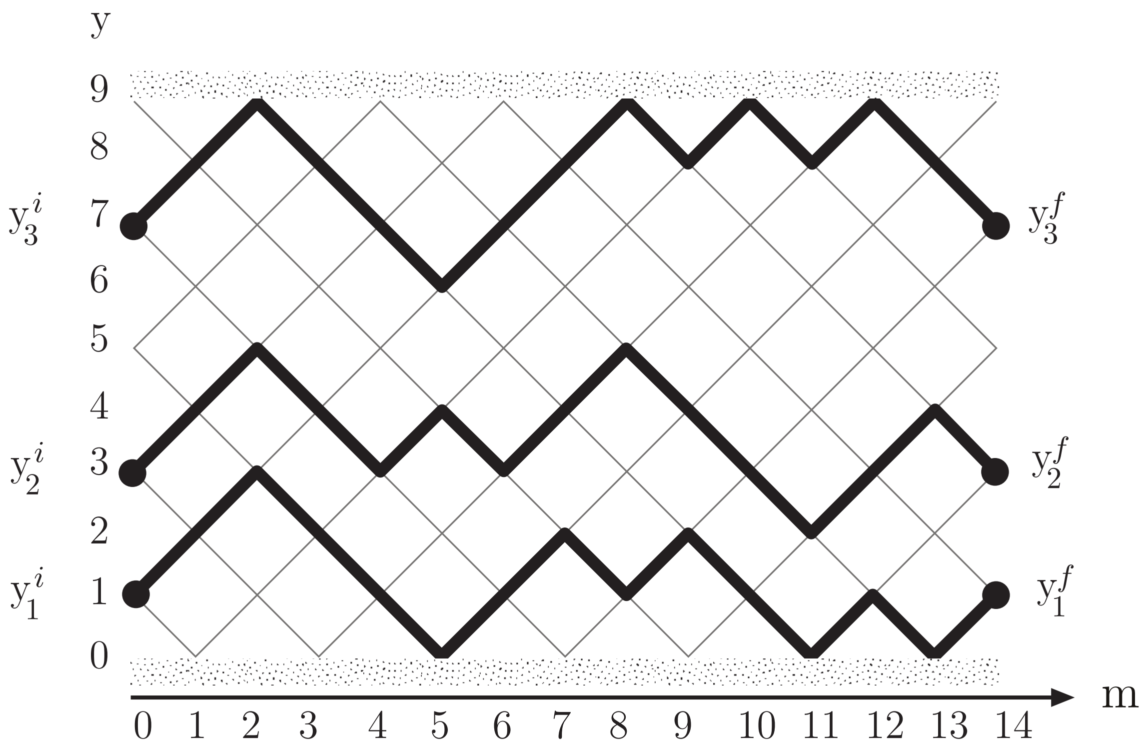

Figure 1: *Three non-intersecting paths of length* $t = 14$ *in a strip of width* $L = 9$. *The variables* x, y_α^i *and* y_α^f *shown. The path closest to the lower "wall" has weight* $v(1)w(1,2)w(2,3)w(3,2)\cdots w(1,0)w(0,1)$.

We will show that $\overset{=\mathcal{N}}{Z_t}(\mathbf{y}^i \to \mathbf{y}^f)$, is given by the following Gessel-Viennot determinant:

$$\overset{=\mathcal{N}}{Z_t}(\mathbf{y}^i \to \mathbf{y}^f) = \begin{vmatrix} \overset{=\mathcal{S}}{Z_t}(y_1^i \to y_1^f) & \overset{=\mathcal{S}}{Z_t}(y_1^i \to y_2^f) & \dots & \overset{=\mathcal{S}}{Z_t}(y_1^i \to y_N^f) \\ \overset{=\mathcal{S}}{Z_t}(y_2^i \to y_1^f) & \overset{=\mathcal{S}}{Z_t}(y_2^i \to y_2^f) & \dots & \overset{=\mathcal{S}}{Z_t}(y_2^i \to y_N^f) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \overset{=\mathcal{S}}{Z_t}(y_N^i \to y_1^f) & \overset{=\mathcal{S}}{Z_t}(y_N^i \to y_2^f) & \dots & \overset{=\mathcal{S}}{Z_t}(y_N^i \to y_N^f) \end{vmatrix} \tag{1.3}$$

where single path the generating function, $\overset{=\mathcal{S}}{Z_t}(y_j^i \to y_k^f)$ is defined as

$$\overset{=\mathcal{S}}{Z_t}(\mathbf{y}^i \to \mathbf{y}^f) = \sum_{\omega \in \Omega_L^1} W(\omega) \tag{1.4}$$

Remark. Note, the determinant (1.3) can be obtained directly from the Gessel-Viennot theorem, however the point of this paper is that the same result can also be obtained from an eigenvector Ansatz.

2 From Bethe Ansatz to determinant

2.1 Transfer matrix formulation

The generating function $\overset{=\mathcal{N}}{Z_t}(\mathbf{y}^i \to \mathbf{y}^f)$ can be written as the matrix element of a product of "transfer matrices". The calculation of the generating function via the transfer matrices then requires we diagonalise the matrices. We will show that if this can be done for the $N = 1$ case then the N-path problem is given by (1.3). First we define the matrices and show how $\overset{=\mathcal{N}}{Z_t}$ can be expressed in terms of the transfer matrices.

Definition 1. Let $y \in \overset{e}{\mathcal{S}}_L$ and $y' \in \overset{o}{\mathcal{S}}_L$. For $N = 1$ the two one-step single path transfer matrices are defined as

$$\left(\overset{eo}{\mathbf{T}}_1\right)_{y,y'} = \begin{cases} 0 & \text{if } |y - y'| > 1 \\ w(y, y') & \text{if } |y - y'| = 1 \end{cases} \tag{2.5a}$$

and

$$\left(\overset{oe}{\mathbf{T}}_1\right)_{y',y} = \begin{cases} 0 & \text{if } |y' - y| > 1 \\ w(y', y) & \text{if } |y - y'| = 1. \end{cases} \tag{2.5b}$$

where $w(y, y')$ is the weight of the edge of $\mathcal{S}$ from (x, y) to (x, y'). The N-path transfer matrices for $N > 1$ are sub-matrices of the direct product of the above $N = 1$ matrices:

$$(\overset{oe}{\mathbf{T}}_N)_{\mathbf{y},\mathbf{y}'} = \left(\bigotimes_{i=1}^{N} \overset{oe}{\mathbf{T}}_1\right)_{\mathbf{y},\mathbf{y}'} \qquad \mathbf{y} \in \overset{o}{\mathcal{U}}_L \text{ and } \mathbf{y}' \in \overset{e}{\mathcal{U}}_L \tag{2.6a}$$

and

$$(\overset{eo}{\mathbf{T}}_N)_{\mathbf{y}',\mathbf{y}} = \left(\bigotimes_{i=1}^{N} \overset{eo}{\mathbf{T}}_1\right)_{\mathbf{y}',\mathbf{y}} \qquad \mathbf{y}' \in \overset{e}{\mathcal{U}}_L \text{ and } \mathbf{y} \in \overset{o}{\mathcal{U}}_L \tag{2.6b}$$

The two "two-step" transfer matrices are then defined as

$$\overset{e\cdot e}{\mathbf{T}}_N = \overset{eo}{\mathbf{T}}_N \overset{oe}{\mathbf{T}}_N \tag{2.7a}$$

$$\overset{o\cdot o}{\mathbf{T}}_N = \overset{oe}{\mathbf{T}}_N \overset{eo}{\mathbf{T}}_N. \tag{2.7b}$$

Remark. (1) Note that only "nearest neighbour" steps are allowed in all cases, and that in general $\left(\overset{oe}{\mathbf{T}}_1\right)_{y',y} \neq \left(\overset{eo}{\mathbf{T}}_1\right)_{y,y'}$. Since we are only considering L odd we have that $\overset{oe}{\mathbf{T}}_N$ and $\overset{eo}{\mathbf{T}}_N$ are square matrices.

The generating function $\overset{=\mathcal{N}}{Z_t}(\mathbf{y}^i \to \mathbf{y}^f)$ of non-intersecting N-paths of length t in a strip is related to $\overset{=\mathcal{N}}{Z}_{t-1}(\mathbf{y}^i \to \mathbf{y})$ by recurrence, the coefficients of which are the elements of one of the two one-step transfer matrices defined above. This relationship is given by the following lemma.

Lemma 1. *For $t > 0$, the generating function $\overset{=\mathcal{N}}{Z_t}(\mathbf{y}^i \to \mathbf{y}^f)$ is given by*

$$\overset{=\mathcal{N}}{Z_t}(\mathbf{y}^i \to \mathbf{y}^f) = \begin{cases} \sum_{\mathbf{y}\in\overset{e}{\mathcal{U}}_L} \overset{=\mathcal{N}}{Z}_{t-1}(\mathbf{y}^i \to \mathbf{y})(\overset{eo}{\mathbf{T}}_N)_{\mathbf{y},\mathbf{y}^f} & \textit{for} \quad \mathbf{y}^f \in \overset{o}{\mathcal{U}}_L \\ \\ \sum_{\mathbf{y}\in\overset{o}{\mathcal{U}}_L} \overset{=\mathcal{N}}{Z}_{t-1}(\mathbf{y}^i \to \mathbf{y})(\overset{oe}{\mathbf{T}}_N)_{\mathbf{y},\mathbf{y}^f} & \textit{for} \quad \mathbf{y}^f \in \overset{e}{\mathcal{U}}_L \end{cases}. \tag{2.8}$$

and for $t = 0$,

$$\overset{=\mathcal{N}}{Z_0}(\mathbf{y}^i \to \mathbf{y}^f) = \delta_{\mathbf{y}^i,\mathbf{y}^f} V(\mathbf{y}^i), \tag{2.9}$$

where

$$V(\mathbf{y}^i) = \prod_{\alpha=1}^{N} v(y^i_\alpha), \tag{2.10}$$

and $v(y^i_\alpha)$ is the weight of vertex $(0, y^i_\alpha)$.

A simple proof of this Lemma can be constructed using induction on t.

Remarks. (1) The restriction of $\overset{eo}{\mathbf{T}}_N$ and $\overset{oe}{\mathbf{T}}_N$ to submatrices of the direct product eliminates the possibility of two paths arriving at the same lattice vertex.
(2) The condition that the single path transfer matrix with elements that vanish for $|y' - y| > 1$ prevents the generation of configurations in which pairs of paths "cross" *without* without having a lattice vertex in common. This the analogue of the non-crossing condition of the Gessel-Viennot Theorem. This "non-crossing" condition is unnecessarily restrictive in the single path case, but necessary for $N > 1$.

Corollary. *For $t = 2r$, r a positive integer,*

$$\overset{=\mathcal{N}}{Z}_{2r}(\mathbf{y}^i \to \mathbf{y}^f) = \begin{cases} V(\mathbf{y}^i)\left((\overset{e\cdot e}{\mathbf{T}}_N)^r\right)_{\mathbf{y}^i,\mathbf{y}^f} & \text{for} \quad \mathbf{y}^i \in \overset{e}{\mathcal{U}}_L \text{ and } \mathbf{y}^f \in \overset{e}{\mathcal{U}}_L \\ V(\mathbf{y}^i)\left((\overset{o\cdot o}{\mathbf{T}}_N)^r\right)_{\mathbf{y}^i,\mathbf{y}^f} & \text{for} \quad \mathbf{y}^i \in \overset{o}{\mathcal{U}}_L \text{ and } \mathbf{y}^f \in \overset{o}{\mathcal{U}}_L \end{cases} \tag{2.11a}$$

and for $t = 2r + 1$,

$$\overset{=\mathcal{N}}{Z}_{2r+1}(\mathbf{y}^i \to \mathbf{y}^f) = \begin{cases} V(\mathbf{y}^i)\sum_{\mathbf{y}\in\overset{e}{\mathcal{U}}_L}\left((\overset{e\cdot e}{\mathbf{T}}_N)^r\right)_{\mathbf{y}^i,\mathbf{y}}(\overset{oe}{\mathbf{T}}_N)_{\mathbf{y},\mathbf{y}^f} & \text{for} \quad \mathbf{y}^i \in \overset{e}{\mathcal{U}}_L \text{ and } \mathbf{y}^f \in \overset{o}{\mathcal{U}}_L \\ V(\mathbf{y}^i)\sum_{\mathbf{y}\in\overset{o}{\mathcal{U}}_L}\left((\overset{o\cdot o}{\mathbf{T}}_N)^r\right)_{\mathbf{y}^i,\mathbf{y}}(\overset{eo}{\mathbf{T}}_N)_{\mathbf{y},\mathbf{y}^f} & \text{for} \quad \mathbf{y}^i \in \overset{o}{\mathcal{U}}_L \text{ and } \mathbf{y}^f \in \overset{e}{\mathcal{U}}_L \end{cases} \tag{2.11b}$$

2.2 From transfer matrices to determinants

The theorems proven below show that the *equivalence* of the Bethe Ansatz for the eigenvectors of (2.7), in the form of equation (2.17) and the result of the Gessel-Viennot Theorem in the form (2.33) rests on showing that for any given set of weights that the resulting one path eigenvectors (conditions of Lemma 2) span the row (or column) space of the corresponding two-step transfer matrix.

In particular, the first theorem states the conditions under which the N-path transfer matrices can be diagonalised: the major condition is that the one-path transfer matrices can be diagonalised — see Lemma 2. The second theorem states that if the Bethe Ansatz gives a complete set of eigenvectors for the N-path problem for $N > 1$, then the N-path generating function is a determinant of one-path generating functions.

Lemma 2.

(a) Let $\overset{e\cdot e}{\mathbf{T}}_1$ and $\overset{o\cdot o}{\mathbf{T}}_1$ be defined by (2.7) then if there exist vectors $\overset{o}{\varphi}{}^R_k$ and $\overset{e}{\varphi}{}^R_k$ such that

$$\overset{eo}{\mathbf{T}}_1\overset{o}{\varphi}{}^R_k = \lambda_k\,\overset{e}{\varphi}{}^R_k \qquad \text{and} \qquad \overset{oe}{\mathbf{T}}_1\overset{e}{\varphi}{}^R_k = \lambda_k\,\overset{o}{\varphi}{}^R_k \tag{2.12}$$

then $\overset{o}{\varphi}{}^R_k$ and $\overset{e}{\varphi}{}^R_k$ are right eigenvectors of $\overset{o\cdot o}{\mathbf{T}}_1$ and $\overset{e\cdot e}{\mathbf{T}}_1$ respectively with eigenvalue λ_k^2.

(b) Let $\{\overset{o}{\varphi}{}^R_k\}_{k\in\mathcal{K}_1}$ and $\{\overset{e}{\varphi}{}^R_k\}_{k\in\mathcal{K}_1}$, where $\mathcal{K}_1$ is some index set, be maximal sets of independent vectors satisfying (2.12). If these sets span the respective column spaces (in which case they are said to be complete) of $\overset{e\cdot e}{\mathbf{T}}_1$ and $\overset{o\cdot o}{\mathbf{T}}_1$, then corresponding sets $\{\overset{o}{\varphi}{}^L_k\}_{k\in\mathcal{K}_1}$ and $\{\overset{e}{\varphi}{}^L_k\}_{k\in\mathcal{K}_1}$ of row vectors may be found such that

$$\overset{p}{\varphi}{}^{L*}_k \cdot \overset{p}{\varphi}{}^R_{k'} = \delta_{k,k'} \qquad \text{and} \qquad \sum_{k\in\mathcal{K}_1}\overset{p}{\varphi}{}^R_k(y)\,\overset{p}{\varphi}{}^{L*}_k(y') = \delta_{y,y'} \tag{2.13}$$

for each $p \in \{e, o\}$, where the $*$ denotes complex conjugation.

(c) Let $\{\overset{o}{\varphi}{}^L_k\}_{k\in\mathcal{K}_1}$ *and* $\{\overset{e}{\varphi}{}^L_k\}_{k\in\mathcal{K}_1}$ *satisfy* (2.13) *then*

$$\overset{o}{\varphi}{}^L_k \overset{oe}{\mathbf{T}}_1 = \lambda_k \overset{e}{\varphi}{}^L_k \qquad \text{and} \qquad \overset{e}{\varphi}{}^L_k \overset{eo}{\mathbf{T}}_1 = \lambda_k \overset{o}{\varphi}{}^L_k \tag{2.14}$$

and also $\overset{o}{\varphi}{}^L_k$ *and* $\overset{e}{\varphi}{}^L_k$ *are left eigenvectors of* $\overset{o\cdot o}{\mathbf{T}}_1$ *and* $\overset{e\cdot e}{\mathbf{T}}_1$ *respectively with eigenvalue* λ_k^2.

The proof of the lemma is elementary linear algebra and we omit it.

Remark. Notice that if λ_k is a solution of (2.12) then so is $-\lambda_k$ with vector $\overset{e}{\varphi}{}^{\mathrm{R}}_k$ replaced by $-\overset{e}{\varphi}{}^{\mathrm{R}}_k$. These vectors are clearly not independent and normally sufficient independent vectors to form a spanning set are obtained by taking only the positive values of λ_k.

From the above left and right one-path vectors we now construct the N-path vectors and hence eigenvectors of $\overset{e\cdot e}{\mathbf{T}}_N$ and $\overset{o\cdot o}{\mathbf{T}}_N$.

Theorem 1. *Let* $\overset{e\cdot e}{\mathbf{T}}_N$ *and* $\overset{o\cdot o}{\mathbf{T}}_N$, $N > 1$, *be given by equations* (2.7). *By imposing an arbitrary ordering on the elements of* $\mathcal{K}_1$ *define*

$$\mathcal{K}_N = \{\mathbf{k} = (k_1, k_2, \dots k_N) | k_i \in \mathcal{K}_1 \text{ and } k_1 < k_2 < \dots < k_N\} \tag{2.15}$$

and

$$\Lambda_{\mathbf{k}} = \prod_{\alpha=1}^{N} \lambda_{k_\alpha}. \tag{2.16}$$

(a) If for $C \in \{L, R\}$ *and* $p \in \{e, o\}$, $\{\overset{p}{\varphi}{}^C_k\}_{k\in\mathcal{K}_1}$ *satisfy the conditions of Lemma 2 then the vectors* $\{\overset{p}{\Phi}{}^C_{\mathbf{k}}\}_{\mathbf{k}\in\mathcal{K}_N}$ *given by the Bethe Ansatz,*

$$\overset{p}{\Phi}{}^C_{\mathbf{k}}(\mathbf{y}) = \sum_{\sigma\in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N} \overset{p}{\varphi}{}^C_{k_{\sigma_\alpha}}(y_\alpha) = \sum_{\sigma\in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N} \overset{p}{\varphi}{}^C_{k_\alpha}(y_{\sigma_\alpha}) \qquad \mathbf{y} \in \overset{p}{\mathcal{U}}_L, \tag{2.17}$$

satisfy

$$\overset{oe}{\mathbf{T}}_N \overset{e}{\Phi}{}^R_{\mathbf{k}} = \Lambda_{\mathbf{k}} \overset{o}{\Phi}{}^R_{\mathbf{k}} \qquad \text{and} \qquad \overset{eo}{\mathbf{T}}_N \overset{o}{\Phi}{}^R_{\mathbf{k}} = \Lambda_{\mathbf{k}} \overset{e}{\Phi}{}^R_{\mathbf{k}}. \tag{2.18}$$

(b) Moreover Lemma 2 holds with $\overset{eo}{\mathbf{T}}_1$ *and* $\overset{oe}{\mathbf{T}}_1$ *replaced by* $\overset{eo}{\mathbf{T}}_N$ *and* $\overset{oe}{\mathbf{T}}_N$, $\mathcal{K}_1$ *replaced by* $\mathcal{K}_N$, *and* λ_k *replaced by* $\Lambda_{\mathbf{k}}$.

Remark. Note, if further neighbour steps are allowed then the crossing condition would be violated and in general, it is *not* possible to use the Bethe Ansatz to obtain a complete set of eigenvectors.

The proofs of part of this theorem and Theorem 2 require the following result.

Proposition 1. *For* $\mathbf{k} \in \mathcal{K}_N$ *and* $\mathbf{y} \in \overset{p}{\mathcal{U}}_L$ *let*

$$\Phi_{\mathbf{k}}(\mathbf{y}) = \sum_{\sigma\in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N} \phi_{k_{\sigma_\alpha}}(y_\alpha) \qquad \text{and} \qquad \Psi_{\mathbf{k}}(\mathbf{y}) = \sum_{\sigma\in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N} \psi_{k_{\sigma_\alpha}}(y_\alpha). \tag{2.19}$$

Also let

$$f(\mathbf{k}) = \prod_{\alpha=1}^{N} f(k_\alpha) \tag{2.20}$$

then

$$\sum_{\mathbf{k}\in\mathcal{K}_N} f(\mathbf{k})\Phi_{\mathbf{k}}(\mathbf{y})\Psi_{\mathbf{k}}(\mathbf{y}') = \sum_{\sigma\in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N}\left(\sum_{k_\alpha\in\mathcal{K}_1} f(k_\alpha)\phi_{k_\alpha}(y_\alpha)\psi_{k_\alpha}(y'_{\sigma\alpha})\right) \tag{2.21}$$

and

$$\sum_{\mathbf{y}\in\overset{p}{\mathcal{U}}_L} \Phi_{\mathbf{k}}(\mathbf{y})\Psi_{\mathbf{k}'}(\mathbf{y}) = \sum_{\sigma\in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N}\left(\sum_{y_\alpha\in\overset{p}{\mathcal{S}}_L} \phi_{k_\alpha}(y_\alpha)\psi_{k'_{\sigma\alpha}}(y_\alpha)\right) \tag{2.22}$$

Proof.

$$\sum_{\mathbf{k}\in\mathcal{K}_N} f(\mathbf{k})\Phi_{\mathbf{k}}(\mathbf{y})\Psi_{\mathbf{k}}(\mathbf{y}') = \sum_{\sigma\in P_N}\epsilon_\sigma \sum_{\mathbf{k}\in\mathcal{K}_N}\sum_{\tau\in P_N}\epsilon_\tau \prod_{\alpha=1}^{N} f(k_\alpha)\phi_{k_{\tau\alpha}}(y_\alpha)\psi_{k_{\sigma\alpha}}(y'_\alpha) \tag{2.23}$$

$$= \sum_{\sigma'\in P_N}\epsilon_{\sigma'} \sum_{\mathbf{k}\in\mathcal{K}_N}\sum_{\tau\in P_N} \prod_{\alpha=1}^{N} f(k_\alpha)\phi_{k_{\tau\alpha}}(y_\alpha)\psi_{k_{\tau\alpha}}(y'_{\sigma'_\alpha}) \tag{2.24}$$

The double sum over permutations, τ and $\mathbf{k}\in\mathcal{K}_N$ is equivalent to summing each k_α independently over $\mathcal{K}_1$ (terms for which two or more components of $\mathbf{k}$ are equal make zero contribution) and the first result follows. The second result follows in the same way by interchanging the roles of k and y. □

Proof. (*of Theorem 1*) (a) We first obtain the cyclic property (2.18) as follows.

$$\left(\overset{oe}{\mathbf{T}}_N\,\overset{e}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}}\right)_{\mathbf{y}} = \sum_{\mathbf{y}'\in\overset{e}{\mathcal{U}}_L}\sum_{\sigma\in P_N}\epsilon_\sigma\left(\overset{oe}{\mathbf{T}}_N\right)_{\mathbf{y},\mathbf{y}'}\prod_{\alpha=1}^{N}\overset{e}{\varphi}{}^{\mathrm{R}}_{k_{\sigma\alpha}}(y'_\alpha) \tag{2.25a}$$

$$= \sum_{\sigma\in P_N}\epsilon_\sigma\sum_{\mathbf{y}'\in\overset{e}{\mathcal{U}}_L}\prod_{\alpha=1}^{N}\left(\overset{oe}{\mathbf{T}}_1\right)_{y_\alpha,y'_\alpha}\overset{e}{\varphi}{}^{\mathrm{R}}_{k_{\sigma\alpha}}(y'_\alpha) \qquad \text{(using (2.6))} \tag{2.25b}$$

$$= \sum_{\sigma\in P_N}\epsilon_\sigma\left[\sum_{y'_1\in\overset{e}{\mathcal{S}}_L}\left(\overset{oe}{\mathbf{T}}_1\right)_{y_1,y'_1}\overset{e}{\varphi}{}^{\mathrm{R}}_{k_{\sigma 1}}(y'_1)\right]\cdots\left[\sum_{y'_N\in\overset{e}{\mathcal{S}}_L}\left(\overset{oe}{\mathbf{T}}_1\right)_{y_N,y'_N}\overset{e}{\varphi}{}^{\mathrm{R}}_{k_{\sigma N}}(y'_N)\right] \tag{2.25c}$$

$$= \sum_{\sigma\in P_N}\epsilon_\sigma\left[\lambda_{k_{\sigma 1}}\,\overset{o}{\varphi}{}^{\mathrm{R}}_{k_{\sigma 1}}(y_1)\right]\cdots\left[\lambda_{k_{\sigma N}}\,\overset{o}{\varphi}{}^{\mathrm{R}}_{k_{\sigma N}}(y_N)\right] \qquad \text{(using (2.12))} \tag{2.25d}$$

$$= \Lambda_{\mathbf{k}}\,\overset{o}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}}(\mathbf{y}) \tag{2.25e}$$

The critical step, and the whole reason for introducing the Bethe Ansatz, is to enable one to go from the restricted sums of (2.25b) to the unrestricted sums in (2.25c). This is justified for two reasons,

1. since $\overset{e}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}}(\mathbf{y}')$ is a determinant, if any of the y_α's are equal then $\overset{e}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}} = 0$ – this allows the restriction $y'_1 < y'_2 \ldots < y'_N$ on the sum to be relaxed to $y'_1 \le y'_2 \ldots \le y'_N$

2. the y_α are in strictly increasing order combined with the fact that the matrix elements of $\overset{oe}{\mathbf{T}}_1$, are only non-zero if $|y_\alpha - y'_\alpha| \le 1$ (the "non-crossing" condition) allows the restriction on the sum to be removed altogether.

The second part of (2.18) follows mutatis mutandis.

Also, $\overset{e}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}}$ is a right eigenvector of $\overset{e\cdot e}{\mathbf{T}}_N$ with eigenvalue $\Lambda^2_{\mathbf{k}}$, since

$$\overset{e\cdot e}{\mathbf{T}}_N \overset{e}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}} = \overset{eo}{\mathbf{T}}_N \overset{oe}{\mathbf{T}}_N \overset{e}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}} = \overset{eo}{\mathbf{T}}_N \Lambda_{\mathbf{k}} \overset{o}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}} = \Lambda^2_{\mathbf{k}} \overset{e}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}}$$

which follows from (2.18). Similarly $\overset{o}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}}$ is a right eigenvector of $\overset{o\cdot o}{\mathbf{T}}_N$ with eigenvalue $\Lambda^2_{\mathbf{k}}$ which completes (a).

(b) First consider orthogonality and normalisation. Using (2.22), for $\mathbf{k}, \mathbf{k}' \in \mathcal{K}_N$

$$\begin{aligned}
\sum_{\mathbf{y} \in \overset{p}{\mathcal{U}}_L} \overset{p}{\Phi}{}^{\mathrm{L}*}_{\mathbf{k}}(\mathbf{y})\, \overset{p}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}'}(\mathbf{y}) &= \sum_{\sigma \in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N} \left(\sum_{y_\alpha \in \overset{p}{\mathcal{S}}_L} \overset{p}{\varphi}{}^{\mathrm{L}*}_{k_\alpha}(y_\alpha)\, \overset{p}{\varphi}{}^{\mathrm{R}}_{k'_{\sigma\alpha}}(y_\alpha) \right) \\
&= \sum_{\sigma \in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N} \delta_{k_\alpha, k'_{\sigma\alpha}} \qquad \text{(using (2.13))} \\
&= \prod_{\alpha=1}^{N} \delta_{k_\alpha, k'_\alpha}
\end{aligned}$$

since the components of $\mathbf{k}$ and $\mathbf{k}'$ are in the same order only the identity permutation gives a non-zero delta function product. Thus

$$\overset{p}{\Phi}{}^{\mathrm{L}*}_{\mathbf{k}} \cdot \overset{p}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}} = \delta_{\mathbf{k},\mathbf{k}'}. \tag{2.26}$$

Our derivation of the "completeness condition" closely parallels the previous derivation but using (2.21). For $\mathbf{y}, \mathbf{y}' \in \overset{p}{\mathcal{U}}_L$

$$\begin{aligned}
\sum_{\mathbf{k} \in \mathcal{K}_N} \overset{p}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}}(\mathbf{y})\, \overset{p}{\Phi}{}^{\mathrm{L}*}_{\mathbf{k}}(\mathbf{y}') &= \sum_{\sigma \in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N} \left(\sum_{k_\alpha \in \mathcal{K}_1} \overset{p}{\varphi}{}^{\mathrm{R}}_{k_\alpha}(y_\alpha)\, \overset{p}{\varphi}{}^{\mathrm{L}*}_{k_\alpha}(y'_{\sigma\alpha}) \right) \\
&= \sum_{\sigma \in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N} \delta_{y_\alpha, y'_{\sigma\alpha}} \qquad \text{(using (2.13))} \\
&= \prod_{\alpha=1}^{N} \delta_{y_\alpha, y'_\alpha}
\end{aligned}$$

so

$$\sum_{\mathbf{k} \in \mathcal{K}_N} \overset{p}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}}(\mathbf{y})\, \overset{p}{\Phi}{}^{\mathrm{L}*}_{\mathbf{k}}(\mathbf{y}') = \delta_{\mathbf{y},\mathbf{y}'} \tag{2.27}$$

Notice that $|\mathcal{K}_N| = \binom{\frac{1}{2}(L+1)}{N}$ which is the row (and column) space dimension, as it should be for completeness.

Using basic linear algebra gives

$$\overset{o}{\Phi}{}^{\mathrm{L}}_{\mathbf{k}} \overset{oe}{\mathbf{T}}_N = \overset{e}{\Phi}{}^{\mathrm{L}}_{\mathbf{k}} \Lambda_{\mathbf{k}} \qquad \overset{e}{\Phi}{}^{\mathrm{L}}_{\mathbf{k}} \overset{eo}{\mathbf{T}}_N = \overset{o}{\Phi}{}^{\mathrm{L}}_{\mathbf{k}} \Lambda_{\mathbf{k}} \tag{2.28}$$

and

$$\overset{e}{\Phi}{}^{\mathrm{L}}_{\mathbf{k}} \overset{e\cdot e}{\mathbf{T}}_N = \overset{e}{\Phi}{}^{\mathrm{L}}_{\mathbf{k}} \Lambda^2_{\mathbf{k}} \qquad \overset{o}{\Phi}{}^{\mathrm{L}}_{\mathbf{k}} \overset{o\cdot o}{\mathbf{T}}_N = \overset{o}{\Phi}{}^{\mathrm{L}}_{\mathbf{k}} \Lambda^2_{\mathbf{k}} \tag{2.29}$$

□

Lemma 3. *Assuming the conditions of Theorem 1 hold then*

$$\overset{=\mathcal{N}}{Z_t}(\mathbf{y}^i \to \mathbf{y}^f) = V(\mathbf{y}^i) \sum_{\mathbf{k}\in\mathcal{K}_N} \overset{p'}{\Phi}{}^{R}_{\mathbf{k}}(\mathbf{y}^i) \Lambda^t_{\mathbf{k}} \overset{p}{\Phi}{}^{L*}_{\mathbf{k}}(\mathbf{y}^f) \qquad \mathbf{y}^i \in \overset{p'}{\mathcal{U}}_L \textit{ and } \quad \mathbf{y}^f \in \overset{p}{\mathcal{U}}_L \tag{2.30}$$

where if t is even, $p' = p$ but otherwise p and p' are of opposite parity.

Proof. If the conditions of Theorem 1 hold then equation (2.13) implies (2.26), (2.27), (2.28) and (2.29) are valid. Using (2.28) and (2.27) it follows that

$$(\overset{oe}{\mathbf{T}}_N)_{\mathbf{y},\mathbf{y}'} = \sum_{\mathbf{k}\in\mathcal{K}_N} \Lambda_{\mathbf{k}} \overset{o}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}}(\mathbf{y}) \overset{e}{\Phi}{}^{\mathrm{L}*}_{\mathbf{k}}(\mathbf{y}') \qquad \text{and} \qquad (\overset{eo}{\mathbf{T}}_N)_{\mathbf{y},\mathbf{y}'} = \sum_{\mathbf{k}\in\mathcal{K}_N} \Lambda_{\mathbf{k}} \overset{e}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}}(\mathbf{y}) \overset{o}{\Phi}{}^{\mathrm{L}*}_{\mathbf{k}}(\mathbf{y}'). \tag{2.31}$$

Also, using (2.29) and (2.27) it follows that

$$(\overset{e\cdot e}{\mathbf{T}}_N)_{\mathbf{y},\mathbf{y}'} = \sum_{\mathbf{k}\in\mathcal{K}_N} \Lambda_{\mathbf{k}} \overset{e}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}}(\mathbf{y}) \overset{e}{\Phi}{}^{\mathrm{L}*}_{\mathbf{k}}(\mathbf{y}') \qquad \text{and} \qquad (\overset{o\cdot o}{\mathbf{T}}_N)_{\mathbf{y},\mathbf{y}'} = \sum_{\mathbf{k}\in\mathcal{K}_N} \Lambda_{\mathbf{k}} \overset{o}{\Phi}{}^{\mathrm{R}}_{\mathbf{k}}(\mathbf{y}) \overset{o}{\Phi}{}^{\mathrm{L}*}_{\mathbf{k}}(\mathbf{y}'). \tag{2.32}$$

Substituting these into (2.11) and using (2.26) gives the result immediately. □

Theorem 2. *If the conditions of Theorem 1 hold then*

$$\overset{=\mathcal{N}}{Z_t}(\mathbf{y}^i \to \mathbf{y}^f) = \det ||\overset{=\mathcal{S}}{Z_t}(y^i_\alpha \to y^f_\beta)||_{\alpha,\beta=1\ldots N}. \tag{2.33}$$

Proof. Using (2.10), (2.21) and (2.30) (which follows from (2.13) by Lemma 3)

$$\begin{aligned} \overset{=\mathcal{N}}{Z_t}(\mathbf{y}^i \to \mathbf{y}^f) &= \sum_{\sigma\in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N} \left(v(y^i_\alpha) \sum_{k_\alpha\in\mathcal{K}_1} \lambda^t_{k_\alpha} \overset{p'}{\varphi}{}^{\mathrm{R}}_{k_\alpha}(y^i_\alpha) \overset{p}{\varphi}{}^{\mathrm{L}*}_{k_\alpha}(y^f_{\sigma\alpha}) \right) \\ &= \sum_{\sigma\in P_N} \varepsilon_\sigma \prod_{\alpha=1}^{N} \overset{=\mathcal{S}}{Z_t}(y^i_\alpha \to y^f_{\sigma\alpha}). \end{aligned} \tag{2.34}$$

which is an expansion of the required determinant. □

2.3 One wall and no wall geometries

When L sufficiently large the path closest to the wall at $y = L$ cannot touch it and so $\overset{=\mathcal{S}}{Z_t}(y^i_\alpha \to y^f_{\sigma_\alpha})$ become the generating function for paths that are affected by only one wall $\bar{Z}^{\mathcal{S}}_t(y^i_\alpha \to y^f_\beta)$. Hence taking the limit $L \to \infty$ gives the corollary,

Corollary. *For* $\mathbf{y}^i \in \overset{p}{\mathcal{U}}_L$ *and* $\mathbf{y}^f \in \overset{p'}{\mathcal{U}}_L$*, the* N*-path generating function with only one wall at height* $y = 0$ *is given by,*

$$\bar{Z}^{\mathcal{N}}_t(\mathbf{y}^i \to \mathbf{y}^f) = ||\bar{Z}^{\mathcal{S}}_t(y^i_\alpha \to y^f_\beta)||_{\alpha,\beta=1\ldots N} \,. \tag{2.35}$$

If we also condition the path closest to the wall at $y = 0$ so that it cannot touch that wall we will end up with the "no boundary" results.

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