Kronecker products, plane partitions and sets of uniqueness

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Abstract

We show that the existence of plane partitions, which are *minimal* in a sense to be defined, yields minimal irreducible summands in the Kronecker product $\chi^\lambda \otimes \chi^\mu$ of two irreducible characters of the symmetric group $\mathsf{S}(n)$. The minimality of the summands refers to the dominance order of partitions of n. We also obtain that the multiplicity of a minimal summand χ^ν equals the number of pairs of Littlewood-Richardson multitableaux of shape (λ, μ), conjugate content and type ν, and give bounds for these numbers. Finally we show how the existence of sets of uniqueness yields information about some multiplicities.

Résumé

Nous montrons que l'existence des partages planes, qui sont *minimaux* dans un certain sens, induit des composantes irréductibles minimaux dans les produit de Kronecker $\chi^\lambda \otimes \chi^\mu$ de deux caractères irréductibles du groupe symétrique $\mathsf{S}(n)$. La minimalité des composantes se réfère à l'ordre de domination des partages de n. Nous obtenons aussi que la multiplicité d'une composante minimal χ^ν est égal au nombre des paires de multitableaux de Littlewood-Richardson de forme (λ, μ), poids conjuguée et type ν, et donnons des bornes pour ces nombres. Finalement nous montrons que l'existence des certaines ensembles, appellés d'unicité, donnent de l'information sur quelques multiplicités.

1 Introduction

This is a summary of the results contained in [21]. The Kronecker product $\chi^\lambda \otimes \chi^\mu$ of two irreducible characters of the symmetric group $\mathsf{S}(n)$ is in general a reducible character of $\mathsf{S}(n)$. The multiplicity $c(\lambda, \mu, \nu)$ of an irreducible character χ^ν in the product

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$\chi^\lambda \otimes \chi^\mu$ can be described by a simple formula which follows from the orthogonality relations, and goes back at least to Murnaghan [10, p.765]

$$c(\lambda,\mu,\nu) = \frac{1}{n!} \sum_{\sigma \in \mathsf{S}(n)} \chi^\lambda(\sigma)\chi^\mu(\sigma)\chi^\nu(\sigma).$$

This formula has the virtue of showing the symmetry of $c(\lambda,\mu,\nu)$ in λ, μ, ν. However, it doesn't help too much when one wants to compute $c(\lambda,\mu,\nu)$ explicitly, or even to decide whether $c(\lambda,\mu,\nu)$ is different from zero for some particular choices of λ, μ, ν. Methods for computing $c(\lambda,\mu,\nu)$ are described in [10, 8, 7, 6, 3, 4, 20]. Explicit formulas for $c(\lambda,\mu,\nu)$ can be obtained, basically from the Littlewood-Richardson rule, for arbitrary λ, μ, and the simplest choices of ν, see for example [10, 8, 16, 22, 20]. Other formulas have been found when each of λ and μ is either a hook partition or a partition with two parts, and ν is arbitrary by Remmel and Whitehead [12, 13, 14]; and when λ, μ, ν are rectangular partitions by Clausen and Meier [2]. They also described an algorithm that produces the maximal summand of $\chi^\lambda \otimes \chi^\mu$ in the lexicographic order, either by rows or by columns.

In this paper we show that the existence of plane partitions which are *minimal* in a certain sense yields minimal summands of $\chi^\lambda \otimes \chi^\mu$ in the dominance order. The multiplicity of the minimal summands has a combinatorial description in terms of pairs of Littlewood-Richardson multitableaux. We also show how a notion coming from discrete tomography, that of set of uniqueness, yields information about some $c(\lambda,\mu,\nu)$'s.

2 Definitions and known results

Let λ be a partition of n, in symbols $\lambda \vdash n$. We denote by $|\lambda|$ the sum of its parts, by $\ell(\lambda)$ the number of its parts, and by λ' its conjugate partition. We use the notation $\lambda \trianglerighteq \mu$ to indicate that λ is greater or equal than μ in the dominance order of partitions, see [1, 7, 15]. For any partition λ of n, let χ^λ denote the irreducible character of $\mathsf{S}(n)$ associated to λ, and ϕ^λ the permutation character associated to λ. They are related by the *Young's rule*

$$\phi^\lambda = \sum_{\alpha \trianglerighteq \lambda} K_{\alpha\lambda}\, \chi^\alpha, \tag{1}$$

where $K_{\alpha\lambda}$ is a Kostka number, that is, the number of semistandard tableaux of shape α and content λ, see [7, 2.8.5], [15, §2.11].

For any tableau T (a skew diagram filled with positive integers) there is a word $w(T)$ associated to T given by reading the numbers in T from left to right, in successive rows, starting with the top row. Let ν be a partition of n of length r. Let $\rho(i) \vdash \nu_i$, $1 \le i \le r$. A sequence $T = (T_1, \dots, T_r)$ of tableaux is called a *Littlewood-Richardson multitableaux* of *shape* λ, *content* $(\rho(1), \dots, \rho(r))$ and *type* ν if

(i) There exists a sequence of partitions

$$0 = \lambda(0) \subset \lambda(1) \subset \cdots \subset \lambda(r) = \lambda,$$

such that $|\lambda(i)/\lambda(i-1)| = \nu_i$ for all $1 \le i \le r$, and

(ii) for all $1 \le i \le r$, T_i is a semistandard tableau of shape $\lambda(i)/\lambda(i-1)$ and content $\rho(i)$ such that $w(T_i)$ is a lattice permutation, see [7, 2.8.13], [9, I.9], [15, §4.9].

For each partition λ of n let $c^{\lambda}_{(\rho(1),\dots,\rho(r))}$ denote the number of Littlewood-Richardson multitableaux of shape λ and content $(\rho(1), \dots, \rho(r))$. Let

$$lr^*(\lambda, \mu; \nu) := \sum_{\rho(1) \vdash \nu_1, \dots, \rho(r) \vdash \nu_r} c^{\lambda}_{(\rho(1),\dots,\rho(r))}\, c^{\mu}_{(\rho(1)',\dots,\rho(r)')} \tag{2}$$

be the number of pairs (S, T) of Littlewood-Richardson multitableaux of shape (λ, μ) and type ν, such that S and T have conjugate content, that is, if S has content $(\rho(1), \dots, \rho(r))$, then T has content $(\rho(1)', \dots, \rho(r)')$. Then

$$lr^*(\lambda, \mu; \nu) = \sum_{\gamma \trianglelefteq \nu'} K_{\gamma' \nu}\, c(\lambda, \mu, \gamma), \tag{3}$$

compare with [7, 2.9.17], [20, Section 3].

Let λ, μ, ν be partitions of n. We denote by $\mathsf{M}(\lambda, \mu)$ the set of matrices with non-negative integer coefficients of size $\ell(\lambda) \times \ell(\mu)$, with row sum vector λ, and column sum vector μ; and by $\mathsf{M}^*(\lambda, \mu, \nu)$ the set of all 3-dimensional matrices $A = (a_{ijk})$ of size $\ell(\lambda) \times \ell(\mu) \times \ell(\nu)$, whose entries are zeros or ones, and have *plane sum vectors* λ, μ, ν, that is,

$$\sum_{jk} a_{ijk} = \lambda_i, \quad 1 \le i \le \ell(\lambda),$$
$$\sum_{ik} a_{ijk} = \mu_j, \quad 1 \le j \le \ell(\mu),$$
$$\sum_{ij} a_{ijk} = \nu_k, \quad 1 \le k \le \ell(\nu).$$

Finally let $m^*(\lambda, \mu, \nu) := |\mathsf{M}^*(\lambda, \mu, \nu)|$.

We now recall some facts about plane partitions. For a positive integer m, let $[\,m\,] := \{1, \dots, m\}$. A subset S of the 3-dimensional box $B(p, q, r) := [\,p\,] \times [\,q\,] \times [\,r\,]$ is called *pyramid* if for all $(a, b, c) \in S$ and for all $(x, y, z) \in B(p, q, r)$ the conditions

$x \leq a$, $y \leq b$ and $z \leq c$ imply $(x, y, z) \in S$. Pyramids were used in [19] to give examples of sets of uniqueness. We will say more about them in Section 4. A *plane partition* with at most p rows, at most q columns, and largest part $\leq r$ is a matrix $A = (a_{ij})$ with non-negative integer coefficients of size $p \times q$, such that $0 \leq a_{ij} \leq r$, and whose rows and columns are weakly decreasing, see [11, Section 421]. There is a simple well-known one-to-one correspondence between pyramids $S \subseteq B(p, q, r)$ and plane partitions $A = (a_{ij})$ with at most p rows, at most q columns, and largest part $\leq r$, see [11, Section 423]. For this reason a pyramid is also called the graph or the diagram of its associated plane partition. The correspondence is given by $S \longmapsto Z(S) = (z_{ij})$, where $z_{ij} := |\{k \in [\,r\,] \mid (i, j, k) \in S\}|$; its inverse is $A \longmapsto S(A)$, where $S(A) := \{(i, j, k) \in B(p, q, r) \mid 1 \leq k \leq a_{ij}\}$. The *slice vectors* λ, μ, ν of any subset S of the box $B(p, q, r)$ are formed by the cardinalities of its slices parallel to the coordinate planes:

$$\begin{aligned} \lambda_i &:= |\{x \in S \mid x_1 = i\}|, \quad 1 \leq i \leq p, \\ \mu_j &:= |\{x \in S \mid x_2 = j\}|, \quad 1 \leq j \leq q, \\ \nu_k &:= |\{x \in S \mid x_3 = k\}|, \quad 1 \leq k \leq r. \end{aligned} \tag{4}$$

If S is a pyramid, then λ, μ, and ν are partitions of $|S|$, and the correspondence $S \longmapsto Z(S)$ satisfies: $Z(S) \in \mathsf{M}(\lambda, \mu)$, and $\pi(Z(S)) = \nu'$. Conversely, if λ, μ, ν are partitions of n, and $A \in \mathsf{M}(\lambda, \mu)$ is a plane partition with $\pi(A) = \nu$, then its associated pyramid $S(A)$ has slice vectors λ, μ, ν'.

Given $A \in \mathsf{M}(\lambda, \mu)$, we denote by $\pi(A)$ the partition of n obtained from A by ordering its entries decreasingly. For any partition ν of n we define

$$a(\lambda, \mu; \nu) := |\{A \in \mathsf{M}(\lambda, \mu) \mid \pi(A) = \nu\}|$$

and

$$p(\lambda, \mu; \nu) := |\{A \in \mathsf{M}(\lambda, \mu) \mid \pi(A) = \nu \text{ and } A \text{ is a plane partition}\}|.$$

We say that a matrix A is *minimal* in $\mathsf{M}(\lambda, \mu)$ if $A \in \mathsf{M}(\lambda, \mu)$, and it does not exist $B \in \mathsf{M}(\lambda, \mu)$ with $\pi(B) \lhd \pi(A)$. We also say that ν is *minimal* for $\chi^\lambda \otimes \chi^\mu$ if $c(\lambda, \mu, \nu) \neq 0$ and $c(\lambda, \mu, \gamma) = 0$ for all $\gamma \lhd \nu$.

3 Minimal matrices and plane partitions

Our main theorem (3.1) shows how the existence of minimal plane partitions in $\mathsf{M}(\lambda, \mu)$ yields minimal summands in $\chi^\lambda \otimes \chi^\mu$ in the dominance order. The theorem will follow from (3.2)-(3.6) below. This sequence of results also shows which consequences follow only from the existence of a minimal matrix, and which require the existence of a minimal plane partition. Proposition 3.2 and Theorem 3.5 may be of interest by themselves.

3.1. Theorem. [21] *Let M be a minimal matrix in $\mathsf{M}(\lambda, \mu)$, and let $\nu = \pi(M)$. Suppose M is a plane partition. Then*

(1) ν is minimal for $\chi^\lambda \otimes \chi^\mu$.

(2) $c(\lambda, \mu, \nu) = lr^*(\lambda, \mu; \nu')$.

(3) $p(\lambda, \mu; \nu) \le c(\lambda, \mu, \nu) \le a(\lambda, \mu; \nu)$.

(4) $c(\alpha, \beta, \gamma) = 0$ for all $\alpha \trianglerighteq \lambda$, $\beta \trianglerighteq \mu$, and $\gamma \triangleleft \nu$.

(5) $c(\lambda, \mu, \nu) = a(\lambda, \mu; \nu)$ if and only if $c(\alpha, \beta, \nu) = 0$ for all $\alpha \trianglerighteq \lambda$, $\beta \trianglerighteq \mu$ such that $(\alpha, \beta) \neq (\lambda, \mu)$.

3.2. Proposition. *Let M be a matrix in $\mathsf{M}(\lambda, \mu)$, and let $\nu = \pi(M)$. Then M is minimal if and only if $m^*(\lambda, \mu, \nu') = a(\lambda, \mu; \nu)$.* □

3.3. Proposition. *Let M be a minimal matrix in $\mathsf{M}(\lambda, \mu)$, and let $\nu = \pi(M)$. Then*

$$c(\alpha, \beta, \gamma) = 0$$

for all $\alpha \trianglerighteq \lambda$, $\beta \trianglerighteq \mu$, and $\gamma \triangleleft \nu$. □

3.4. Corollary. *Let M be a minimal matrix in $\mathsf{M}(\lambda, \mu)$, and let $\nu = \pi(M)$. Then*

(1) $c(\lambda, \mu, \nu) = lr^*(\lambda, \mu; \nu')$.

(2) $a(\lambda, \mu; \nu) = c(\lambda, \mu, \nu) + \sum K_{\alpha\lambda} K_{\beta\mu} c(\alpha, \beta, \nu)$. Here the sum is over all pairs (α, β) such that $\alpha \trianglerighteq \lambda$, $\beta \trianglerighteq \mu$ and $(\alpha, \beta) \neq (\lambda, \mu)$. □

Under the assumptions of Proposition 3.3 we know that $c(\lambda, \mu, \gamma) = 0$ for all $\gamma \triangleleft \nu$, but still $c(\lambda, \mu, \nu)$ could be zero. For example if $\lambda = \mu = (3^2)$, the minimal matrices in $\mathsf{M}(\lambda, \mu)$ are $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$. However, for $\nu = (2^2, 1^2)$, we have $c(\lambda, \mu, \nu) = lr^*(\lambda, \mu; \nu') = 0$. Therefore, we need to impose an extra condition in M in order to assure the positivity of $c(\lambda, \mu, \nu)$. One such condition, as we shall see below, is that M is a plane partition.

3.5. Theorem. *Let λ, μ, ν be partitions of n. Then*

$$p(\lambda, \mu; \nu) \le lr^*(\lambda, \mu; \nu').$$

□

3.6. Corollary. *Let M be a minimal matrix in $\mathsf{M}(\lambda, \mu)$, and let $\nu = \pi(M)$. Then*

(1) $p(\lambda, \mu; \nu) \le c(\lambda, \mu, \nu) \le a(\lambda, \mu; \nu)$.

(2) If M is a plane partition, then ν is minimal for $\chi^\lambda \otimes \chi^\mu$. □

3.7. Example. Let $\lambda = (8,7,4,2)$, $\mu = (11,6,4)$. With the aid of a computer we generated all minimal matrices in $\mathsf{M}(\lambda,\mu)$. They are

$$A = \begin{bmatrix} 3 & 3 & 2 \\ 3 & 2 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 & 2 \\ 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 2 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 4 & 2 & 2 \\ 3 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 3 & 3 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 4 & 2 & 2 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 4 & 3 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

Since A, B, C and D are plane partitions, then by Theorem 3.1.1 the partitions $\pi(A) = (3^4, 2^4, 1)$, $\pi(B) = (4, 3, 2^7)$, $\pi(C) = (4^2, 2^4, 1^5)$, $\pi(D) = (4, 3^2, 2^4, 1^3)$ are minimal for $\chi^\lambda \otimes \chi^\mu$. If ν is any of the first three partitions, then $1 = p(\lambda,\mu;\nu) = c(\lambda,\mu,\nu) = a(\lambda,\mu;\nu)$. If $\nu = \pi(D)$, then $1 \le c(\lambda,\mu,\nu) \le 4$, by Theorem 3.1.3, and $c(\lambda,\mu,\nu) = lr^*(\lambda,\mu;\nu')$, by Theorem 3.1.2. Using (2) we get easily that $c(\lambda,\mu,\nu) = 3$. This shows that the inequalities in Theorem 3.1.3 may be strict. Of course Theorem 3.1.4 applies to our four partitions. In particular, for $\sigma = (9,6,4,2)$ and $\nu = \pi(D)$ we have $c(\sigma,\mu,\gamma) = 0$ for all $\gamma \lhd \nu$; therefore $c(\sigma,\mu,\nu) = lr^*(\sigma,\mu;\nu')$. This last number is easily seen to be 1. Then, it follows from Corollary 3.4.2 that $c(\alpha,\beta,\nu) = 0$ for all $\alpha \trianglerighteq \lambda$, $\beta \trianglerighteq \mu$ such that $(\alpha,\beta) \neq (\lambda,\mu), (\sigma,\mu)$.

3.8. Remark. We consider again the example after Corollary 3.4: let $\lambda = \mu = (3^2)$, then no minimal matrix in $\mathsf{M}(\lambda,\mu)$ is a plane partition, so we cannot apply Theorem 3.1 to obtain minimal summands in $\chi^\lambda \otimes \chi^\mu$. The only partition associated to minimal matrices in $\mathsf{M}(\lambda,\mu)$ is $\nu = (2^2, 1^2)$ and we have $c(\lambda,\mu,\nu) = 0$. It turns out that the partitions covering ν, namely $\sigma = (2^3)$ and $\tau = (3, 1^3)$ are the only partitions corresponding to minimal summands in $\chi^\lambda \otimes \chi^\mu$. This leads us to the following natural questions: When does all minimal summands in $\chi^\lambda \otimes \chi^\mu$ come from minimal plane partitions? How do we determine all minimal summands not comming from a minimal plane partition? These are questions we will address in a future paper.

4 Sets of uniqueness and minimal matrices

In this section we show that the existence of sets of uniqueness puts severe restrictions on some $c(\lambda,\mu,\nu)$'s.

Let S be a subset of $B(p,q,r)$. Its slice vectors $\lambda = (\lambda_1,\ldots,\lambda_p)$, $\mu = (\mu_1,\ldots,\mu_q)$, and $\nu = (\nu_1,\ldots,\nu_r)$ are compositions of $|S|$, that is, vectors of non-negative integers whose coordinates sum $|S|$, see (4). The set S is called a *set of uniqueness* if it is the only set with slice vectors λ, μ, ν. Sets of uniqueness were introduced in [5], where a geometric characterization of them was given by the absence of certain configurations in $B(p,q,r)$. Note that, as long as we are concerned with properties of S which depend on the cardinalities of its slices, we may and will assume that λ, μ, and ν are weakly decreasing, namely, that they are partitions of $|S|$. If this were not the case, we just permute some slices of S. Thus a set S is a set of uniqueness if and only if $m^*(\lambda,\mu,\nu)=1$.

The starting point of this work was the attempt to use (1), a result of Snapper [17], which expresses $m^*(\lambda,\mu,\nu)$ as an inner product of characters, and some knowledge on the numbers $c(\lambda,\mu,\nu)$ in order to find conditions on λ, μ, ν which would imply that S is a set of uniqueness. However, these numbers are hard to compute; it proved more fruitful to try to get information about the $c(\lambda,\mu,\nu)$'s from the existence of sets of uniqueness. This is the content of Corollary 4.2 which eventually developed into Theorem 3.1. In [18] an algebraic characterization of sets of uniqueness was given. Here, we need only one implication, which can be reformulated in the following way:

4.1. Theorem. *Let S be a set of uniqueness and suppose that its slice vectors λ, μ, ν are partitions of $|S|$. Then S is a pyramid, its associated plane partition $Z(S)$ is minimal in $\mathsf{M}(\lambda,\mu)$, and $a(\lambda,\mu;\nu')=1$.* □

From this and from Theorem 3.1 we obtain

4.2. Corollary. *Let S be a set of uniqueness and suppose its slice vectors λ, μ, ν are partitions of $|S|$. Then*
(1) ν' is minimal for $\chi^\lambda\otimes\chi^\mu$.
(2) $c(\lambda,\mu,\nu')=1$,
(3) For all $\alpha\trianglerighteq\lambda$, $\beta\trianglerighteq\mu$, $\gamma\trianglelefteq\nu'$ such that $(\alpha,\beta,\gamma)\neq(\lambda,\mu,\nu')$, we have $c(\alpha,\beta,\gamma)=0$. □

4.3. Examples. The simplest example of set of uniqueness is the box $B(a,b,c)$. It has slice vectors $\lambda=((bc)^a)$, $\mu=\big((ac)^b\big)$, and $\nu=((ab)^c)$. Then from the previous corollary we recover Satz 2.3 in [2]: $c(\lambda,\mu,\nu')=1$ and obtain new identitites $c(\alpha,\beta,\gamma)=0$ for all $\alpha\trianglerighteq\lambda$, $\beta\trianglerighteq\mu$, $\gamma\trianglelefteq\nu'$ such that $(\alpha,\beta,\gamma)\neq(\lambda,\mu,\nu')$. Another simple example of set of uniqueness is the *hook set* $H=H(a,b,c)$ associated to the plane partition of size $(a+1)\times(b+1)$

$$Z(H)=\begin{bmatrix} c+1 & 1 & \cdots & 1\\ 1 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & \cdots & 0\end{bmatrix},$$

see Section 1 in [18]. It has slice vectors $\lambda = (b+c+1, 1^a)$, $\mu = \big(a+c+1, 1^b\big)$, and $\nu = (a+b+1, 1^c)$. Then the previous corollary yields $c(\lambda, \mu, \nu') = 1$, and $c(\lambda, \mu, \gamma) = 0$ for all $\gamma \lhd \nu'$, which are contained in [12], as well as some new identities.

4.4. Remark. The notion of minimal matrix seems to be important. It was used in [18] to characterize sets of uniqueness, and in this paper to obtain information about some $c(\lambda, \mu, \nu)$'s. Proposition 3.2 provides a characterization for minimal matrices. It would be desirable, however, to have more practical ways for deciding whether a given matrix M is minimal in $\mathsf{M}(\lambda, \mu)$. This is a problem we propose for further study.

Note added in proof. After submitting this paper for publication I learned from A. Zelevinsky and M. Kapranov that the inequality $p(\lambda, \mu; \nu) \leq c(\lambda, \mu, \nu)$ in our Theorem 3.1.3, has been proved for all λ, μ and ν by L. Manivel. His methods, however, are quite different from ours. See Proposition 3.1. in Ann. Inst. Fourier, Grenoble 47, 3(1997), 715-773.

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