RIBBON TILE INVARIANTS

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ABSTRACT. Let $\mathbf{T}$ be a finite set of tiles, $\mathcal{B}$ be a set of regions Γ tileable by $\mathbf{T}$. We introduce a *tile counting group* $\mathbb{G}(\mathbf{T}, \mathcal{B})$ as a group of all linear relations for the number of times each tile $\tau \in \mathbf{T}$ can occur in a tiling of a region $\Gamma \in \mathcal{B}$. We compute the tile counting group for a large set of *ribbon tiles* also known as rim hooks in a context of representation theory of the symmetric group.

The tile counting group is presented by its set of generators which consists of certain new *tile invariants.* In a special case these invariants generalize Conway-Lagarias invariant for tromino tilings and a height invariant which is related to computation of characters of the symmetric group.

The heart of the proof is the known bijection betwen rim hook tableaux and certain standard skew Young tableaux. We also discuss signed tilings by the ribbon tiles and apply our results to the tileability problem.

0. TRIVIA

Suppose we are given a set of the *tiles* on a plane. We are allowed to use translations of the tiles to arrange them in a geometric shape (each tile may occur several times). This arrangement is called *tiling* of that shape. One can ask whether a given region can be tiled by a given set of tiles, and if it can, how many different tilings there are.

For example, with a set of tiles shown in Figure 0.1 one can make four different tilings of the 4-by-6 rectangle. Two of them are shown in Figure 0.2. Now one can try to find a criteria for when you can tile a rectangle. Observe that each of these tiles alone can tile the whole plane.

Our personal favorite example is given by a set of tiles shown in Figure 0.3. One can show that there exist only one tiling of the fourth quadrant (see Fig. 0.4). The proof is left to the reader.

It turns out that there are certain nice sets of tiles for which it is not clear whether a given region can be tiled. Here is an example. Consider 8 tiles shown in Figure 0.5. One can show that the 25-by-25 square cannot be tiled by these tiles. Generally,

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Theorem 0.1 If an a-by-b rectangle can be tiled tiles shown in Figure 0.5, then $10 \mid a \cdot b$.

Of course, if $a = b = 25$ one can prove the claim by an exhaustive search. Another example. Consider 6 tiles shown in Figure 0.6. We again have

Theorem 0.2 If an a-by-b rectangle can be tiled by tiles shown in Figure 0.6, then $10 \mid a \cdot b$.

Of course, the area argument show $5 \mid a \cdot b$.

Consider now a different region. Let Δ_N be a triangular shape shown in Figure 0.7. One can check that Δ_{24} can be tiled by tiles shown in Figure 0.6 while Δ_{25} cannot. Generally,

Theorem 0.3 If Δ_N can be tiled tiles shown in Figure 0.6, then $N \equiv 0, 4, 15, 19 \pmod{20}$.

It turns out that all three theorems can be proved by use of the same kind of argument. Heuristically, the reason for untileability arises from a completely different question:

- Given a set of tiles and a tileable region, are there any linear relations for the number of times each tile occurs in a tiling?

The rest of the paper explains the relevance of this question. All three theorems are proved in section 7.

1. Introduction

Let $\mathbb{Z}^2$ be a square lattice, $\mathcal{R}$ be a set of all compact simply connected regions in $\mathbb{Z}^2$. We think of these regions as of disjoint unions of (1×1)-squares. Sometimes they are called *polyominoes*. Fix a finite set of *tiles* $\mathbf{T} = \{\tau_1, \dots, \tau_N\}$, $\tau_i \in \mathcal{R}$, $i = 1, \dots, N$. Let tiles be invariant under translations. We say that a region $\Gamma \in \mathcal{R}$ is *tileable by* $\mathbf{T}$ if it can be presented as a disjoint union of the regions

$$\Gamma = \coprod_{1 \le j \le l} \tau'_j$$

where each region τ'_j, $1 \le j \le l$ is a translation of some τ_{i_j}. Such a disjoint union is called a *tiling* s of Γ. Denote $\mathcal{S} = \mathcal{S}(\Gamma, \mathbf{T})$ a set of all tilings of Γ by the set of tiles $\mathbf{T}$.

Fix a set of tiles $\mathbf{T}$ and a region $\Gamma \in \mathcal{R}$. There are two basic questions one can ask:

- Is Γ tileable by $\mathbf{T}$?
- If Γ is tileable by $\mathbf{T}$, what do the tilings look like ?

The first question is very classical and well studied (see e.g. [G]). It is usually not hard to find a tiling if Γ is tileable by $\mathbf{T}$, while proving the opposite can be exteremely difficult. Except for the ad hoc examples, there are basically two techniques for proving that a region can not be tiled: coloring arguments and Conway group analysis (see [CL, T]). Note also that the case when $\mathbf{T}$ contains a 1-by-1 square is trivial: every region is tileable.

While the first questions admits only "yes" or "no" answer, the second question could be posed in many ways, each of them giving us some information about the structure of the tiling set $\mathcal{S}(\Gamma, \mathbf{T})$. One can find the following two questions in the literature (see e.g. [G, CEP]) :

- How many tilings of Γ are there ?
- What do random tilings $s \in \mathcal{S}(\Gamma, \mathbf{T})$ look like ?

It turns out that the answers to these questions heavily depend on the geometry of Γ and can be very complicated even in very simple cases. In particular, finding the number of tilings $|\mathcal{S}(\Gamma, \mathbf{T})|$ is a more general problem than just finding whether a certain regions has a tiling. In some cases this problem is known to be NP-complete, and probably cannot be solved by means other than exhaustive enumeration (see [GJ, p. 257]).

To avoid these difficulties we propose another approach to the problem. We fix only $\mathbf{T}$ and ask about properties of tilings of all regions at once. We would like to ask the following two questions:

- Are there any relations for the number of times each tile occurs in a tiling of a given region ?
- Is there a finite set of *local replacement rules* (we also call them *local moves* or just *moves*) such that for every region $\Gamma \in \mathcal{R}$, any tiling of Γ can be changed into any other tiling by a sequence of moves ?

Before we give formal definitions, let us illustrate what happens in the case of *dominoes*. Although small, this example will illustrate variety of approaches as well as the complexity of a problem.

Let $\mathbf{T}_2$ be a set of two tiles: horizontal domino τ_1 and vertical domino τ_2 (see Fig. 1.1). It is easy to come up with a necessary condition for tileability (see e.g. [G]). Color the region in a checkerboard fashion. Since each domino must contain one black and one white square, the total number of black squares must be equal to the total number of white squares. For example, the region shown in Figure 1.2 cannot be tiled by dominoes since it has 8 black squares and only 6 white squares. Unfortunately there exist untileable regions with equal number of the black and white squares (see e.g. Fig. 1.3). This means that we need a stronger condition for the tileability criteria.

Now suppose we have a region Γ that is known to be tileable. We want to compute the number of tilings Γ has. It turns out to be an interesting and nontrivial question. In case of a 2-by-m rectangle the number of tilings is a Fibonacci number $F(m) = F(m-1) + F(m-2)$ (see Fig. 1.4). In case of a general rectangle the problem was solved by Kastelyn and Temperley & Fisher (see [Ka, TF]). In case of an Aztec diamond (see Fig. 1.5) the domino tilings were enumerated by Elkies, Kuperberg, Larsen and Propp (see [EKLP]). Both results gave rise to many other questions about domino tiling (see e.g. [CEP]). In this work we do not further consider any numerical results of this type.

Let Γ be a region tileable by dominoes τ_1, τ_2. Consider $s \in \mathcal{S}(\Gamma, \mathbf{T}_2)$ a domino tiling of Γ. Suppose s consists of $a_1 = a_1(s)$ copies of the horizontal domino τ_1, and of $a_2 = a_2(s)$ copies of the vertical domino τ_2. Of course, $2\,(a_1 + a_2)$ is equal to the area $|\Gamma|$ of the region (see Fig. 1.6). This gives us the first relation. There is one more relation which is less obvious: $a_2 = Const \pmod 2$. To see this, let us color black every other column of the region Γ (see Fig. 1.7). Denote by c_1, c_2 the number of the black and white regions respectively, and by $d = c_1 - c_2$. Since horizontal dominoes contain exactly one black and one white square, and vertical dominoes contain two squares of the same color, we immediately get $a_2 = d/2 \pmod 2$ (see Fig. 1.7).

Here is another way to look at the set of tilings $\mathcal{S}(\Gamma, \mathbf{T}_2)$. Let us allow the following *local replacement rules* (or simply *moves*): take two adjacent horizontal or vertical dominoes and flip them (see Fig 1.8). Of course, this move gives us a new tiling of Γ (see Fig 1.9). It is known that by a sequence of such moves we can get from any tiling $s \in \mathcal{S}(\Gamma, \mathbf{T}_2)$ of a simply connected region Γ to any other tiling $s' \in \mathcal{S}(\Gamma, \mathbf{T}_2)$ (see e.g. [T]). From here we immediately get $a_1(s) + a_2(s) = a_1(s') + a_2(s')$ and $a_2(s) = a_2(s') \pmod 2$ since these identities are trivial for any single move. This also implies that for a general region there are no other relation for the numbers a_1, a_2 that do not follow from these two. We will use a similar logic when proving our main results.

Now we are ready to introduce *tile counting group* and *tile invariants*. Let $\mathbf{T} = \{\tau_1, \ldots, \tau_N\}$ be a set of tiles. Denote by $\mathcal{R}_{\mathbf{T}} \subset \mathcal{R}$ set of regions tileable by $\mathbf{T}$. Let $\mathcal{B} \subset \mathcal{R}_{\mathbf{T}}$ be a fixed subset of tileable regions. Consider a tileable region $\Gamma \in \mathcal{B}$. We identify each tiling $s \in \mathcal{S}(\Gamma, \mathbf{T})$ with its multiset of tiles $s \simeq \{\tau_{i_1}, \ldots, \tau_{i_l}\}$. Of course, by doing so we loose some information about the geometric structure of the tilings since there could be many tilings of Γ with the same multiset of tiles (see e.g. Fig. 1.4). As before, by $|\Gamma|$ we denote the area of Γ.

Let $\mathbb{Z}\langle\mathbf{T}\rangle$ be a group of formal integer linear combinations of $\mathbf{T}$. With each pair of tilings $s_1, s_2 \in \mathcal{S}(T,\Gamma)$ of a region $\Gamma \in \mathcal{B}$ we associate a relation:

$$(\tau_{i_1} + \cdots + \tau_{i_l} = \tau_{j_1} + \cdots + \tau_{j_r})$$

Let I be linear span of such relations for all regions $\Gamma \in \mathcal{B}$ and for all pairs of tilings $s_1, s_2 \in \mathcal{S}(T,\Gamma)$. Define *tile counting group* to be a quotient group

$$\mathbb{G}(\mathbf{T};\mathcal{B}) = \mathbb{Z}\langle\mathbf{T}\rangle/I$$

This will be the main object of our study. Since both groups in the quotient are abelian, one can think of a tile counting group $\mathbb{G}(\mathbf{T};\mathcal{B})$ as of a subgroup of $\mathbb{Z}\langle\mathbf{T}\rangle$. Thus it is reasonable to describe $\mathbb{G}(\mathbf{T};\mathcal{B})$ by its set of independent generators (or the *basis*) given in $\mathbb{Z}\langle\mathbf{T}\rangle$.

For example, let $\mathbf{T}_2$ be a set of dominoes (see Fig. 1.1), and $\mathcal{B}$ be a set of simply connected regions. Two tilings in Figure 1.8 correspond to a relation $2\cdot\tau_1 = 2\cdot\tau_2$. Since every domino tiling of a simply connected region can be obtained from every other domino tiling, I in this case is generated by the relation above. Therefore

$$\mathbb{G}(\mathbf{T}_2;\mathcal{B}) = \mathbb{Z}^2/I \simeq \mathbb{Z}\times\mathbb{Z}_2$$

The basis can be given as $\tau_1+\tau_2,\ \tau_1-\tau_2\ \in \mathbb{Z}\langle\mathbf{T}_2\rangle$. Note that the second generator has an order 2 as an element in $\mathbb{G}(\mathbf{T}_2;\mathcal{B})$, while it has an infinite order as an element of $\mathbb{Z}\langle\mathbf{T}_2\rangle$.

Here is another way to describe at the tile counting group. Let G be an abelian, not necessarily finite group. A map $f : \mathcal{B} \to G$ is called a *tile invariant* (or just an *invariant*) if for any tileable region $\Gamma \in \mathcal{B}$ and for any its tiling $s \in \mathcal{S}(\Gamma,\mathbf{T})$, $s \simeq \{\tau'_{i_1},\dots,\tau'_{i_l}\}$, where τ'_j is a translation of a tile τ_j, we have

$$f(\Gamma) = f(\tau_{i_1}) + \cdots + f(\tau_{i_l})$$

The problem is to find all the tile invariants for a fixed set of tiles $\mathbf{T}$. Clearly a tile invariant is determined by its values on $\mathbf{T}$, so the problem of finding invariant is equivalent to finding maps $f : \mathbf{T} \to G$ which can be extended to the set of all regions $\mathcal{B}$.

Let $\sum_{\tau\in\mathbf{T}} a(\tau) \in \mathbb{Z}\langle\mathbf{T}\rangle$ be an element of a tile counting group $\mathbb{G} = \mathbb{G}(\mathbf{T};\mathcal{B})$. Suppose m is its order in $\mathbb{G}$ (m could be infinity). Then a map $f : \mathbf{T} \to \mathbb{Z}_m$, $m < \infty$ or $f : \mathbf{T} \to \mathbb{Z}$, $m = \infty$ defined by $f(\tau) = a(\tau) \pmod m$ or $f(\tau) = a(\tau)$, is a tile invariant, where by $\mathbb{Z}_m$ we mean the additive group of integers modulo m. Conversely, every tile invariant can be lifted to an element of the tile counting group. Thus the problem of computing the tile counting group $\mathbb{G}(\mathbf{T})$ is equivalent to description of all invariants. We say that tile invariants $f_1, f_2, \dots$ form an *independent basis of invariants* if they correspond to an independent generating set in a tile counting group. An invariant $f : \mathbf{T} \to G$ is called *trivial* if $f(\tau) = 0$ for all $\tau \in \mathbf{T}$, where $0 \in G$ is an identity element. Otherwise the invariant is called *nontrivial.* An invariant $f : \mathbf{T} \to G$ is called *primitive* is $G \simeq \mathbb{Z}$ or $\mathbb{Z}_m$ for some m. For the rest of the paper we will be considering only primitive invariants.

Note when $\mathcal{B} = \mathbf{T}$ every map $f : \mathcal{B} \to G$ is a tile invariant, i.e $\mathbb{G}(\mathbf{T}, \mathbf{T}) \simeq \mathbb{Z}^{|\mathbf{T}|}$. Generally, the bigger is our set of regions $\mathcal{B}$, the more equations on f we have, and the fewer tile invariants we get.

The obvious example of a nontrivial tile invariant is given by the area of tiles:

$$f_0 : \mathbf{T} \to \mathbb{Z}\,, \quad f_0(\tau_i) = |\tau_i|$$

which can be extended to all tileable regions: $f_0(\Gamma) = |\Gamma|$. This implies that the tile counting group has $\mathbb{Z}$ as a subgroup. In the case of domino tiles $\mathbf{T}_2$ we also get another invariant (see above):

$$f_* : \mathbf{T}_2 \to \mathbb{Z}_2\,, \quad f_*(\tau_1) = 0,\ f_*(\tau_2) = 1 \mod 2$$

The main result of this paper is a description of a tile counting group for the following set of tiles.

Let the axis on a plane be as shown in Figure 1.10. We say that squares (i, j) and (i', j') lie on the same diagonal if $i - j = i' - j'$. For example, two squares $(2, 4)$ and $(5, 7)$ lie on the same diagonal (see Fig. 1.10). A *ribbon tile* is a simply connected region with no two squares lying on the same diagonal. An example of a ribbon tile is shown in Figure 1.11. Denote by $\mathbf{T}_n$ the set of all ribbon tiles τ with n squares: $|\tau| = n$. Obviously, set $\mathbf{T}_2$ is the set of dominos (see Fig 1.1). Sets $\mathbf{T}_3$, $\mathbf{T}_4$ and $\mathbf{T}_5$ are shown in Figures 1.12-14.

Note that $|\mathbf{T}_n| = 2^{n-1}$. Indeed, we can code each ribbon tile by a sequence $(\varepsilon_1, \dots, \varepsilon_{n-1})$ of $n - 1$ zeroes and ones as follows. Call the lower left square the *starting square.* Begin with the starting square and move along the tile. Write **0** when going right, and write **1** when going up. See Figures 1.12 – 1.14 for sequences of all tiles in $\mathbf{T}_3$, $\mathbf{T}_4$ and $\mathbf{T}_5$.

Definition 1.1 Consider a sequence of maps $f_1, \dots, f_m : \mathbf{T}_n \to \mathbb{Z}$, $m = \left\lfloor \frac{n-1}{2} \right\rfloor$ defined as follows

$$f_i(\varepsilon_1, \dots, \varepsilon_{n-1}) = \varepsilon_i - \varepsilon_{n-i}$$

We call the map f_i the *i-convexity invariant.*

Definition 1.2 A constant map $f_0 : \mathbf{T}_n \to \mathbb{Z}$ defined as

$$f_0(\varepsilon_1, \dots, \varepsilon_{n-1}) = 1$$

is called the *area invariant.*

Definition 1.3 If n is even, a map $f_* : \mathbf{T}_n \to \mathbb{Z}_2$ defined as

$$f_*(\varepsilon_1, \dots, \varepsilon_{n-1}) = \varepsilon_{n/2} \pmod 2$$

is called the *parity invariant.*

Before we state our main results, we need to specify the set of regions $\mathcal{B} \in \mathcal{R}_{\mathbf{T}_n}$. A region $\Gamma \in \mathcal{R}$ is called *row-convex* (*column-convex*) if every horizontal (vertical) line intersects Γ by an interval or does not intersect it at all (see Fig. 1.15). Let $\mathcal{B}_{rc}$ be a set of tileable row-convex simply connected regions.

The main result of this paper is the following Theorem.

Theorem 1.4 Let $\mathcal{B} = \mathcal{B}_{rc}$ be as above. Then
1) when $n = 2m+1$, $\mathbb{G}(\mathbf{T}_n, \mathcal{B}) \simeq \mathbb{Z}^{m+1}$ and the maps $f_0, f_1, \dots, f_m$ form an independent basis of invariants.
2) when $n = 2m$, $\mathbb{G}(\mathbf{T}_n, \mathcal{B}) \simeq \mathbb{Z}^m \times \mathbb{Z}_2$ and the maps $f_0, f_1, \dots, f_{m-1}, f_*$ form an independent basis of invariants.

When $n = 2$ Theorem 1.4 says that the area and parity invariants form an independent basis. Analogously when $n = 3$ the Theorem 1.4 says that beside the area invariant f_0 there exists one other nontrivial tile invariant $f_1 : \mathbf{T} \to \mathbb{Z}$, where

$$f_1(\mathbf{10}) = 1\,, \quad f_1(\mathbf{01}) = -1\,, \quad f_1(\mathbf{00}) = f_1(\mathbf{11}) = 0$$

(see Fig. 1.12). In a different form this invariant was discovered by Conway and Lagarias in [CL] (see also [T]). To say that f_1 is an invariant is equivalent to saying that:

- $\#\,\mathbf{10} - \#\,\mathbf{01} = Const$

This means that the number of times the **10** tromino occurs in a tiling minus the number of times the **01** tromino occurs in the same tiling of a region Γ depends only on a region Γ and not on a tiling.

Here is another nontrivial invariant that exist for all $n > 1$.

Definition 1.5 Consider a map $f_\bullet : \mathbf{T}_n \to \mathbb{Z}_2$ defined as follows

$$f_\bullet(\varepsilon_1, \dots, \varepsilon_{n-1}) = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{n-1} \pmod 2$$

We call $f_\bullet$ the *height invariant.*

The reason why $f_\bullet$ is called the *height invariant* can be easily seen from the picture. Consider the smallest rectangular box the ribbon tile τ can fit in (see Fig 1.16). Then $f_\bullet(\tau) = a - 1 \pmod 2$, where a is the height of the rectangle. This invariant was considered earlier in connection with the certain characters of the symmetric group S_N (see [R, JK, St]). Namely, it corresponds to signs in the Murnagama–Nakayama summation formula for computing the character values on the conjugacy classes (n^a), where $N = a \cdot n$ (see [JK, M] for details). Observe that

$$f_\bullet = \begin{cases} f_1 + \cdots + f_{m-1} + f_m \mod 2, & n = 2m+1 \\ f_1 + \cdots + f_{m-1} + f_* \mod 2, & n = 2m \end{cases}$$

This proves that $f_\bullet$ is indeed an invariant provided Theorem 1.4 holds.

Now let us say a few words about how Theorem 1.4 is proved. We shall present a finite set of moves which preserve the invariants but enable us to get to any tiling starting with any other. Formally, let $\mathcal{B}_y$ be the set of row and column-convex regions such that when fit into the smallest possible box contain the upper right, upper left and the lower left corner of the box (see Fig. 1.17).

Theorem 1.6 Let $\mathcal{B} = \mathcal{B}_y$ be as above. For every $n > 1$ there is a finite set of at most $n\,4^n$ moves such that any tiling of $\Gamma \in \mathcal{B}$ by $\mathbf{T}_n$ can be transformed by a sequence of moves to any other such a tiling.

When $n = 2$ we need only one move (see Fig 1.8). When $n = 3$ we already need 6 moves (see Fig 1.18). Together with Theorem 1.6 we prove it in section 3.

To finish the introduction, let us compare the definition of tile invariants with the *generalized coloring arguments* introduced by Conway and Lagarias (see [CL]) .

Let G be an abelian, not necessarily finite group, and e be its identity element. A map $f : \mathcal{R} \to G$ is called a *coloring map* if for every region $\Gamma \in \mathcal{R}$ we have

$$f(\Gamma) = f(x_1) + \cdots + f(x_{|\Gamma|})$$

where $x_1, \ldots, x_{|\Gamma|}$ are the squares in Γ. Of course, f is defined by its values on all (1×1) squares.

Coloring map $f : \mathcal{R} \to G$ is called $\mathbf{T}$-*coloring* if $f(\Gamma) = e$ for all $\Gamma \in \mathcal{R}_{\mathbf{T}}$. In a sense the $\mathbf{T}$-coloring maps are complementary to tile invariants which are defined only on tileable regions.

It is clear that to show that f is a $\mathbf{T}$-coloring maps all we need is to check that $f(\tau') = e$ for all translations τ' of a tile $\tau \in \mathbf{T}$. If this is the case, we say that f gives a *coloring argument* for a set of tiles $\mathbf{T}$. The idea is that we show that now $f(\Gamma) = e$ becomes a necessary condition for the $\mathbf{T}$-tileability which is usually easy to check. Various arguments can be found in [G].

To give an example, recall the argument we used to prove untileability by dominoes (see Fig. 1.2). It was, basically, a map $f : \mathcal{R} \to \mathbb{Z}$, defined by

$$f(i,j) = \begin{cases} 1\,, & \text{if } i + j = 1 \mod 2 \\ 0\,, & \text{if } i + j = 0 \mod 2 \end{cases}$$

The map f is an example of a $\mathbf{T}$-coloring map. Indeed, observe that by definition for both vertical and horizontal dominoes τ_1, τ_2 (see Fig. 1.1) we have $f(\tau_1) = f(\tau_2) = 0$.

It turns out that the coloring arguments cannot be used to prove Theorems $0.1 - 3$. In particular, we have the following result.

Theorem 1.7 Let $\mathbf{T}$ be a set of tiles shown in Figure 0.5. Consider a rectangle $\Gamma = [5 \cdot a \times 5 \cdot b]$, where a, b – odd. Then for any $\mathbf{T}$-coloring map $f : \mathcal{R} \to G$ we have $f(\Gamma) = e$.

Recall that by Theorem 0.1 the region Γ in Theorem 1.7 is not $\mathbf{T}$-tileable. Thus Theorem 0.1 can not be proved by use of only coloring arguments. Theorem 1.7 and its analogs for othe sets of tiles will be proven in section 8.

Let us note that coloring maps can be used not to prove untileability but also to find some tile invariants. Here is how it can be done.

Let $\mathbf{T}$ be a set of tiles, G be an abelian group. Consider a map $g : \mathbf{T} \to G$. Suppose $f : \mathcal{R} \to G$ is a coloring map such that $f(\tau') = g(\tau)$ for all translations τ' of a tile $\tau \in \mathbf{T}$. Then there exist a tile invariant $\widehat{g} : \mathcal{R}_{\mathbf{T}} \to G$ such that $\widehat{g}(\tau) = g(\tau)$ for all $\tau \in \mathbf{T}$. We call this an *extended coloring argument* corresponding to the map f.

For example, let $\mathbf{T} = \mathbf{T}_2$ be the set of dominoes. Consider a coloring map $f : \mathcal{R} \to \mathbb{Z}_2$ defined by $f(i,j) = j \mod 2$. In a different form the map f was considered earlier (see Figure 1.7). We have $g(\tau_1) = 1$ and $g(\tau_2) = 0$ (see Figure 1.1) which proves that the map $f_\bullet$ (see Definition 1.5) is indeed an invariant if $n = 2$.

One can try to use the extended coloring argument to get all the tile invariants for a given set of tiles $\mathbf{T}$ and a set of regions $\mathcal{B} \subset \mathcal{R}_{\mathbf{T}}$. It is easy to see that this is impossible if the tile counting group $\mathbb{G}(\mathbf{T}_n, \mathcal{B}) \supsetneq \mathbb{G}(\mathbf{T}_n, \mathcal{R}_{\mathbf{T}})$ i.e. if there exist a tile invariant for the set of regions $\mathcal{B}$ that is not an invariant for the set of all $\mathbf{T}$-tileable regions $\mathcal{R}_{\mathbf{T}}$. It turns out that in case of the ribbon tiles $\mathbf{T}_n$, $n > 2$ neither convexity nor parity invariants follow from the extended coloring arguments. Analogously, for the height invariant we have the following result.

Theorem 1.8 Let $\mathbf{T}_n$, $n > 1$ be a set of ribbon tiles. Then the height invariant $f_\bullet$ follows from the extended coloring argument if and only if $n = 2$.

The proof of Theorem 1.8 is given in section 9. Of course, the "if" part is already proven. Incidently, proving this theorem was the original goal of this work.

The rest of the paper is constructed as follows. In section 2 we define a rim hook correspondence which is used in section 3 to prove Theorem 1.6. In section 4 we check that the coloring maps defined in Definition $1.1 - 1.3$ are invariant under the local moves defined in Theorem 1.5. In section 5 we present a technique for work with tile invariants which enables us to extend a set of regions. In section 6 we show that there are no other ribbon tile invariants other than given in Theorem 1.4. Then we prove the main theorem itself.

The second part of the paper contains several applications of the main result. In section 7 we use ribbon tile invariants to find necessary conditions for tileability. We prove Theorems $0.1 - 0.3$ and few other similar results. In section 8 we define and analyze signed tilings and prove Theorem 1.7 along with other related results. This section is motivated by the work [CL] of Conway and Lagarias although we were able to avoid the use of combinatorial group theory. In section 9 we prove the Theorem 1.8 and explore the connection between tile extended arguments and tile invariants. Finally, in section 10 we present several conjectures and open problems.

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2. The rim hook bijection

Let us recall standard notation in combinatorics related to the representation theory of the symmetric group (see e.g [JK, M]).

A *partition* is a nonincreasing integer sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0$. With each partition λ we associate a region Γ_λ called *Young diagram* or *Ferrers shape* defined as follows:

$$\Gamma_\lambda = \{(i,j) \in \mathbb{Z}^2 \,|\, 1 \leq i \leq l,\, 1 \leq j \leq \lambda_i\}$$

See Figure 2.1 for a Young diagram associated with partition $(5,5,4,3,1)$. Denote $|\lambda| = \lambda_1 + \dots + \lambda_l = |\Gamma_\lambda|$.

A *skew Young diagram* $\Gamma_{\lambda\setminus\mu}$ is a set theoretic difference of the Young diagrams associated with partitions λ and μ:

$$\Gamma_{\lambda\setminus\mu} = \Gamma_\lambda \setminus \Gamma_\mu$$

For example, a skew Young diagram $\Gamma_{(5,5,4,3,1)\setminus(3,2)}$ is shown in the Figure 2.2.

To simplify the notation we will use simbol λ to denote bothe the partition and the corresponding region Γ_λ, which we also call Young diagram shape λ. By $\mathcal{R}_y$, $\mathcal{R}_{sy}$ denote the set of all Young diagrams and skew Young diagrams respectively.

By $\lambda \circ \mu$ we denote a skew Young diagram obtained as a disjoint union of the Young diagrams λ and μ where μ is located to the right and above λ. An example of $(3,3,1) \circ (2,1) = (5,4,3,3,1) \setminus (3,3)$ is shown in Figure 2.2.

A *Young tableau* is a Young diagram λ filled with integer numbers which increase in rows and columns (see Fig. 2.3). Young tableaux is called *standard* if these numbers are $1, \dots, |\lambda|$. We can think of a Young tableau as of a flag of Young diagrams $\emptyset = \lambda^1 \subset \lambda^2 \subset \dots \subset \lambda^n = \lambda$. A *skew Young tableau* is defined analogously.

Recall that by $\mathbf{T}_n$ we denote a set of ribbon tiles with n squares (see Fig. 1.12-14). A *rim hook tableau* is a tiling of a Young diagram Γ_λ by ribbon tiles $\tau \in \mathbf{T}_n$ filled with numbers $1, 2, \dots, |\lambda|/n$ such that squares of tiles with greater numbers are located either to the right or below squares of tiles with smaller numbers (see Fig. 2.4). Again we can think of a rim hook tableau as of a flag of Young diagrams.

The *rim hook bijection* φ maps Young diagrams λ, $|\lambda| = m \cdot n$, tileable by $\mathbf{T}_n$ into a n-tuples of Young diagrams $(\nu^1, \dots, \nu^n)$, $|\nu^1| + \dots + |\nu^n| = m$. The bijection φ designed in such a way that whenever we add a ribbon tile to the diagram λ on the outside, there exist i, $1 \leq i \leq n$ such that ν^i gets a square on the outside (see Fig. 2.5). If we think of rim hook tableaux as of the flag sequences of tileable Young diagrams, the rim hook bijection maps these flag sequences into the n-tuples of Young tableaux filled with numbers $1, \dots, m$ which are increasing in rows and columns in each tableau. It is known that this establishes a bijection between rim hook tableau of a fixed tileable Young diagram λ and the n-tuples of Young tableaux with shapes $\varphi(\lambda) = (\nu^1, \dots, \nu^n)$ which are filled with numbers $1, \dots, m$, where $|\lambda| = m \cdot n$ (see e.g. [JK, SW]).

The easiest way to understand the rim hook bijection is to look at the Figure 2.6. Take a rim hook tableau tiled with 14 tiles $\tau_i \in \mathbf{T}_3$ and rotate it counterclockwise by $135°$ degrees. Then project all hooks on the horizontal axis preserving their labels and relative order. Split the "shadows" into three (n in general case) separate sets of "shadows" depending on their horizontal coordinate $\mod 3$. Then simply shorten the shadows and reverse the procedure. At the end we get three Young tableaux filled with numbers $1, \dots, 14$ (see Figure 2.6).

Theorem 2.1 The map φ defined above is a one-to-one correspondence.

The theorem goes back to Nakayama and Robinson (see [R, JK]). In modern times it was rediscovered by Stanton and White (see [SW, FS]) and is sometimes attributed to them.

Another way to think of the rim hook bijection is to say that it establishes a bijection between rim hook tableaux of shape λ and standard Young tableaux of the skew shape $\nu^1 \circ \cdots \circ \nu^n$. We shall be using this interpretation in the next section. Various proofs and applications of the Theorem can be found in [JK, FS, S].

3. Local moves

Let $\lambda \setminus \mu$ be a skew Young diagram. We define *local moves* on a set of standard Young tableaux of shape $\lambda \setminus \mu$ as follows. Take a pair of numbers i and $i+1$, $1 \le i < |\lambda|$ and exchange them if they lie in different rows and columns. We claim that using these moves one can start with any standard Young tableau of shape $\lambda \setminus \mu$ and get any other such a tableau.

Formally, let $\Omega(\lambda \setminus \mu)$ be a graph with vertices all standard Young tableaux of shape $\lambda \setminus \mu$ and edges obtained by applying local moves. An example with $\lambda = (3, 2)$ and $\mu = \emptyset$ is shown in Figure 3.1.

Theorem 3.1 Let $\lambda \setminus \mu$ is a skew Young diagram. Then graph $\Omega(\lambda \setminus \mu)$ is connected.

This result is known and not hard to prove. Some generalizations and applications can be found in [BW, BK].

Sketch of Proof. Introduce an orientation of edges of graph $\Omega(\lambda \setminus \mu)$ by distinguishing situations when a local move exchanges i and $i+1$ with $i+1$ lying to the right and above of i from those where $i+1$ is lying to the left and below of i (see Figure 3.1). Observe that the orientation is acyclic and has exactly one sink. This proves the result. $\square$

Consider what happens if we apply the bijection φ to the vertices of a graph $\Omega(\lambda \setminus \mu)$ in Theorem 3.1. Fix a skew Young diagram $\nu = \nu^1 \circ \cdots \circ \nu^n$. Define *local moves* on a set of rim hook tableaux of shape $\lambda = \varphi^{-1}(\nu^1, \dots, \nu^n)$ by the image of the corresponding local moves on a standard skew Young tableaux. Observe that squares in a Young tableau of shape ν correspond to rim hooks in a rim hook tableau of shape λ. Thus the corresponding local moves on a set of rim hook tableaux will be preserving all the rim hooks except two. Since these rim hooks have consequent labels, together they form a skew shape which has exactly two rim hook tableaux (see Figure 3.2). Note that the two rim hooks may lie far from each other, in which case the local move is just relabelling of their numbers. When $n = 1$ this is the only case that occurs.

Now we are ready to state an analog of Theorem 3.1. Denote $\Omega_n(\lambda \setminus \mu)$ a graph with rim hook tableaux as vertices and edges connecting those pairs of tableaux which have the same set of all but two rim hooks.

Theorem 3.2 Let λ be a Young diagram tileable by $\mathbf{T}_n$. Then graph $\Omega_n(\lambda)$ is connected.

Note that here we do not claim that $\Omega_n(\lambda \setminus \mu)$ is connected for any tileable skew Young diagram $\lambda \setminus \mu$. It is true and is proven in [Pa]. We can avoid use this natural generalization for the purpose of studying ribbon tile invariants.

Proof. Define $(\nu^1, \dots, \nu^n) = \varphi(\lambda)$ to be an image of λ under the rim hook correspondence. Consider $G = \Omega(\nu^1 \circ \cdots \circ \nu^n)$. The correspondence φ^{-1} maps vertices of G onto $\Omega_n(\lambda)$ and edges onto edges. In other words, $\varphi^{-1}(G)$ is a subgraph of $\Omega_n(\lambda)$. By Theorem 3.1 graph G is connected. Therefore graph $\Omega_n(\lambda)$ is also connected. $\square$

Now let us make a graph on the ribbon tilings of λ. The idea is to erase the labels in the rim hook tableaux and connect those that were connected before.

Formally, consider a graph $\Theta_n(\lambda)$ with vertices to be all ribbon tilings $s \in \mathcal{S}(\Gamma_\lambda, \mathbf{T})$ of a fixed shape λ. Define the edges to be the pairs of tilings that differ by exactly two tiles. Two examples of such graphs $\Theta_2(3,3,2)$ and $\Theta_3(3,3,3,3)$ are shown in Figure 3.3 and Figure 3.4 respectively.

Theorem 3.3 Let λ be a Young diagram tileable by $\mathbf{T}_n$. Then graph $\Theta_n(\lambda)$ is connected.

Proof Consider a map $\iota : \Omega_n(\lambda) \to \Theta_n(\lambda)$ which maps rim hook tableaux to ribbon tilings by erasing labels of tiles. By definition ι maps edges of $\Omega_n(\lambda)$ into edges of $\Theta_n(\lambda)$. Therefore in order to prove that $\Theta_n(\lambda)$ is connected all we need to show is that each vertex has a preimage. Indeed, if this is true, in order to find a path between two vertices in $\Theta_n(\lambda)$ simply take their preimages, find a path between them in $\Omega_n(\lambda)$ and then map it back to $\Theta_n(\lambda)$.

In other words, the Theorem in now reduced to the following lemma.

Lemma 3.4 Every ribbon tiling of a Young diagram λ admits a labelling which makes it rim hook tableaux of shape λ.

Proof. Prove the lemma by induction on the number of squares $|\lambda|$. Base of induction is obvious. Fix a Young diagram λ. By the *border strip* of λ define set of squares $(i,j) \in \lambda$ such that $(i+1, j+1) \notin \lambda$. A tile τ is called *border tile* if it lies in the border strip (see Figure 3.5). We claim that every ribbon tiling of λ must contain at least one border tile. If we find such a tile τ, label it with the largest number. Then there are no tiles that lie to the right or below τ and we can proceed by induction with $\lambda \setminus \tau$.

In order to find a border ribbon start with the lower left corner. It must belong to some tile. This tile has this square as its starting square. If it is not a border tile, find the first border square that is not in that tile. It must belong to some other tile. This tile also has this square as its starting square. Keep on doing so until we find the border tile. It must always exist since the top right corner must also belong to some tile and this square cannot be a starting square of any tile unless $n = 1$ in which case it is a border tile by definition (see Figure 3.6).

This proves the step of induction together with the lemma. The lemma respectively implies Theorem 3.3. □

Proof of Theorem 1.6. Observe that the set or regions $\mathcal{B} = \mathcal{B}_y$ is exactly the set of all Young diagram shapes. Take the moves to be as described above. The number of different moves is bounded by the the number of pairs of ribbon tiles aligned to each other. The latter number is easily bounded by $n \cdot 4^n$ which proves the Theorem. □

Finally, we would like to note that in the proof of Lemma 3.4 rim hooks need not be of the same length.

4. Tile Invariants

Let $\mathcal{B} = \mathcal{B}_y$ be a set of tileable Young diagram shapes. In the previous section we showed that all the tilings of $\lambda \in \mathcal{B}_y$ can be obtained from each other by a finite set of moves. Here we show that the maps f_i, $0 \le i < n/2$ are constant along these moves. In other words, we shall prove Theorem 1.4 for the set of regions $\mathcal{B}_y$.

Let us look at the structure of the moves we introduced in section 3. Consider a large enough example shown in Figure 3.2. Recall that each ribbon tile is coded by a sequence $(\varepsilon_1, \dots, \varepsilon_{n-1})$ of $n-1$ zeroes and ones. In these notation our pair of tiles is mapped into a similar pair:

$$\mathbf{00110101}\ ,\ \mathbf{01011110}\ \to\ \mathbf{00100101}\ ,\ \mathbf{01010110}$$

Subtracting the sequences as vectors we get vectors $(0, 0, 0, -1, 0, 0, 0, 0)$ and $(0, 0, 0, 0, -1, 0, 0, 0)$. In other words, first tiles in a pair differ at the fourth place where 1 becomes 0. Respectively, second tiles in a pair differ at the fifth place where 1 becomes 0 as well. Note that all tiles contain $n = 9$ squares and $5 + 4 = 9$. We claim that this is a general observation.

Lemma 4.1 In every move defined above one tile sequence changes from 1 to 0 (or from 0 to 1) at some place i while another has exactly the same change at a place $n - i$.

Proof. The proof is done by the following observation. Note that each skew Young diagram shape which corresponds to a move, can be broken into three parts by the number of squares in a diagonal parallel to $y = -x$ line (see Figure 4.1). After the move, the south-west and north-east part remain the same, while the middle part remain divided into two identical small ribbon tiles which get switched now. Therefore the only difference in tile sequences occur only in places where south-west and north-east part touch the middle part. If the south-west part was touching the upper of two small ribbon tiles, it now touches the lower one. This means that at a place i the number in a sequence changed from 1 to 0 (see Figure 4.1). Respectively, the north-east part was touching the lower of two small ribbon tiles and now is touching the upper one. This means that at a place $n-i$ the number in a sequence changed from 1 to 0.

The second case when the south-west part was touching the upper of two small ribbon tiles before the move, and the lower tile after the move, is analogous. This proves the lemma. □

Now recall definitions of the convexity invariants:

$$f_i(\varepsilon_1, \ldots, \varepsilon_{n-1}) = \varepsilon_i - \varepsilon_{n-i}$$

where $1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor$. Analogously the parity invariants are defined by

$$f_*(\varepsilon_1, \ldots, \varepsilon_{n-1}) = \varepsilon_m \mod 2$$

where n is even, $m = n/2$.

Lemma 4.2 The maps f_i, f_* are invariant under the local moves.

Proof. The lemma follows easily from Lemma 4.1. Indeed, as we showed above if a move changes tiles τ_1, τ_2 into tiles τ_1', τ_2', then

$$f_i(\tau_1') = f_i(\tau_1) \pm 1, \quad f_i(\tau_2') = f_i(\tau_2) \mp 1$$

where the different sign in the second equation comes from the minus sign in $f_i = \varepsilon_i - \varepsilon_{n-i}$.

Therefore

$$f_i(\tau_1') + f_i(\tau_2') = f_i(\tau_1) + f_i(\tau_2)$$

which proves that maps f_i are invariant under the local moves.

The case of a parity invariant is slightly different since here we do not have opposite signs. Instead we have

$$f_*(\tau_1') = f_*(\tau_1) \pm 1, \quad f_*(\tau_2') = f_*(\tau_2) \pm 1$$

Therefore

$$f_i(\tau_1') + f_i(\tau_2') = f_i(\tau_1) + f_i(\tau_2) \pm 2 = f_i(\tau_1) + f_i(\tau_2) \pmod 2$$

which proves that the map f_* is invariant under the local moves. This finishes proof of the lemma. □

Recall that by f_0 we denote the area invariant.

Corollary 4.3 Let $\mathcal{B} = \mathcal{B}_y$ be a set of tileable Young diagram shapes. Then, when $n = 2m+1$, maps $f_0, f_1, \dots, f_m$ are the ribbon tile invariants. Analogously, when $n = 2m$, maps $f_0, f_1, \dots, f_{m-1}, f_*$ are the ribbon tile invariants.

Proof. By Lemma 4.2 maps f_i, f_* are invariant under the local moves. By Theorem 3.3 we can get any ribbon tiling of a Young diagram from any other. Therefore the above mentioned maps are indeed invariants on a set of tileable Young diagram shapes. This proves the corollary. $\square$

Note that Corollary 4.3 proves only one part of Theorem 1.4 for the set of regions $\mathcal{B} = \mathcal{B}_y$. The second part which states that these invariants form a basis will be proven in section 6.

5. Increasing the set of regions

In this section we will generalize Corollary 4.3 from the set of Young diagram shapes to the set of all row-convex regions $\mathcal{B}_{rc}$. As an intermediate step we use a set of all skew Young diagram shapes.

Our approach is based on the following general observation.

Lemma 5.1 Let $\mathbf{T}$ be a set of tiles, $\mathcal{B}_1 \subset \mathcal{B}_2$ be two sets of $\mathbf{T}$-tileable regions. Suppose for each region $\Gamma_2 \in \mathcal{B}_2$ there is a region $\Gamma_1 \in \mathcal{B}_1$, such that $\Gamma_1 \supset \Gamma_2$ and $\Gamma_1 \setminus \Gamma_2$ is $\mathbf{T}$-tileable. Then if $f : \mathcal{B}_2 \to G$ is an invariant on $\mathcal{B}_1$, it is also an invariant on $\mathcal{B}_2$

Proof. We need to show that for any $\Gamma_2 \in \mathcal{B}_2$ all tilings $s \in \mathcal{S}(\Gamma_2, \mathbf{T})$ have the same G-value of f. We know that for any $\Gamma_2 \in \mathcal{B}_2$ there is a region $\Gamma_1 \in \mathcal{B}_1$, $\Gamma_1 \supset \Gamma_2$ such that $\Gamma_1 \setminus \Gamma_2$ is $\mathbf{T}$-tileable. Therefore the set of tilings $\mathcal{S}(\Gamma_2, \mathbf{T})$ is in a correspondence with a subset $\mathcal{S}' \subset \mathcal{S}(\Gamma_1, \mathbf{T})$ such that all the tilings $s \in \mathcal{S}'$ have the same fixed tiling of $\Gamma_1 \setminus \Gamma_2$. By definition value of f is the same on all tilings of $\mathcal{S}(\Gamma_1, \mathbf{T})$. Since it is fixed on $\Gamma_1 \setminus \Gamma_2$, it must be also the same on $\mathcal{S}(\Gamma_2, \mathbf{T})$. This finishes the proof. $\square$

We call the set $\mathcal{B}_2$ of $\mathbf{T}$-tileable regions *reducible* to $\mathcal{B}_1$, if $\mathcal{B}_1 \subset \mathcal{B}_2$ and they satisfy the conditions of Lemma 5.1. Of course, if $\mathcal{B}_3$ is reducible to $\mathcal{B}_2$ and $\mathcal{B}_2$ is reducible to $\mathcal{B}_1$, then $\mathcal{B}_3$ is reducible to $\mathcal{B}_1$.

Lemma 5.2 Let $\mathcal{B}_{sy}$ be the set of $\mathbf{T}_n$-tileable skew Young diagram shapes, $\mathcal{B}_y$ be the set of $\mathbf{T}_n$-tileable ordinary Young diagram shapes. Then $\mathcal{B}_{sy}$ is reducible to $\mathcal{B}_y \subset \mathcal{B}_{sy}$.

Proof. Indeed, all we need to prove is that every skew Young diagram can be imbedded in an ordinary Young diagram such that their difference is tileable by the ribbon tiles. There is an easy way to do that just by using the horizontal and the vertical tiles.

The idea is shown in Figure 5.1. We start with the rightmost column of a skew Young diagram shape and move to the left. Whenever we move left, add on top a column of vertical tiles until they level or exceed the column on the right. If they do exceed the column on the right, for each exceeding square add to the right a row of horizontal tiles until they level or exceed the row below. In example shown in Figure 5.1 we do nothing for the first two columns. For the third column from the right we add one vertical tile and two horizontal. We add just one vertical tile for each of the next two columns. For the sixth column we are forced to add three vertical and two horizontal tiles, etc.

We stop when we are finished with the last column (ninth in case of the Figure 5.1). By construction we always have a Young diagram shape to the right of the building column. Therefore the resulting shape is also a Young diagram shape. this proves the lemma. □

Remark 5.3 In [Pa] we use this construction to define a generalization of the rim hook bijection for skew shapes. Note also that Lemma 5.2 can be generalized for *any* set of tiles $\mathbf{T}$ which contains a horizontal and a vertical tile, not necessarily of the same length.

Lemma 5.4 Let $\mathcal{B}_{sy}$ be the set of $\mathbf{T}_n$-tileable skew Young diagram shapes, $\mathcal{B}_{rc}$ be the set of $\mathbf{T}_n$-tileable row-convex regions. Then $\mathcal{B}_{rc}$ is reducible to $\mathcal{B}_{sy} \subset \mathcal{B}_{rc}$.

Proof. Indeed, all we need to prove is that every row-convex region can be imbedded in a skew Young diagram such that their difference is tileable by the ribbon tiles. There is an easy way to do that just by using just the horizontal tiles.

The idea is shown in Figure 5.2 and similar to the idea used in Lemma 5.2. We start with the top row of our row-convex region and move to the bottom row. Each time we move down we add a row of horizontal tiles to the left such that they level or exceed the row above. When get to the bottom we start adding rows of the horizontal tiles to the right of the region in such a way that each row level or exceed the row below (see Figure 5.2). At the end we get a skew Young diagram shape which proves the lemma. □

Corollary 5.5 The statement of Corollary 4.3 holds for the set of row-convex $\mathbf{T}_n$-tileable regions $\mathcal{B}_{rc}$.

Proof. From Lemmas 5.2, 5.4 we have $\mathcal{B}_{rc}$ is reducible to $\mathcal{B}_y$. By Lemma 5.1 this implies that every ribbon tile invariant for the set of regions $\mathcal{B}_y$ is also an invariant for the set of regions $\mathcal{B}_{rc}$. Together with Corollary 4.3 this proves the result. □

6. The tile counting group

Here we will prove that there are no invariants other than those which follow from convexity, parity and area invariants. Together with Corollary 5.5 this implies the main result of the paper Theorem 1.4.

The idea is to show that every invariant is completely defined by its values on the horizontal and two-row ribbon tiles.

Denote $\tau_0 = \mathbf{0000} \ldots \ \mathbf{0}$, $\tau_i = \mathbf{0} \ldots \ \mathbf{010} \ldots \ \mathbf{0}$ ($\mathbf{1}$ is on i-th place), $1 \le i \le n-1$. Let $\mathbf{B}_n \subset \mathbf{T}_n$ be the set of tiles τ_i, where $0 \le i \le m = \left\lfloor \frac{n}{2} \right\rfloor$.

Lemma 6.1 Let $f_1, f_2 : \mathcal{B}_{rc} \to G$ be two ribbon tile invariants. We claim that if for every $\tau \in \mathbf{B}_n$ we have $f_1(\tau) = f_2(\tau)$, then $f_1 \equiv f_2$.

Proof. We need to show that for every ribbon tile $\tau \in T_n$ we have $f_1(\tau) = f_2(\tau)$. This would immediately imply that $f_1 \equiv f_2$. Prove it by induction on the height ht of a ribbon tile (see the Introduction) :

$$ht(\tau) = 1 + \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{n-1}$$

First, prove the base of induction. If $ht(\tau) = 1$ then τ is a horizontal tile $\tau_0 \in \mathbf{B}_n$. If $ht(\tau) = 2$, then $\tau \in \mathbf{B}_n$ or $\tau = \tau_i$, $m < i \le n-1$. Observe that τ_i and τ_{n-i} form a two-row skew Young diagram shape which can be also divided into two horizontal tiles (see Figure 6.1). Therefore if invariants f_1, f_2 agree on $\mathbf{B}_n$ they must also agree on all the two-row ribbon tiles.

Now suppose the claim holds for all tiles $\tau \in \mathbf{T}_n$ with $ht(\tau) < k$. Let $\tau \in \mathbf{T}_n$ be a ribbon tile and $ht(\tau) = k$. The sequence corresponding to τ can be presented in the following form $(\varepsilon_1, \dots, \varepsilon_i, 1, 0, 0, \dots, 0)$. The two-row ribbon tile τ_{n-i-1} can be aligned with the top two rows of τ to form a skew Young diagram shape (see Figure 6.2). This region can be also divided into a horizontal tile and a tile τ' with a sequence $(\varepsilon_1, \dots, \varepsilon_i, 0, 0, 0, \dots, 0)$ (see Figure 6.2). Note that $ht(\tau') = ht(\tau) - 1 = k-1$. Therefore if invariants f_1, f_2 agree on all tiles $\tau \in \mathbf{T}_n$, $ht(\tau) < k$ they must also agree on all tiles $\tau \in \mathbf{T}_n$, $ht(\tau) = k$. This proves the step of induction and finishes proof of the lemma. □

Proof of Theorem 1.4. Corollary 5.5 implies that our maps are indeed invariants. All we need to prove now is that they are independent and generate the whole tile counting group.

To show independence consider values our invariants take on $\mathbf{B}_n \subset \mathbf{T}_n$. The area invariant f_0 is a constant on all $\mathbf{B}_n$, including τ_0. The convexity invariant f_i is nonzero only on the tile τ_i. Analogously, the parity invariant is nonzero only on the tile τ_m, $n = 2m$. This immediately implies independence.

Now, Lemma 6.1 proves that a tile invariant is completely determined by its values on $\mathbf{B}_n$. This implies that in case $n = 2m+1$ there can be no invariants that are not generated by $f_0, f_1, \dots, f_m$. Therefore $\mathbb{G}(\mathbf{T}; \mathcal{B}) \simeq \mathbb{Z}^{m+1}$.

We still have a little room left in case $n = 2m$ since the parity invariant takes values in $\mathbb{Z}_2$ rather than in $\mathbb{Z}$. Recall that in the proof of Lemma 6.1 we showed that if f is an invariant, then

$$f(\tau_i) + f(\tau_{n-i}) = 2\, f(\tau_0)$$

(see Figure 6.1). When $i = m$ this gives $2 \cdot f(\tau_m) = 2 \cdot f(\tau_0)$ which implies that there can be no invariants that are not generated by $f_0, f_1, \dots, f_{m-1}$ and f_*. Therefore $\mathbb{G}(\mathbf{T}; \mathcal{B}) \simeq \mathbb{Z}^m \times \mathbb{Z}_2$. This finishes the proof of Theorem 1.4 □

7. Applications to tileability

In this section we use ribbon tile invariants to give new tileability criteria for certain sets of tiles. Among the results we prove Theorems $0.1 - 3$.

We use the following logic. Let $\mathbf{T} \subset \mathbf{T}_n$ be a subset of ribbon tiles. Suppose Γ is a region which is tileable by $\mathbf{T}_n$. Then we can use tile invariants to find diophantine equations for the number of times each tile $\tau \in \mathbf{T}_n$ occurs in the tilings. When restricted to a smaller set of tiles $\mathbf{T}$ sometimes these equations do not have an integer solution. This would imply that Γ is untileable by $\mathbf{T}$. Generally, having solution for these equations becomes a *tileability test* which is easy to use in practice.

Here are few examples when we apply the above logic successfully.

1. Let $\mathbf{T} \subset \mathbf{T}_4$ be a set of four tiles shown in Figure 7.1. We ask for which values (a, b) a rectangle $[a \times b]$ can be tiled by $\mathbf{T}$. Note that $\mathbf{T}$ is asymmetric, so in our notation where a is a height and b is a width of a rectangle.

Theorem 7.1 Let $\mathbf{T}$ be a set of tiles in Figure 7.1. Then a rectangle $[a \times b]$ can be tiled by $\mathbf{T}$ if and only if (a, b) satisfies one of the following:
1) $4|a$
2) $8|b,\ \ a \geq 3$
3) $2|a,\ \ 4|b$

Proof. The tileability in cases $1) - 3)$ follows from the existence of tilings of the rectangles $[4 \times 1]$, $[2 \times 4]$ and $[3 \times 8]$ (see Figure 7.2).

To prove untileability in all the other cases we need several observations. First, it is obvious that no rectangle $[1 \times b]$ can be tiled.

Suppose now a is odd, $a \geq 3$. Then $4|b$ and the rectangle $[a \times b]$ can be tiled by a horizontal tile $[1 \times 4] \in \mathbf{T}_4$. Thus the height invariant (see Definition 1.5) $f_\bullet = f_1 + f_2 \pmod 2$, $f_\bullet : \mathcal{B} \to \mathbb{Z}_2$ has a value

$$f_\bullet([a \times b]) = 0$$

However $f_\bullet(\tau) = 1$ for all $\tau \in \mathbf{T}$. Therefore in order to be tileable, the rectangle $[a \times b]$ must contain an even number of tiles, i.e. $8|b$ and we are in case 2).

Now we need to show that the no rectangle $[a \times b]$, with $a, b = 2 \mod 4$ can be tiled by $\mathbf{T}$. This can be done by use of the coloring arguments (see the Introduction). We will do it in the next section (see Corollary 8.5). $\square$

2. Let $\mathbf{T} \subset \mathbf{T}_5$ be a set of eight tiles shown in Figure 0.5. Note that in this case $\mathbf{T}$ is symmetric under the transposition. Theorem 0.1 claims that a rectangle $[a \times b]$ can be tiled by $\mathbf{T}$ only if $10|a \cdot b$. We claim that even stronger statement is true.

Theorem 7.2 Let $\mathbf{T}$ be a set of tiles in Figure 0.5. Then a rectangle $[a \times b]$ can be tiled by $\mathbf{T}$ if and only if $10|a \cdot b$ and $a, b > 1$.

Proof. The tileability follows immediately from the existence of tilings of the rectangles $[2 \times 5]$, $[3 \times 10]$ and of the rectangles $[5 \times 2]$, $[10 \times 3]$ that are transposed to them (see Figure 7.3).

The other direction is similar to the one used in the proof of Theorem 7.1. Observe that if a rectangle $[a \times b]$ can be tiled by $\mathbf{T}$, then either a or b can be divided by 5. But then $[a \times b]$ is tileable by either horizontal tile τ_0 or a vertical tile τ_0', $\tau_0, \tau_0' \in \mathbf{T}_5$. Observe that

$$f_\bullet(\tau_0) = f_\bullet(\tau_0') = 0 \mod 2$$

where $f_\bullet : \mathcal{B} \to \mathbb{Z}_2$ is a height invariant. Therefore $f_\bullet([a \times b]) = 0$ for any tileable rectangle $[a \times b]$. On the other hand for any tile $\tau \in \mathbf{T}$ we have $f_\bullet(\tau) = 1$. Therefore there must be even number of tiles in $[a \times b]$ i.e. $10|a \cdot b$. This proves the theorem.
□

3. Let $\mathbf{T} \subset \mathbf{T}_5$ be a set of six tiles shown in Figure 0.6. Note that in this case $\mathbf{T}$ is also symmetric under the transposition.

Theorem 7.3 Let $\mathbf{T}$ be a set of tiles in Figure 0.5. Then a rectangle $[a \times b]$ can be tiled by $\mathbf{T}$ if and only if $10|a \cdot b$ and $a, b \neq 1, 3$.

Proof. The tileability follows immediately from the existence of tilings of the rectangles $[2 \times 2]$, $[7 \times 10]$ and of the rectangles $[5 \times 2]$, $[10 \times 7]$ that are transposed to them (see Figure 7.4).

To show the other direction, use the same reasoning as in the proof of Theorem 7.2. If a rectangle $[a \times b]$ can be tiled by $\mathbf{T}$, then its area must be divisible by 5 and therefore it is tileable by either horizontal or a vertical tile in $\mathbf{T}_5$. Thus the 1-convexity invariant f_1 is 0 on any tileable $[a \times b]$. On the other hand, $f_1(\tau) = \pm 1$ for each $\tau \in \mathbf{T}$. Therefore $[a \times b]$ has to be tiled by an even number of tiles and $10|a \cdot b$. This finishes proof of the theorem. □

Theorem 7.4 The triangular shape Δ_N can be tiled by tiles shown in Figure 0.6 if and only if $N \equiv 0, 4, 15, 19 \pmod{20}$.

Proof. First prove the tileability part. Observe that Δ_4 and Δ_{15} are both tileable (see Figure 7.5). Now, if we have tilings of the regions Δ_M, Δ_N we can construct a tiling of the region Δ_{M+N+1} assuming the rectangle $[(M+1) \times (N+1)]$ is tileable. (see Figure 7.6). Analogously, if we have tilings of the regions Δ_M, Δ_N we can construct a tiling of the region Δ_{M+N} assuming the rectangle $[M \times N]$ is tileable (see Figure 7.7). Since both $[5 \times 16]$ and $[4 \times 15]$ are tileable (see Theorem 7.3), this gives us tilings of $\Delta_{20} = \Delta_{15+5+1}$ and $\Delta_{19} = \Delta_{15+5}$. Going further, if Δ_M if tileable, then Δ_{20+M} is also tileable. This covers all the values $N \equiv 0, 4, 15, 19 \pmod{20}$.

To prove "only if" part, start with computing the area. We have

$$f_0(\Delta_N) = \frac{N\,(N+1)}{2}$$

Since the area must be divisible by 5, this gives us $5|N\,(N+1)$ and $N \equiv 0, 4 \pmod{5}$. Now, in each of these cases it is easy to see that Δ_N can be tiled by $\mathbf{T}_5$. Indeed, both Δ_4 and Δ_5 can be tiled by $\mathbf{T}_5$ (see Figure 7.8). Since any rectangle

$[a \times b]$, is tileable by either horizontal or a vertical tile in $\mathbf{T}_5$, we can use construction in the Figure 7.7 repeatedly and tile Δ_N for all $N \equiv 0, 4 \pmod 5$.

Now we can compute the value of 1-convexity invariant f_1 on Δ_N, $N \equiv 0, 4 \pmod 5$. We have

$$f_1(\Delta_4) = f_1(\Delta_5) = 0\ ,\ f_1(\Delta_{N+5}) = f_1(\Delta_N)$$

which gives us

$$f_1(\Delta_{5\,m+4}) = f_1(\Delta_{5\,m+5}) = 0$$

Since f_1 takes only ± 1 values on $\mathbf{T}$, this implies that in order to be tileable Δ_N must have an even number of tiles. Thus $2|f_0 = \frac{N\,(N+1)}{10}$, and $20|N\,(N+1)$. This immediately implies $N \equiv 0, 4, 15, 19 \pmod{20}$. □

4. Let $\mathbf{T} \subset \mathbf{T}_5$ be a set of eight tiles shown in Figure 7.9. As before we solve the tileability problem for rectangular and triangular shapes.

Theorem 7.5 Let $\mathbf{T}$ be a set of tiles in Figure 7.9. Then a rectangle $[a \times b]$ can be tiled by $\mathbf{T}$ if and only if (a, b), $a \le b$ satisfies one of the following:
1) $10|a \cdot b$, $a \ge 8$
2) $a = 4, 6$, $5|b$
3) $a = 5$, $2|b$,

Proof. The tileability follows from the existence of tilings of the rectangles $[4 \times 5]$, $[5 \times 6]$ and of the rectangulars $[5 \times 4]$, $[6 \times 5]$ that are transposed to them (see Figure 7.10). Indeed, then we can construct rectangles $[4 \times 5\,m]$, $[6 \times 5\,m]$, $[5 \times 2\,r]$, $r \ge 3$ which cover cases 2) and 3). An easy check shows that this also implies tileability in case 1).

Now let us prove the "only if" direction. First, we show that $10|a\,b$ in order for $[a \times b]$ to be tileable by $\mathbf{T} \subset \mathbf{T}_5$. Indeed, the area $f_0 = a \cdot b$ must be divisible by 5 which implies that $[a \times b]$ is tileable by either a horizontal or a vertical tile in $\mathbf{T}_5$. Now, it implies that 2-convexity invariant takes zero value on a rectangle, while its value is ± 1 on each of the eight tiles in $\mathbf{T}$. Therefore $[a \times b]$ must contain an even number of tiles in order to be tileable by $\mathbf{T}$ which proves the claim.

Observe that since $10|a \cdot b$, $a, b > 7$ covers by case 1) we are left with cases when a or b is at most 7. It is easy to see that there are no tileable rectangle with $a = 2, 3$. An elaborate search of about a dozen of beginnings shows that there are no tileable rectangle with $a = 7$. These cover all the untileable cases and finishes proof of the Theorem. □

Theorem 7.6 The triangular shape Δ_N can be tiled by tiles shown in Figure 7.9 if and only if $N \equiv 0, 5, 14, 19 \pmod{20}$.

Proof. First prove the tileability part. Observe that Δ_5 and Δ_{14} are both tileable (see Figure 7.11). Recall that by Theorem 7.5 rectangles $[14 \times 5]$, $[15 \times 6]$ and $[20 \times N]$, $N \ge 5$ are all tileable by $\mathbf{T}$. Using the same construction as in the proof of Theorem 7.4 (see Figure 7.6,7) we get tilings on shapes Δ_{19}, Δ_{20}. We also get a tiling of Δ_{20+N} from a tiling of Δ_N. This covers all the values $N \equiv 0, 5, 14, 19 \pmod{20}$ and proves the "if" part.

To prove the "only if" part, first recall that the area is divisible by 5 if and only if $N \equiv 0, 4 \pmod 5$ and in all these cases Δ_N is tileable by $\mathbf{T}_5$ (see proof of Theorem 7.4). Compute the 2-covexity invariant $f_2(\Delta_N)$. We have (see Figure 7.7,8):

$$f_2(\Delta_4) = f_2(\Delta_5) = -1 \ , \ f_2(\Delta_{N+5}) = f_2(\Delta_N) - 1$$

which gives us

$$f_2(\Delta_{5\,m+4}) = f_2(\Delta_{5\,m+5}) = -m$$

Since f_2 takes only ± 1 values on $\mathbf{T}$, this implies that in order to be tileable the number of tiles in $\Delta_{5\,m+4,5}$ must have the same parity as m. This immediately implies $N \equiv 0, 4, 15, 19 \pmod{20}$ and finishes the proof. $\square$

8. Signed tilings and coloring arguments

Instead of ordinary tilings one can try using *signed tilings* (see [CL]) which are basically placements of tiles on a plane with weights $+1$ or -1 assigned to each of them. We say that they tile a region if the sum of weights of tiles is 1 for every square inside a region and 0 elsewhere.

Even with the small sets of tiles it often happens that there are untileable regions which have signed tilings. For example, a region in Figure 1.3 has no ordinary domino tiling but has a signed domino tiling. Indeed, simply tile a rectangle $[3 \times 4]$ by dominoes and add two horizontal dominoes on the top and on the bottom with negative signs.

It turns out that signed tilings are easier to study because of their connection with coloring arguments. By analogy with tile invariants, a coloring map $f : \mathcal{R} \to G$ is *trivial* if $f(\Gamma) = 0$, for every $\gamma \in \mathcal{R}$, where 0 is an identity element in an abelian group G. Otherwise f is called *nontrivial.*

Theorem 8.1 A region Γ has a signed tiling if and only if there is no abelian group G and a nontrivial $\mathbf{T}$-coloring map $f : \mathcal{R} \to G$.

Proof. Indeed, consider a group $\mathfrak{G}$ obtained as a free abelian group generated by elements $x_{i,j}$ that correspond to squares of a square grid. Let $\mathfrak{I} \subset \mathfrak{G}$ be a subgroup generated by sums $x_{i_1,j_1} + x_{i_2,j_2} + \dots$ that correspond to translations of tiles.

Observe that the $\mathbf{T}$-coloring maps correspond to the elements of the quotient group $\mathfrak{G}/\mathfrak{I}$. Therefore for every region Γ there exist a $\mathbf{T}$-coloring map $f : \mathcal{R} \to G$ such that $f(\Gamma) \neq 0$ unless sum of squares of Γ lies in $\mathfrak{I}$. On the other hand, a region Γ has a signed tiling if and only if the sum of its squares lies in $\mathfrak{I}$. This proves the result. $\square$

By analogy with tile counting group we define a *coloring group* $\mathbb{O}(\mathbf{T}) = \mathfrak{G}/\mathfrak{I}$. It follows from the proof of the theorem that signed tilings and $\mathbf{T}$-coloring maps are basically dual to each other. However, it is convenient to separate them since they give different visualization of the subject.

It is important to note that whenever we have a $\mathbf{T}$-coloring map which proves that a certain region Γ is untileable this also implies that Γ has no signed tiling. Therefore in an event when Γ is untileable but has a signed tiling (like a region in

Figure 1.3 mentioned above), it also means that this fact cannot be proved by use of the **T**-coloring maps. Traditionally the coloring maps were a major instrument in proving untileability results (see [G]) so it is often desirable to check whether they can be used to prove any of our negative results. Below we will show that the results we obtained in the previous section in fact *cannot* be proved by use of the coloring arguments.

Finally, let us point out where the difference between signed and ordinary tilings comes from. Instead of an abelian group G one can take a monoid M (commutative semigroup with an identity element). One can also define semigroup morphism from all regions to M which is identity on all tiles. It is not hard to prove a theorem similar to Theorem 8.1 which says that a region Γ has a tiling if and only if there exist no such a morphism which is not identity on Γ (cf. [CL]). We skip the details.

Theorem 8.2 Let $\mathbf{T}_n$ be the set of ribbon tiles, $\mathbb{O}(\mathbf{T}_n)$ is its coloring group. Then

$$\mathbb{O}(\mathbf{T}_n) \simeq \mathbb{Z}^{n-1}$$

Proof. Let Z be an abelian group generated by elements $z_0, z_1, \dots, z_{n-1}$. Define a coloring map

$$\zeta : \mathcal{R} \to Z/(z_0 + \dots + z_{n-1})$$

which acts on generators as follows:

$$\zeta(x_{i,j}) = z_{(j-i \mod n)}$$

To prove that ζ is indeed a $\mathbf{T}_n$-coloring map note that by definition ribbon tiles in $\mathbf{T}_n$ contain no squares lying on the same diagonal. Therefore they must contain squares lying in n subsequent diagonals: i, $i+1$, ... , $i+n-1$, where $i \in \mathbb{Z}$. Thus for all $\tau \in \mathbf{T}_n$

$$\zeta(\tau) = z_i + \dots + z_n + z_1 + \dots + z_{i-1} = 0$$

and ζ is indeed a $\mathbf{T}_n$-coloring map.

We can now prove that the coloring group for the set of ribbon tiles $\mathbf{T}_n$ is $\mathbb{Z}^{n-1}$. Let us prove that for any region Γ, $\zeta(\Gamma) = 0$ there exist a signed tiling of Ga. Then, by Theorem 8.1, it will imply that the coloring group is isomorphic to $Z/(z_0 + \dots + z_{n-1}) \simeq \mathbb{Z}^{n-1}$ which proves the result.

Let d_1 be the difference of ribbon tiles **00** ... **01** and **00** ... **00** which have the same starting square. What adding d_1 to a region does, it adds a square $x_{i,j}$ and subtracts a square $x_{i+1,j+1}$ lying on the same diagonal. Let d_2 be the difference of two horizontal tiles **00** ... **0** one of them having the starting square $x_{i,j}$ and the other in $x_{i,j+1}$. By analogy with d_1, d_2 adds a square $x_{i,j}$ and subtracts a square $x_{i,j+n}$ lying on the same diagonal modulo n. Note that all the differences d_1 and d_2 are equivalent up to translation.

Now, let Γ be a region such that $\zeta(\Gamma) = 0$. We show that by adding enough many differences d_1 and d_2 we can get an empty region. Indeed, by using d_2's we can move all the squares of Γ into the first n diagonals. Furthermore, by using d_2's we can move all the squares of Γ into the squares $x_{0,0}$, $x_{0,1}$, ... , , $x_{0,n-1}$. Since $\zeta(\Gamma) = 0$ it means that we get the same weight m at each of these squares. When subtract m copies of the horizontal tile, we get an empty region which is exactly what we needed. This finishes proof of the theorem. □

Note that if we use only d_1 without d_2 we get a sequence of numbers $\dots, m_{-1}$, m_0, m_1, $m_2, \dots$ which are numbers of squares in diagonals. This sequence gives a rise to a coloring argument for the ordinary tilings that is more general than the $\mathbf{T}_2$-coloring map ζ. For example, it proves that the region in Figure 1.3 is untileable by dominoes. Indeed, the sequence is $1, 1, 2, 2, 2, 1, 1$. The first two ones imply that there is a domino lying in the first two diagonals. Now, the two in the third place imply that there must be two dominoes that lie in the the third and fourth diagonal. Consequently, there must be two dominoes that lie in the the fifth and sixth diagonal. But this is impossible since sixth number in the sequence is one. Therefore the region in Figure 1.3 is indeed untileable by dominoes.

When we restrict our set of tiles to its subset, all the coloring maps remain. However, a priori other coloring maps may appear. We show that in the cases considered in the previous section it does not happen.

Theorem 8.3 For the set of tiles shown in Figure 7.1 the coloring group is $\mathbb{Z}^3$. For sets of tiles shown in Figures 0.5, 0.6 and 7.9 the coloring group is $\mathbb{Z}^4$.

Proof. All we need to show that our sets of tiles generate differences d_1 and d_2. This would imply that their coloring group is the same as of the ribbon tiles.

1) For the set of tiles in Figure 7.1, d_1 comes from subtracting **010** and **001** with the same starting square. The difference d_2 comes from subtracting **100** and **001** with starting squares $x_{i,j}$ and $x_{i-1,j}$ respectively.

2) For the set of tiles in Figure 0.5, d_1 comes from subtracting **0010** and **0001** with the same starting square. The difference d_2 comes from subtracting **0111** and **1110** with starting squares $x_{i,j}$ and $x_{i,j+1}$ respectively.

3) For the set of tiles in Figure 0.6, denote d_2 a difference d_2 between **0111** and **1110** with starting squares $x_{i,j}$ and $x_{i,j+1}$ respectively. It adds $x_{i,j}$ and subtracts $x_{i-3,j+2}$. Analogously define a difference d_2' between **0001** and **1000** with starting squares $x_{i-1,j}$ and $x_{i,j}$ respectively. It subtracts $x_{i,j}$ and adds $x_{i-2,j+3}$. Now the difference d_1 comes from adding d_2 and d_2'.

4) For the set of tiles in Figure 7.9, d_1 comes from subtracting **0010** and **0011** with the same starting square. The difference d_2 comes from subtracting **0010** and **0100** with starting squares $x_{i,j}$ and $x_{i,j+1}$ respectively. □

Theorem 8.3 basically tells us that $\zeta(\Gamma) = 0$ is a criteria for a region Γ to have a signed tiling in each of those cases. Before we can conclude, we need the following technical result.

Lemma 8.4
1) Let $n = 5$, $\Gamma = \Delta_N$. Then $\zeta(\Gamma) = e$ if and only if $N \equiv 0, 4 \pmod 5$.
2) Let $n = 5$, $\Gamma = [a \times b]$. Then $\zeta(\Gamma) = e$ if and only if $5 | a \cdot b$.
3) Let $n = 4$, $\Gamma = [a \times b]$. Then $\zeta(\Gamma) = e$ if and only if $4 | a$ or $4 | b$.

Proof. Parts 1) and 2) follow immediately from the area being divisible by 5 and existence of tilings by T_5 in all these cases (see proof of Theorem 7.4).

Part 3) is analogous except for the case when $a, b \equiv 2 \pmod 4$. In this case a simple direct computation of zeta shows that $\zeta \neq e$. □

Corollary 8.5

1) Let $n = 5$, $\mathbf{T}$ be any of the sets of tiles shown in Figures 0.5, 0.6 and 7.9. Then Δ_N has a signed tiling by $\mathbf{T}$ if and only if $N \equiv 0, 4 \pmod 5$.

2) Let $n = 5$, $\mathbf{T}$ be any of the sets of tiles shown in Figures 0.5, 0.6 and 7.9. Then $[a \times b]$ has a signed tiling by $\mathbf{T}$ if and only if $5 | a \cdot b$.

3) Let $n = 4$, $\mathbf{T}$ be the sets of tiles shown in Figure 7.1. Then $[a \times b]$ has a signed tiling by $\mathbf{T}$ if and only if $4|a$ or $4|b$.

Proof. Follows immediately from Lemma 8.4, Theorem 8.3 and 8.1. □

Corollary 8.6 Neither of the Theorems $0.1 - 3$ can be proved by the coloring arguments.

9. Extended coloring arguments and tile invariants

Recall the definition of a *coloring group*:

$$\mathbb{O}(\mathbf{T}) = \mathfrak{G}/\mathfrak{I}$$

where $\mathfrak{G}$ is a free abelian group generated by elements $x_{i,j}$ that correspond to squares of a square grid, and $\mathfrak{I} \subset \mathfrak{G}$ be a subgroup generated by sums $x_{i_1,j_1} + x_{i_2,j_2} + \dots$ that correspond to translations of tiles.

By analogy define an *extended coloring group* as follows:

$$\overline{\mathbb{O}}(\mathbf{T}) = \mathfrak{G}/\mathfrak{I}'$$

where $\mathfrak{I}' \subset \mathfrak{G}$ is a subgroup generated by relations

$$x_{i_1,j_1} + x_{i_2,j_2} + \dots = x_{p_1,q_1} + x_{p_2,q_2} + \dots$$

where tiles $\tau_1 = \coprod_k (i_k, j_k)$, $\tau_2 = \coprod_k (p_k, q_k)$ are transltaions of the same tile $\tau \in \mathbf{T}$. Of course, $\mathbb{O}(\mathbf{T}) \subset \overline{\mathbb{O}}(\mathbf{T})$.

By definition, every extended coloring argument for a set of tiles $\mathbf{T}$ corresponds to an element of the extended coloring group $\overline{\mathbb{O}}(\mathbf{T})$. In other words, there is a map $\nu : \overline{\mathbb{O}}(\mathbf{T}) \to \mathbb{G}(\mathbf{T})$. Since ν is homomorphic, the image $\mathbb{E}(\mathbf{T}) = \nu(\overline{\mathbb{O}}(\mathbf{T}))$ is a subgroup in the tile counting group $\mathbb{G}(\mathbf{T})$. Call this image $\mathbb{E}(\mathbf{T}) \subset \mathbb{G}(\mathbf{T})$ a *torsion group*.

Theorem 9.1 $\mathbb{E}(\mathbf{T}) \simeq \overline{\mathbb{O}}(\mathbf{T})/\mathbb{O}(\mathbf{T})$

Proof By definition, $\mathbb{O}(\mathbf{T})$ is the kernel of ν. This implies the resut. □

Now we can show that in case of ribbon tiles most of invariants can not be proved by the extended coloring arguments. We shall give a complete description of the torsion group $\mathbb{E}(\mathbf{T}_n)$ and compare it with the previously computed tile counting group $\mathbb{G}(\mathbf{T}_n) \supset \mathbb{E}(\mathbf{T}_n)$.

Definition 9.2 Let $\mathbf{T}_n$ be a set of all ribbon tiles. A map $f_\blacktriangledown : \mathbf{T}_n \to \mathbb{Z}_n$ defined as

$$f_\blacktriangledown(\varepsilon_1, \dots, \varepsilon_{n-1}) = \sum_{i=1}^{n-1} i \cdot \varepsilon_i \pmod n$$

is called the *shade invariant.*

An easy geometric interpretation of $f_\blacktriangledown$ is given in Figure 9.1. Imagine a wall behind our ribbon tile τ and the light is coming from southeast. Then $f_\blacktriangledown(\tau)$ is equal to the shaded area modulo n.

First, observe that $f_\blacktriangledown$ is a tile invariant. Indeed, if n is odd we have

$$f_\blacktriangledown = f_1 + 2\, f_2 + \dots + m\, f_m \pmod n$$

where f_i is the i-convexity invariant and $n = 2\,m + 1$. Analogously if n is even we have

$$f_\blacktriangledown = f_1 + 2\, f_2 + \dots + (m-1)\, f_{m-1} + f_*|^{\mathbb{Z}_n} \pmod n$$

where $n = 2\,m$ and $g = f_*|^{\mathbb{Z}_n} \mod n$ is a parity invariant lifted to $\mathbb{Z}_{2\,m}$:

$$g(\varepsilon_1, \dots, \varepsilon_{n-1}) = m\varepsilon_m \pmod{2\,m}$$

This gives $f_\blacktriangledown \in \mathbb{G}(\mathbf{T})$. Let us show that $f_\blacktriangledown \in \mathbb{E}(\mathbf{T})$, i.e. that the shade invariant can be obtained by the extended coloring argument. Indeed, consider a coloring map $f : \mathcal{R} \to \mathbb{Z}$, defined by $f(i,j) = i \mod n$. It is easy to see that $\nu(f) = f_\blacktriangledown$ and therefore $f_\blacktriangledown \in \mathbb{E}(\mathbf{T}_n)$. Analogously the area invariant $f_0 = \nu(g) \in \mathbb{E}(\mathbf{T}_n)$, where $g : \mathcal{R} \to \mathbb{Z}$ is a coloring map defined by

$$f(i,j) = \begin{cases} 1\,, & i - j = 0 \mod n \\ 0\,, & i - j \neq 0 \mod n \end{cases}$$

We claim that except for f_0 and $f_\blacktriangledown$ and their linear combinations, no other non-trivial invariant can be obtained by the extended coloring argument.

Theorem 9.3 Let $\mathbf{T}_n$ be a set of all ribbon tiles, f_0 be the are invariant and $f_\blacktriangledown$ be the shade invariant. Then $\mathbb{E}(\mathbf{T}_n) \simeq \mathbb{Z} \times \mathbb{Z}_n$ and maps f_0, $f_\blacktriangledown$ form an independent basis of invariants.

Proof We already showed that maps $f_0, f_\blacktriangledown \in \mathbb{E}(\mathbf{T}_n)$. Since they are independent, together they generate $\mathbb{Z} \times \mathbb{Z}_n$.

In the other direction, recall the computation of the coloring group $\mathbb{O}(\mathbf{T}_n) \simeq \mathbb{Z}^{n-1}$ given in Theorem 8.2. Let us compute $\overline{\mathbb{O}}(\mathbf{T}_n)$. By Theorem 9.1 this is all we need to find $\mathbb{E}(\mathbf{T})$.

Denote $\tau_0 = \mathbf{00} \dots \mathbf{0}$, $\tau_1 = \mathbf{11} \dots \mathbf{1}$ and $\tau_2 = \mathbf{10} \dots \mathbf{0}$. Let us find all coloring maps $f : \mathcal{R} \to \mathfrak{G}$ (see proof of Theorem 8.1), $f(i,j) = x_{i,j}$ such that the sums of squares in translations of τ_{0-2} are constant. Note here that a priori $\nu(f)$ does not have to be a tile invariant since we do not check the relations for other ribbon tiles.

We claim that f is determined by values $x_{1,1}, x_{1,2}, \ldots, , x_{1,n}$ and $x_{2,1}$. Indeed, translations of τ_0 and τ_1 give us $x_{i,j} = x_{i\pm n,j} = x_{i,j\pm n}$. Now, given $x_{i,j}, \ldots x_{i,j+n-2}$ and value $f(\tau_2) = x_{1,1} + \cdots + x_{1,n-1} + x_{2,1}$ we get $x_{i+1,j} = f(\tau_2) - x_{i,j} - \cdots - x_{i,j+n-2}$. Therefore given $x_{1,1}, \ldots, , x_{1,n}$ and $x_{2,1}$ we first determine $x_{1,j}$, for all $j \in \mathbb{Z}$, then $x_{2,j}$, then $x_{3,j}$, etc. For the negative rows use $x_{i,j} = x_{i+n,j}$. This proves the claim.

Now, by taking a quotient $\overline{\mathbb{O}}(\mathbf{T}_n)/\mathbb{O}(\mathbf{T}_n)$ we can make zero all values $x_{1,1}$, $x_{1,2}, \ldots, , x_{1,n-1}$ (see proof of Theorem 8.2). Let $x_{1,n} = a$, $x_{2,1} = a + z$. The computations above give us $x_{i+1,j} = a + i\,z$ and $n\,z = 0$. Therefore $f = a \cdot f_0 + z \cdot f_\blacktriangledown$ and f_0, $f_\blacktriangledown$ generate the whole torsion group $\mathbb{E}(T_n)$. $\square$

As a corollary from Theorem 9.3 we immediately get Theorem 1.8. As it was noted in the introduction, another way to prove Theorem 1.8 would be to find a tileable region $\Gamma \in \mathcal{R}_{\mathbf{T}_n} \setminus \mathcal{R}_{rc}$, $n \geq 3$ such that $f_\bullet$ is not constant on the set $\mathcal{S}(\Gamma, \mathbf{T}_n)$ of ribbon tilings. When $n = 3$ one such example is shown in Figure 9.2. The value of $f_\bullet$ is 1 on first tiling and 0 on the second tiling. Note also such a region Γ probably must have at least one hole inside (see Conjecture 10.1 in the next section).

10. Conclusion

Let us summarize what we did in the paper and compare with what is yet to be done.

The main result of the paper is a description of the tile counting group for a set of ribbon tiles. Note, however that we only considered row convex or column convex regions (which probably include all the interesting one). However, Conway and Lagarias in [CL] were able to prove that for $n = 3$ the map f_1 is an invariant for all simply connected regions. All the available evidence points to the following conjecture.

Conjecture 10.1 The i-convexity and parity maps are the group invariants for all simply connected regions.

The major midpoint of our proof is Theorem 1.6 which claims that there is a finite set of moves which can change any tiling of a given Young shape region to any other tiling. This result has been generalized by the author for any skew shape (see [Pa]). In fact we believe in the following conjecture.

Conjecture 10.2 Let Γ be any simply connected region tileable by $\mathbf{T}_n$. Then graph $\Theta_n(\Gamma)$ is connected.

In other words we claim that the moves defined in section 3 suffice. Of course, the results of section 4 imply that Conjecture 10.1 follows from Conjecture 10.2.

Let us move now to other sets of tiles. Unfortunately it is not always true that there always exist a finite number of moves or local replacement rules as they are also called. For example, let $\mathbf{T} \supset \mathbf{T}_3$ be a set of all trominoes (see [G], Figure 10.1). There are infinitely many regions with exactly two tilings that are not local in any sense (see Figure 10.2). Even though there are no finite number of local moves, there still can be some invariants other than the area. Here is an example.

Let $\mathbf{T}$ be a set of four trominoes $\tau_1, \dots, \tau_4$ in Figure 10.1. Let $\mathcal{B} = \mathcal{R}_{\mathbf{T}}$ be a set of all $\mathbf{T}$-tileable regions. Define maps $f_{1,2}, f_{2,3} : \mathbf{T} \to \mathbb{Z}_3$ as follows:

$$f_{1,2}(\tau_1) = f_{1,2}(\tau_2) = 1, \;\; f_{1,2}(\tau_3) = f_{1,2}(\tau_4) = 0$$

$$f_{2,3}(\tau_2) = f_{2,3}(\tau_3) = 1, \;\; f_{2,3}(\tau_1) = f_{2,3}(\tau_4) = 0$$

Theorem 10.3 Let $\mathbf{T}$, $\mathcal{B} = \mathcal{R}_{\mathbf{T}}$ be as above, f_0 be an area invariant. Then tile counting group

$$\mathbb{G}(\mathbf{T}, \mathcal{B}) \simeq \mathbb{Z} \times \mathbb{Z}_3^2$$

and the maps f_0, $f_{1,2}$, $f_{2,3}$ form an independent basis of invariants.

Theorem 10.3 basically claims that there is one nontrivial tile invariant $f_{1,2}$ which can be stated as follows:

- For any convex region Γ the number of times *modulo 3* the tiles τ_1 and τ_2 occur in a tiling of Γ depends only on Γ.

Theorem 10.3 also claims that rotations of $f_{1,2}$ and the area invariant generate the whole tile invariant group.

Proof of Theorem 10.3 The idea is similar to the one used in section 6. By definition $\mathbb{G}(\mathbf{T}; \mathcal{B}) = \mathbb{Z}^{\mathbf{T}}/I$, where I is a linear span or relations obtained from different tilings of the same region. Thus having enough many relations that generate $I' \subset I$ will imply that $\mathbb{G}(\mathbf{T}; \mathcal{B}) \subset \mathbb{Z}^{\mathbf{T}}/I'$. In this particular case two types of relations will suffice.

First, there is a relation obtained from two tilings of $[2 \times 3]$ (see Figure 10.3):

$$\tau_1 + \tau_3 = \tau_2 + \tau_4$$

Then there is another relation which comes from two tilings in Figure 10.4:

$$4 \cdot \tau_4 + \tau_2 = 4 \cdot \tau_3 + \tau_1$$

In combination with the first relation and rotations this gives $3 \cdot \tau_1 = \dots = 3 \cdot \tau_4$. Simple further computations show that maps f_0, $f_{1,2}$, $f_{2,3}$ are independent and generate the whole tile counting group $\mathbb{G}(\mathbf{T}, \mathcal{R}_{\mathbf{T}})$. Therefore we have $\mathbb{G}(\mathbf{T}, \mathcal{R}_{\mathbf{T}}) \subset \mathbb{Z} \times \mathbb{Z}_3^2$.

Now, we have yet to prove that $f_{1,2}$ is indeed an invariant. This in consequence would imply that $f_{1,2}$ is an invariant and prove the theorem. This can be done by the tile extended coloring argument.

Let $g : \mathcal{R} \to \mathbb{Z}_3$ be a coloring map defined by $g(i, j) = j - i \pmod 3$. Observe that $g \in \overline{\mathbb{O}}(\mathbf{T})$. Compute the corresponding tile invariant $f = \nu(g) : \mathcal{R}_{\mathbf{T}} \to \mathbb{Z}_3$. We have

$$f(\tau_1) = f(\tau_3) = 0, \;\; f(\tau_2) = -1, \;\; f(\tau_2) = 1 \quad \mod 3$$

By the symmetry, we also have another tile invariant $f' : \mathcal{R}_{\mathbf{T}} \to \mathbb{Z}_3$ given by

$$f'(\tau_2) = f'(\tau_4) = 0, f'(\tau_1) = -1,\ f'(\tau_3) = 1 \mod 3$$

From here we have

$$f_{1,2} = f + f' - f_0 \mod 3$$

Therefore both $f_{1,2}$ and $f_{2,3}$ are $\mathbf{T}$-invariants. This finishes the proof. □

Note that in the proof of Theorem we used nothing but the coloring arguments. Of course, with a smaller set of tileable regions, when the tile counting group becomes bigger there is no hope for such a luck. That was the case with ribbon tiles. Indeed, in section 9 we showed that the height invariant can not be extended to set of all tileable regions, so the tile counting group $\mathbb{G}(\mathbf{T}_n, \mathcal{R}_{\mathbf{T}_n}) \subsetneq \mathbb{G}(\mathbf{T}_n, \mathcal{B}_{rc})$. Thus finding a tile counting group is probably hard in a general case unless all invariants follow from the extended coloring arguments.

Even in the case of all $\mathbf{T}$-tileable regions it is still possible to have invariants which do not follow from any extended coloring arguments. Indeed, consider a set $\mathbf{T}$ of two tiles shown in Figure 10.5. It is easy to see that *any* region is either untileable or has a unique tiling. On the other hand extended coloring arguments can prove only the area invariant. Note that in this case every region can has a sign tiling.

We believe that having $\mathbb{E}(\mathbf{T}) \simeq \mathbb{G}(\mathbf{T}, \mathcal{R}_{\mathbf{T}})$, is a rather rare event. However, that it might occur for certain nice sets of tiles such as finite set of rectangles. Withought making a precise conjecture, let us state the following problem.

Problem 10.4 Let $\mathbf{T}$ be a finite set of rectangles, and let $\mathcal{B}$ be set of convex regions tileable by $\mathbf{T}$. Find tile invariant group $\mathbb{G}(\mathbf{T}; \mathcal{B})$ and torsion group $\mathbb{E}(\mathbf{T})$.

For example, let $\mathbf{T}$ consist of two rectangles $[2 \times 1]$ and $[1 \times 3]$. It is not hard to see that $\mathbb{G}(\mathbf{T}; \mathcal{B}) \simeq \mathbb{E}(\mathbf{T}) \simeq \mathbb{Z} \times \mathbb{Z}_5$. There is some recent literature relevant to the problem (see [Ke] for details).

In this paper we always considered tiles to be identical if they can be obtained by translation. Following some recent literature (see e.g. [Pr]) one can try to distinguish between ribbon tiles with different starting points. It seems to us that at least theoretically the whole analysis of the paper can be generalized for this case, though some computations may become messier. We challenge the reader to find tileability applications of this generalizations.

Our proof of the main theorem was based on an ad hoc method which probably cannot be generalized in full for other sets of tiles. The heart of the proof is the rim hook bijection. There are shifted, tree, and skew analogs of this bijection (see [FS, Pa], but they all can be reduced to the original bijection in one way or another.

Problem 10.5 Find a three-dimensional analog of the rim hook bijection.

Of course, there are infinitely many open questions and problems, but the reader is probably too tired already to be bothered by whatever is left.

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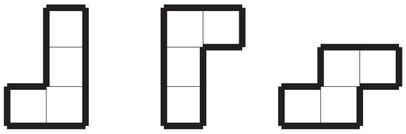

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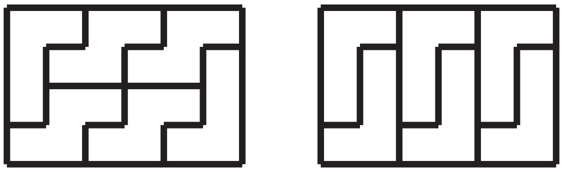

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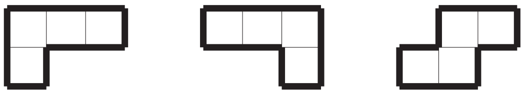

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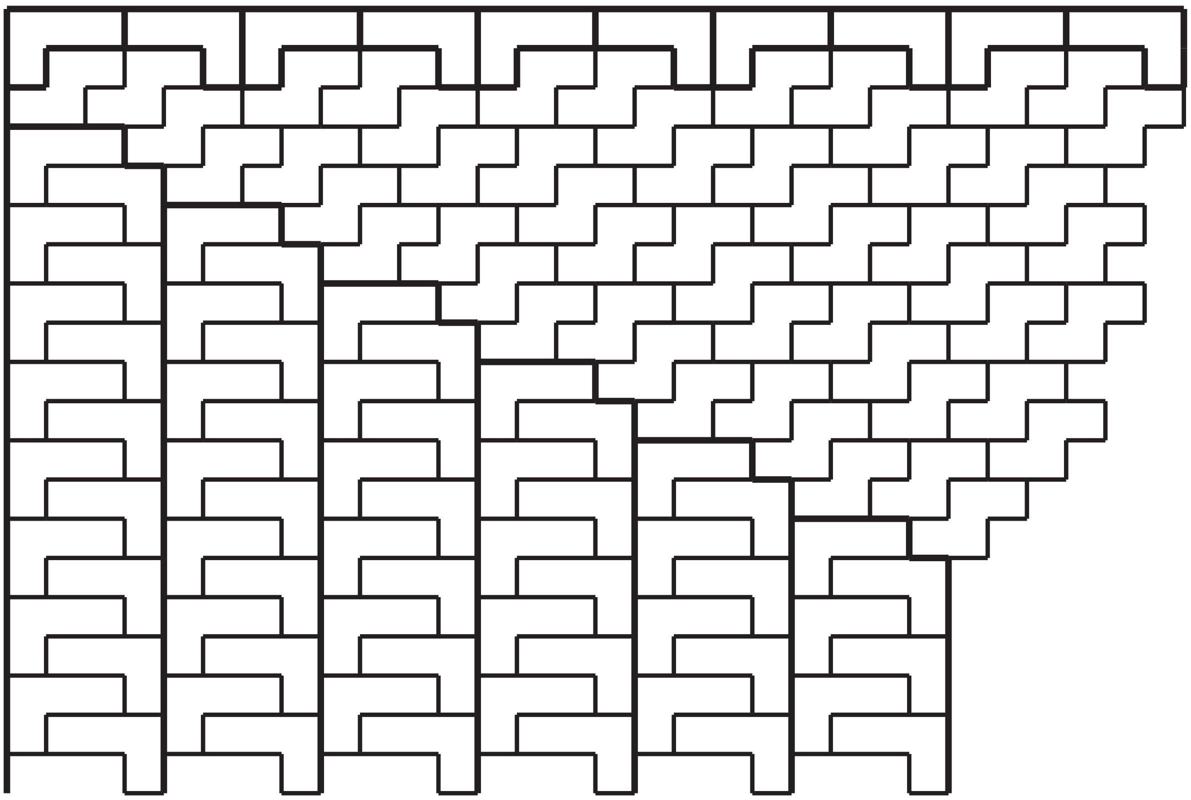

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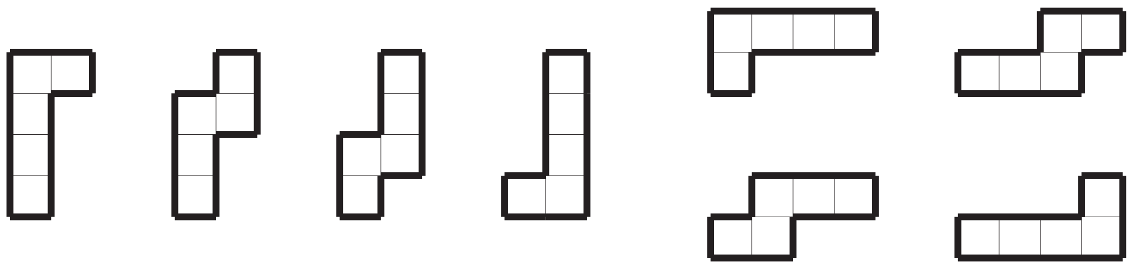

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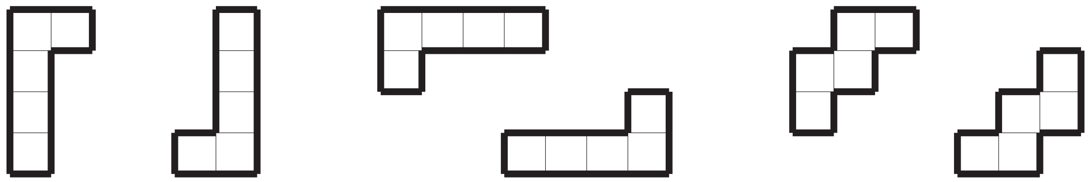

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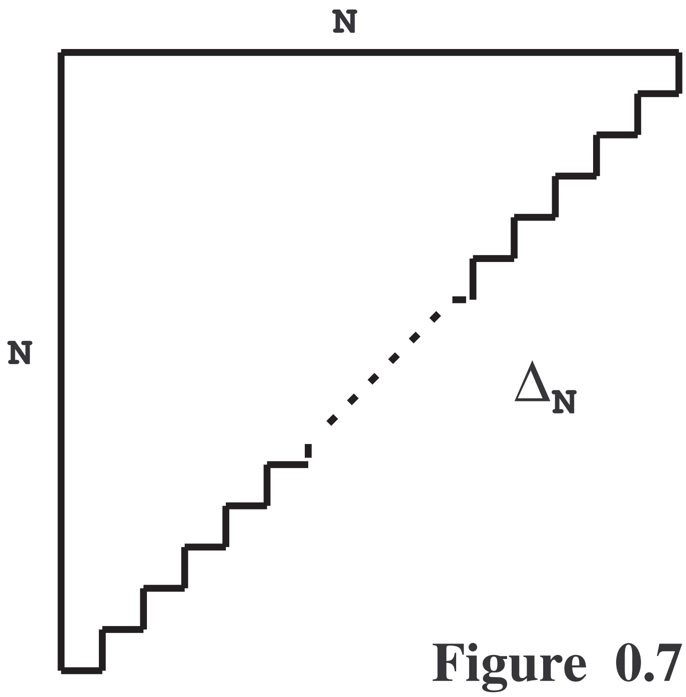

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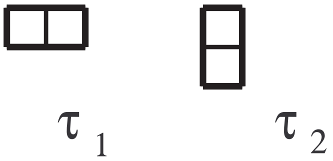

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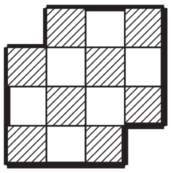

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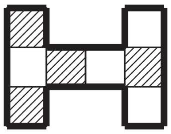

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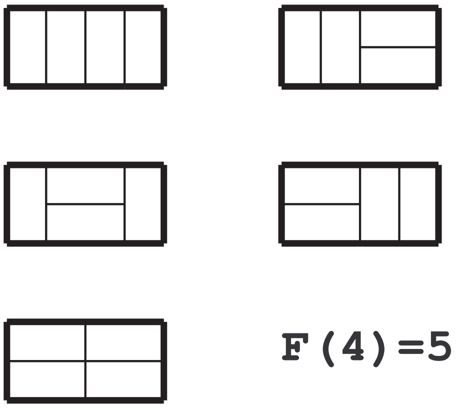

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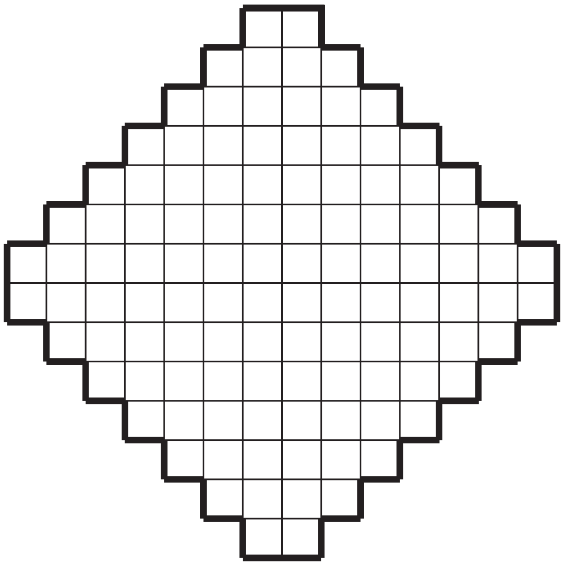

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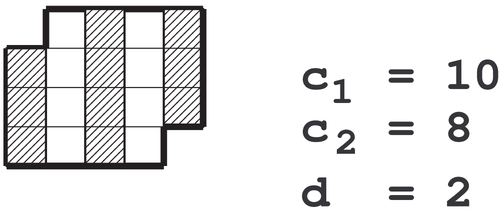

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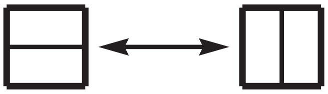

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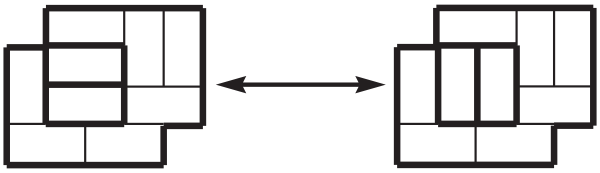

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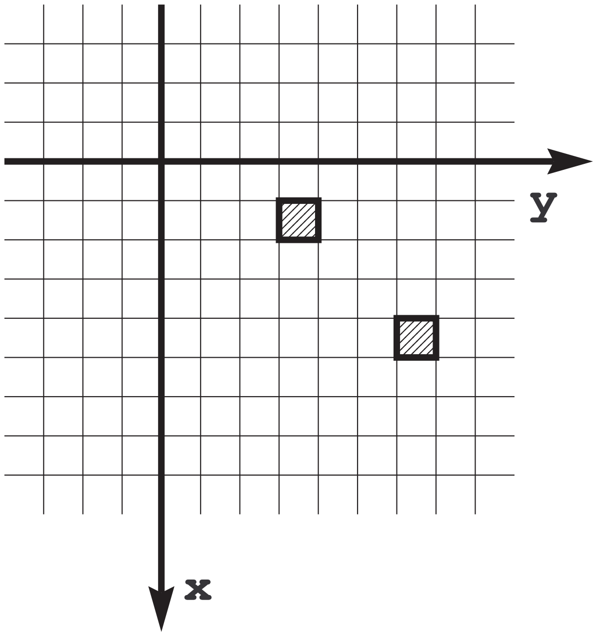

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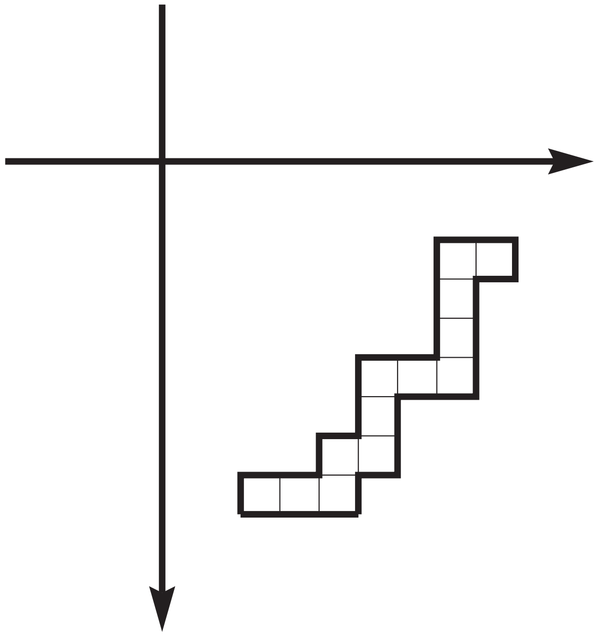

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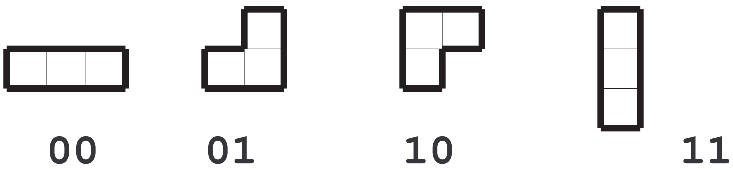

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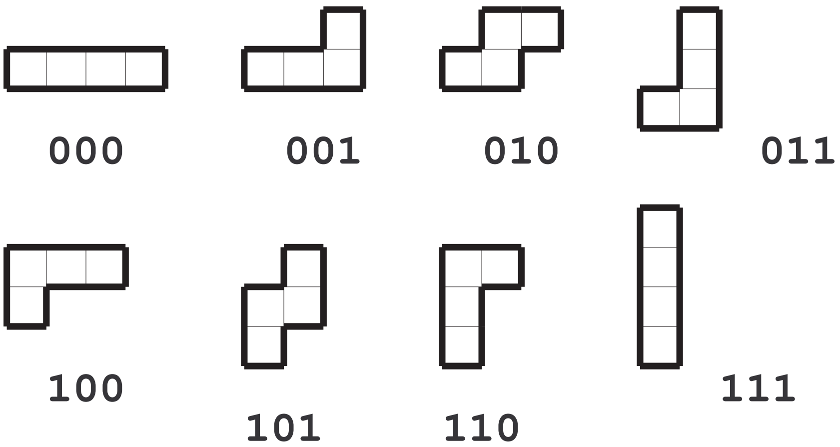

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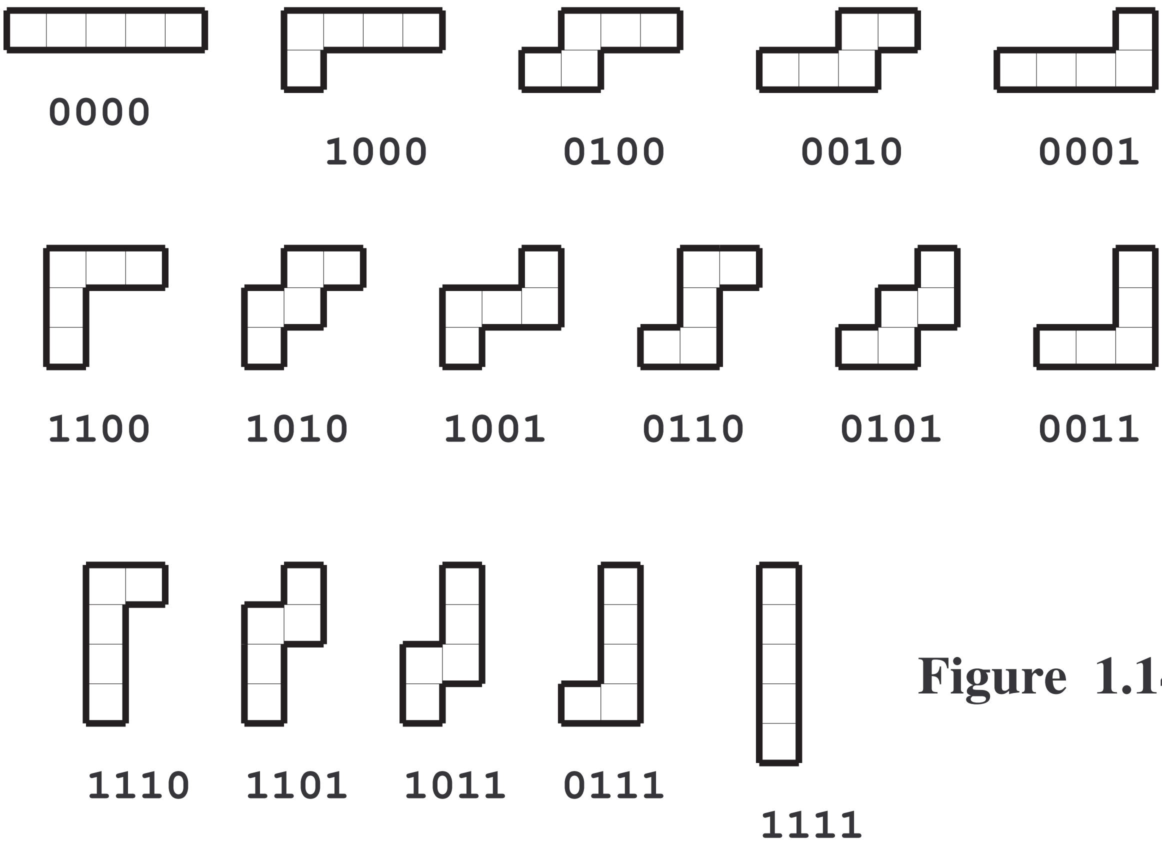

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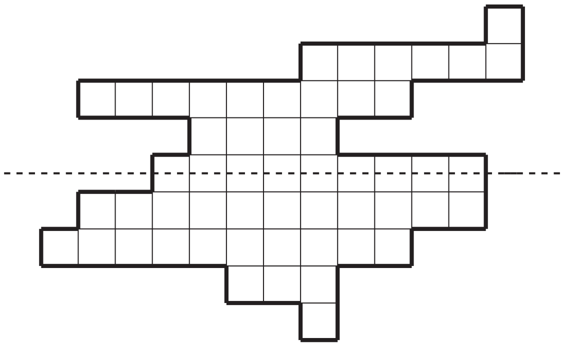

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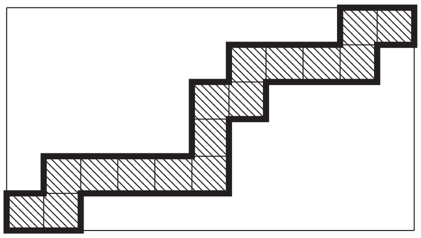

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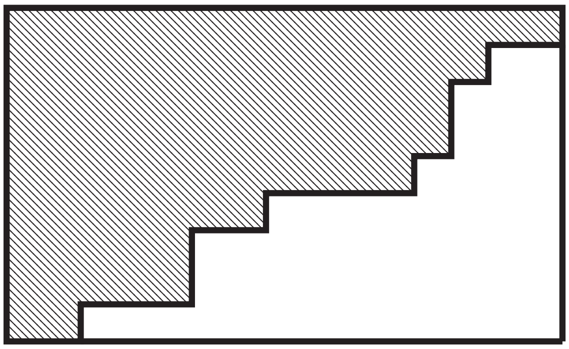

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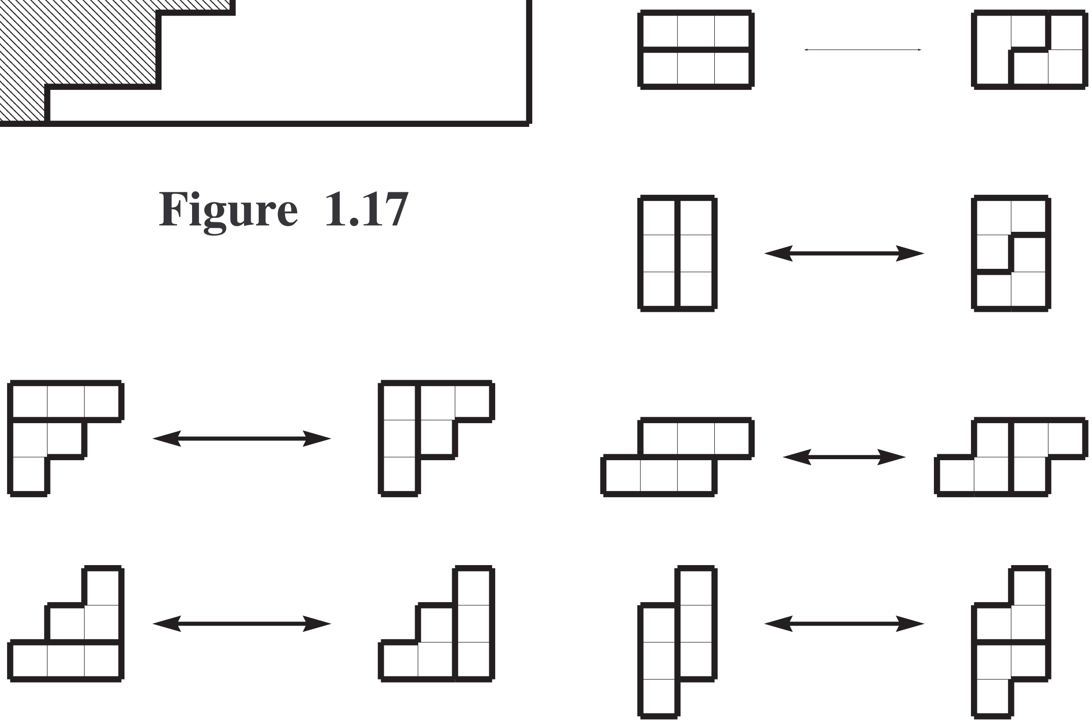

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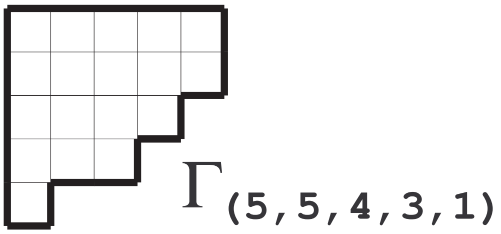

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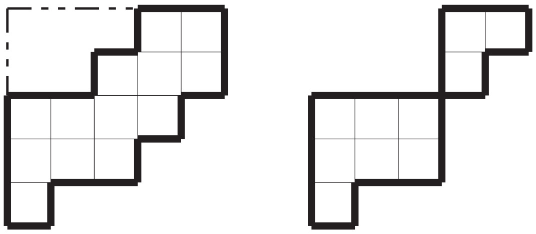

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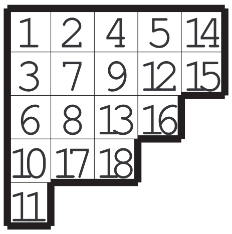

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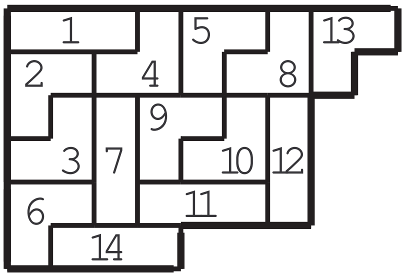

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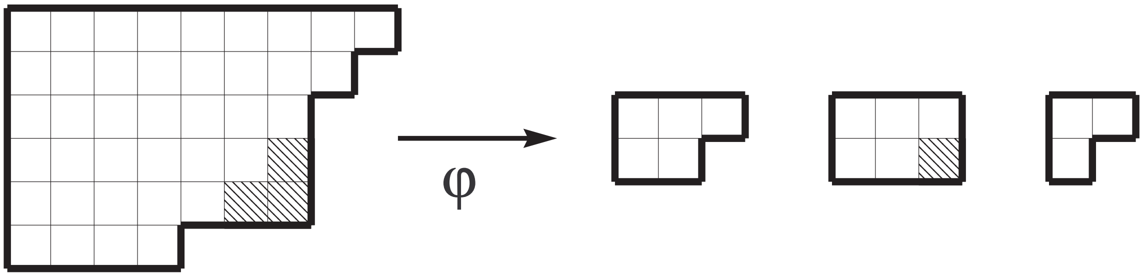

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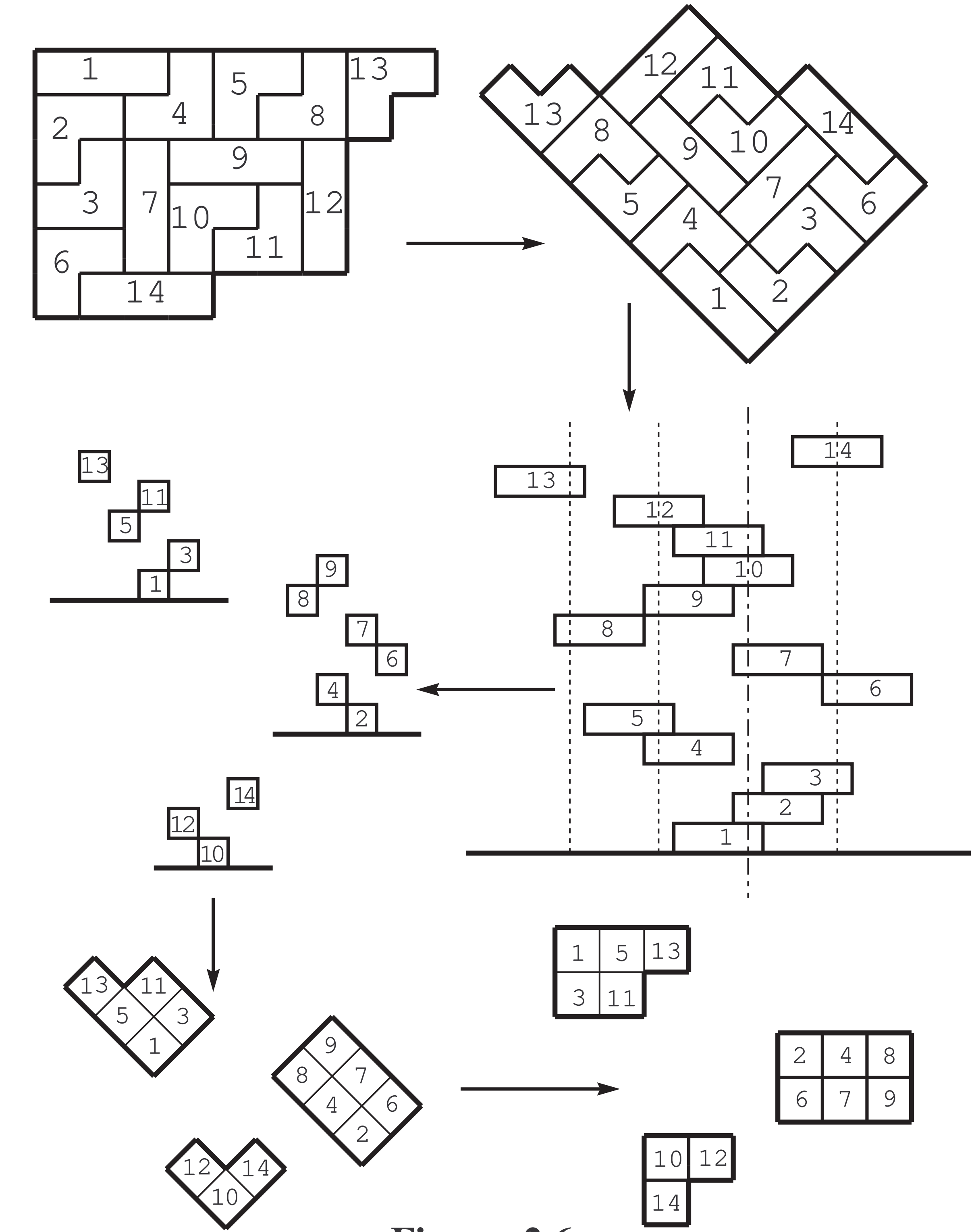

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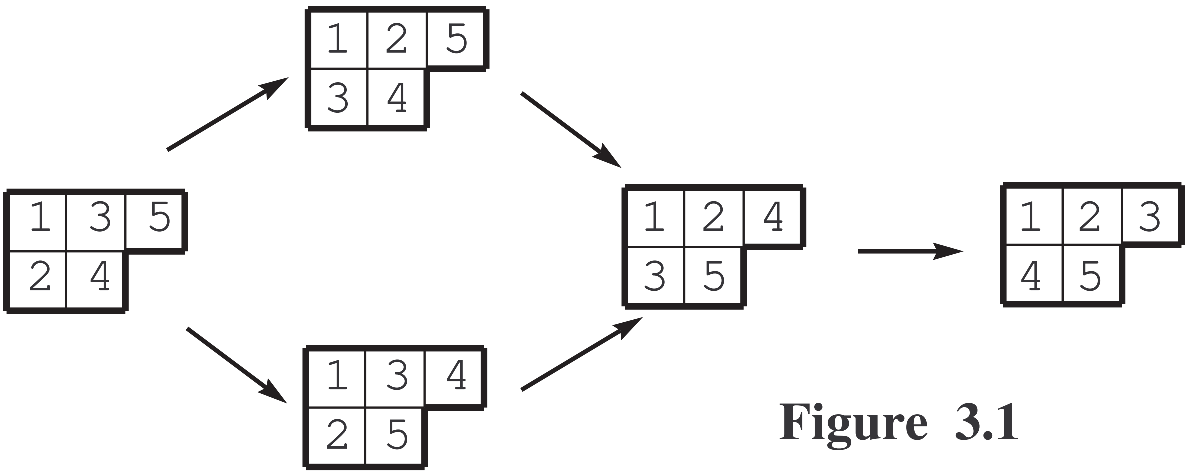

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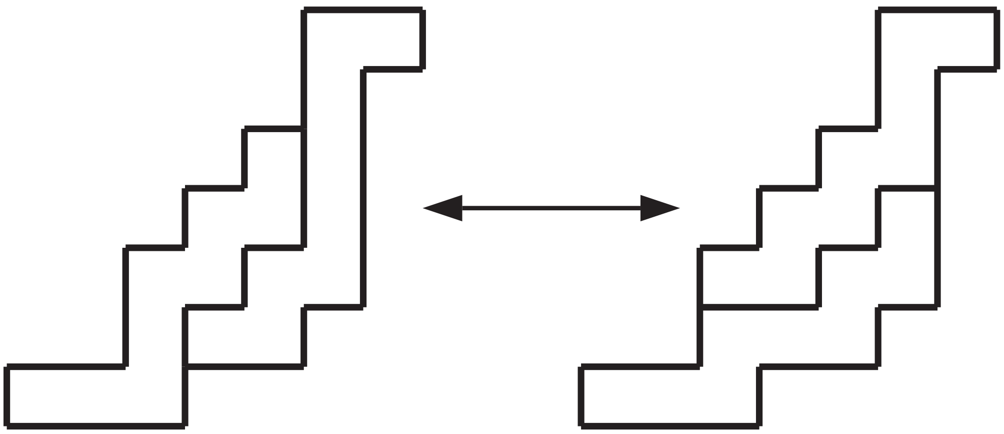

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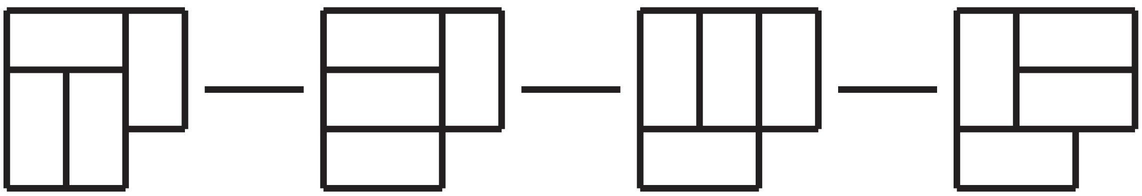

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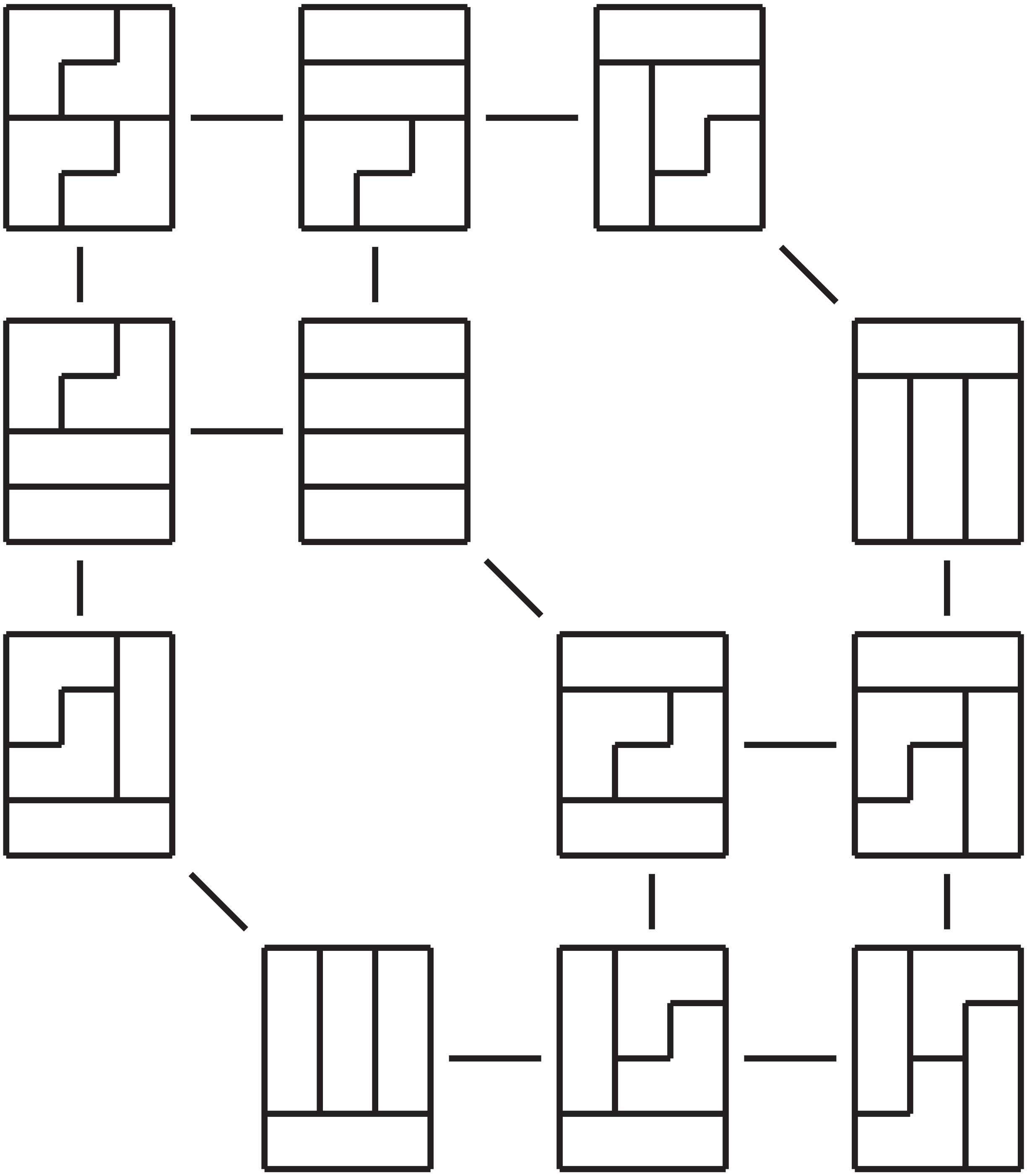

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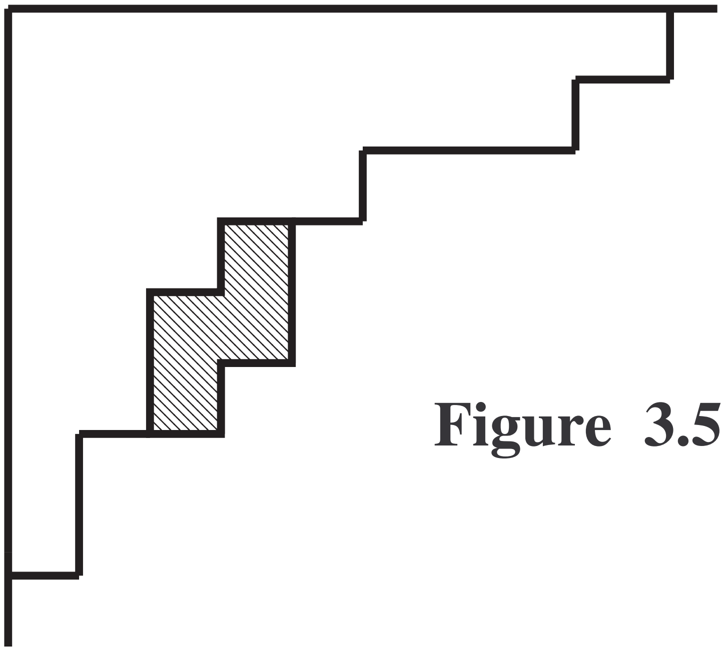

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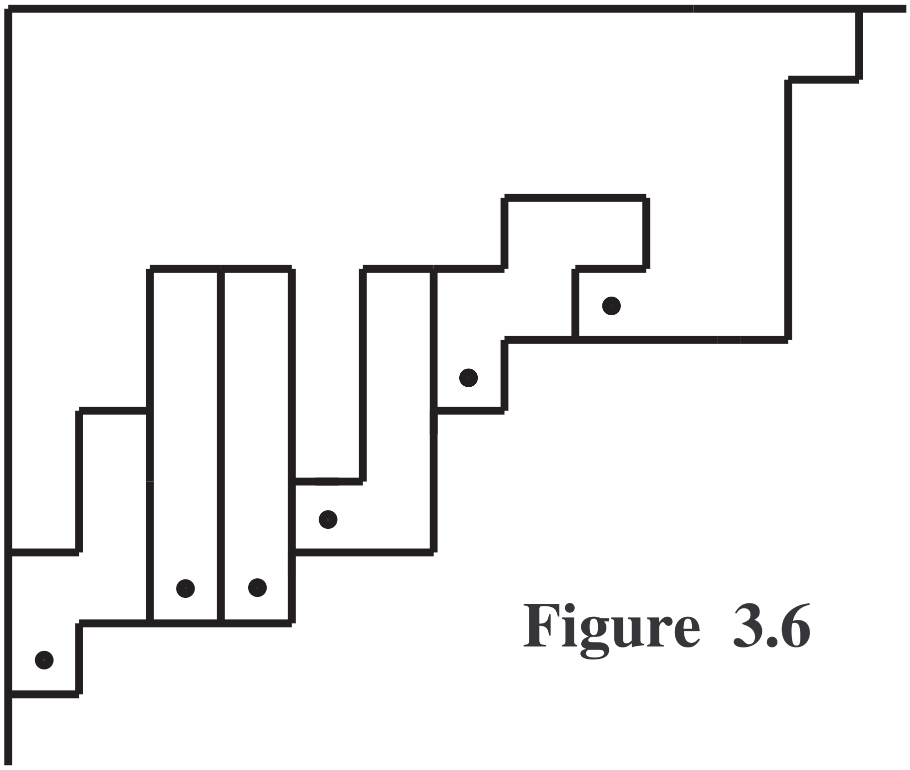

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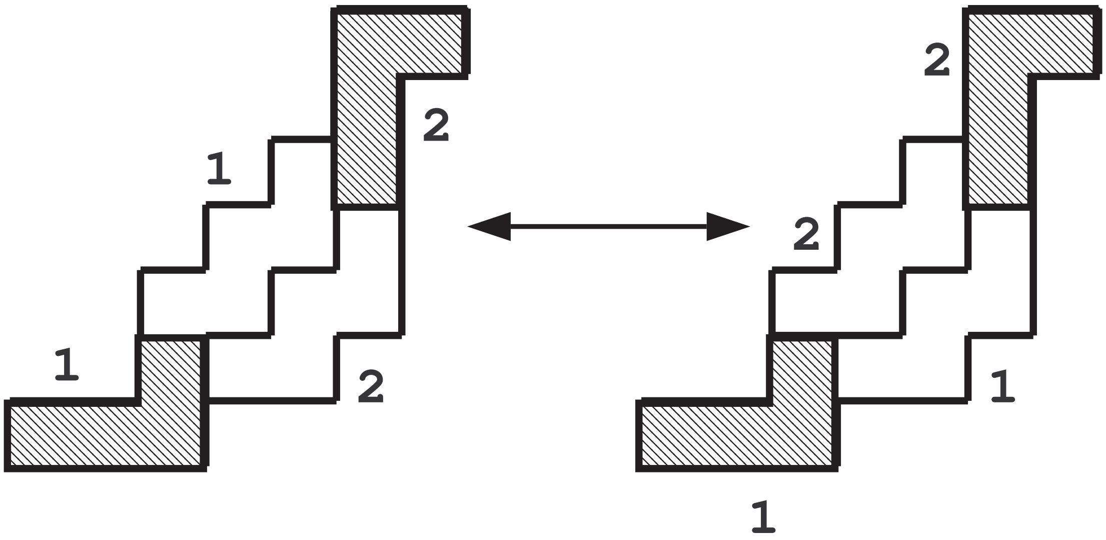

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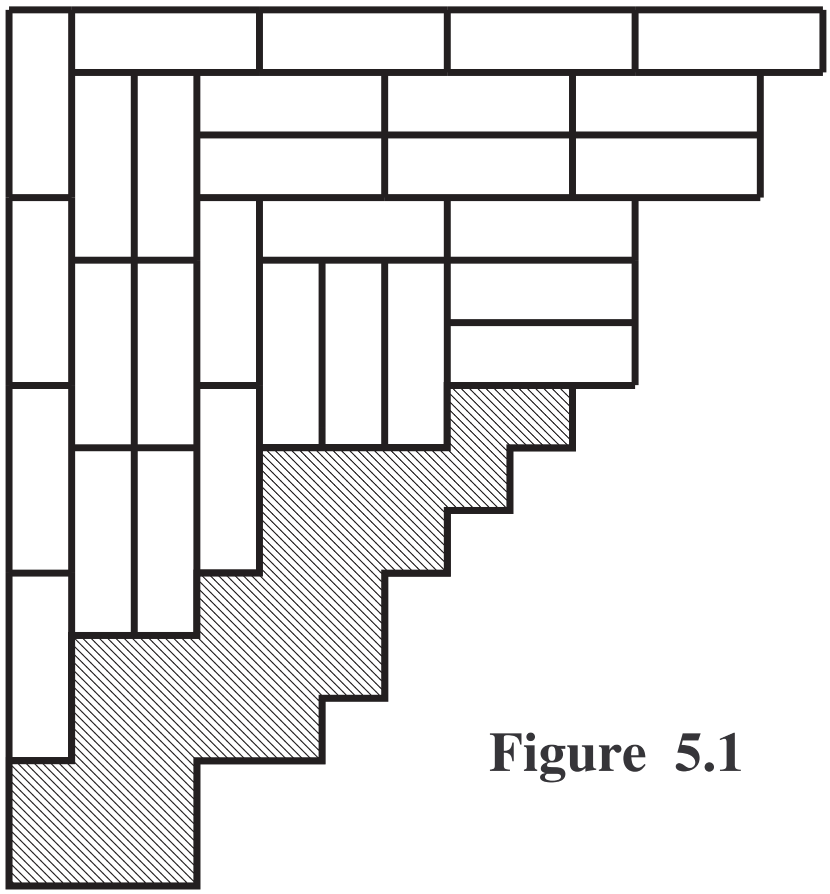

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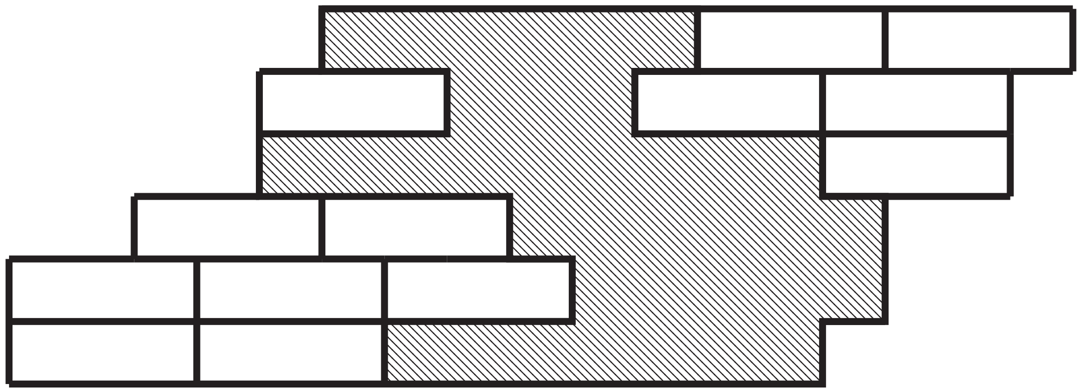

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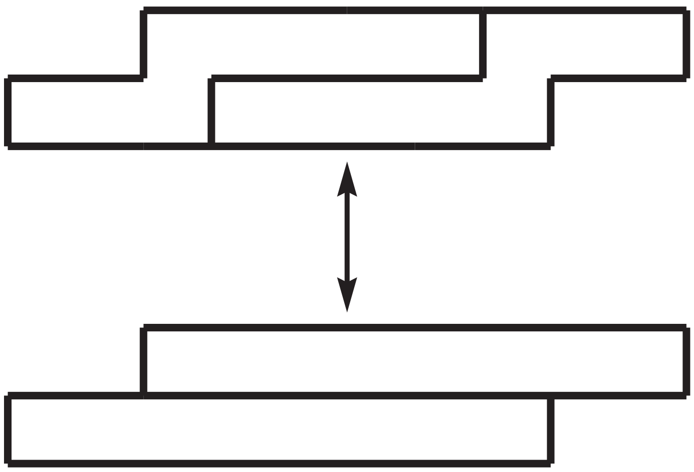

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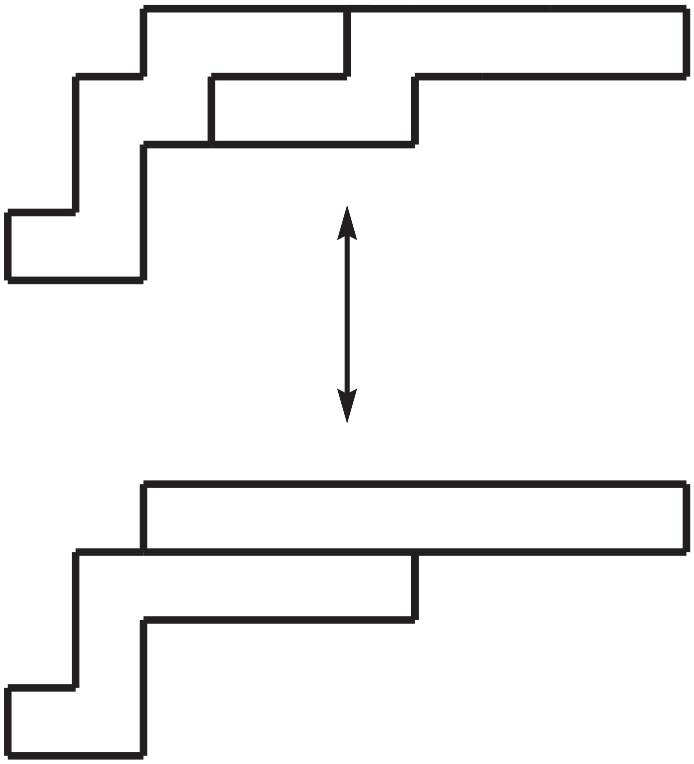

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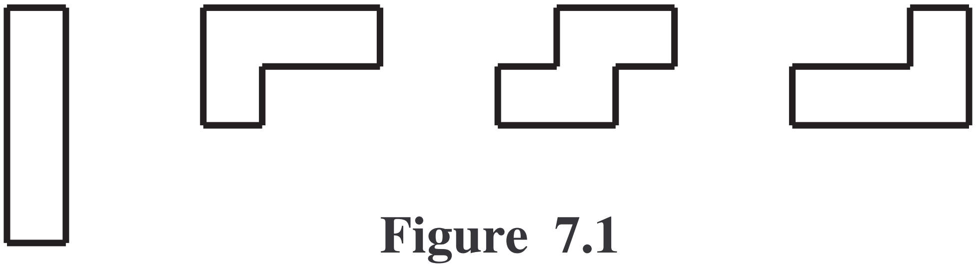

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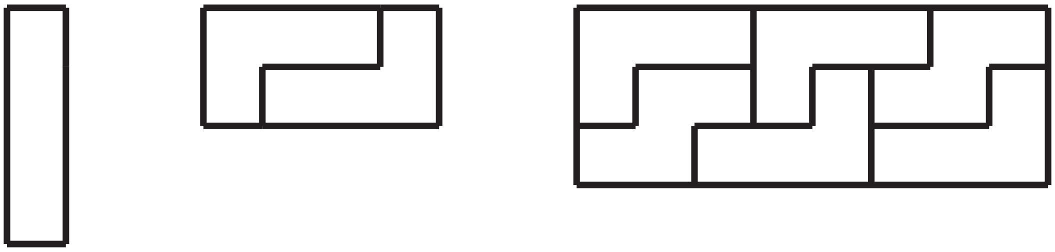

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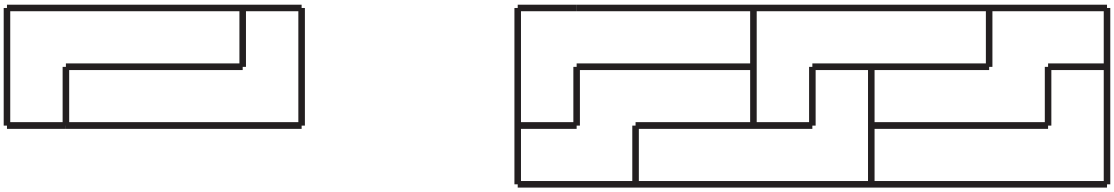

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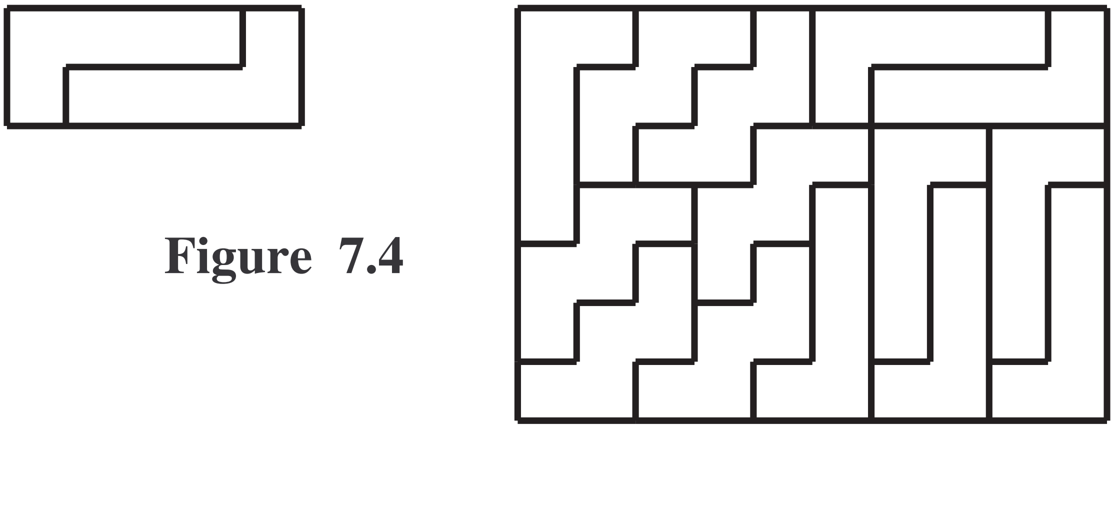

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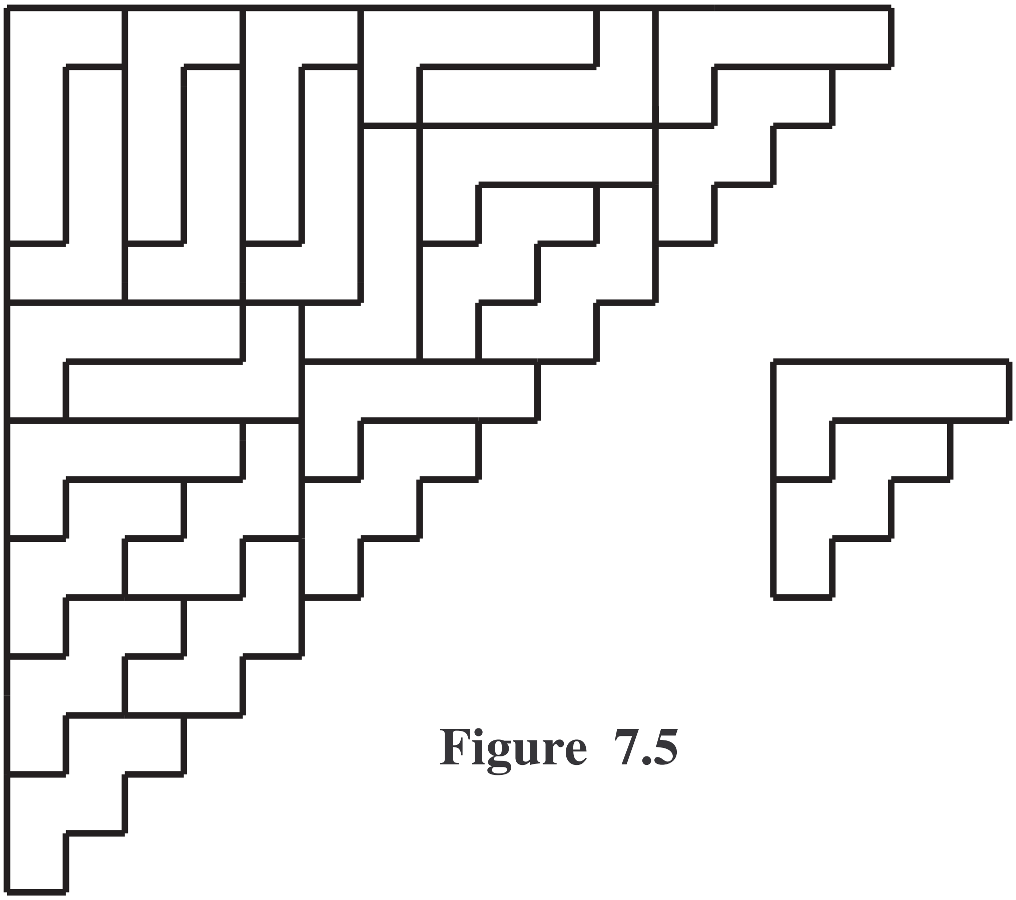

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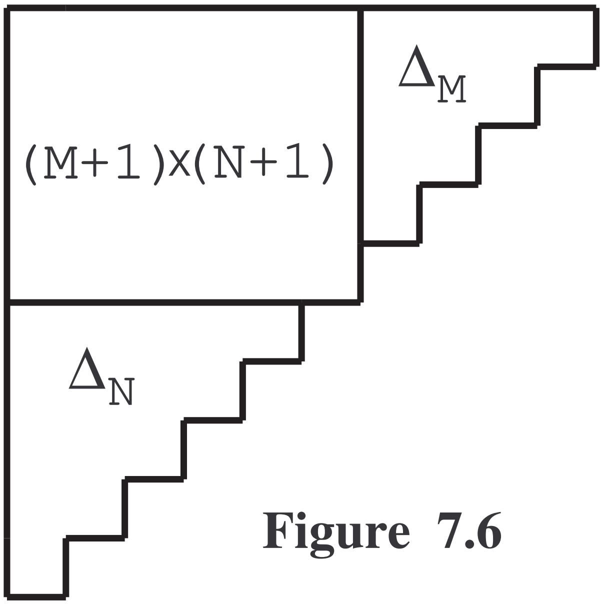

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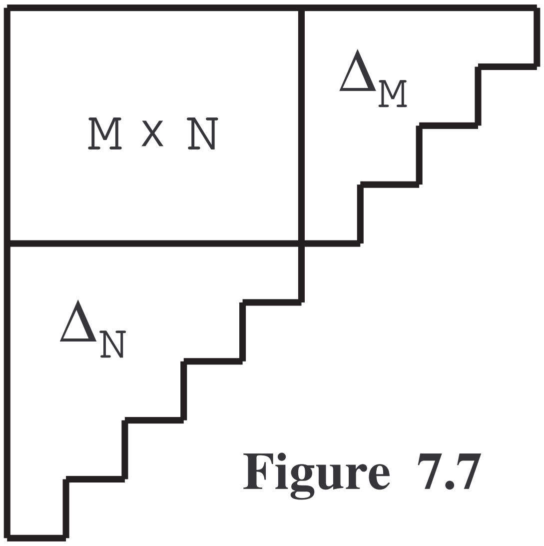
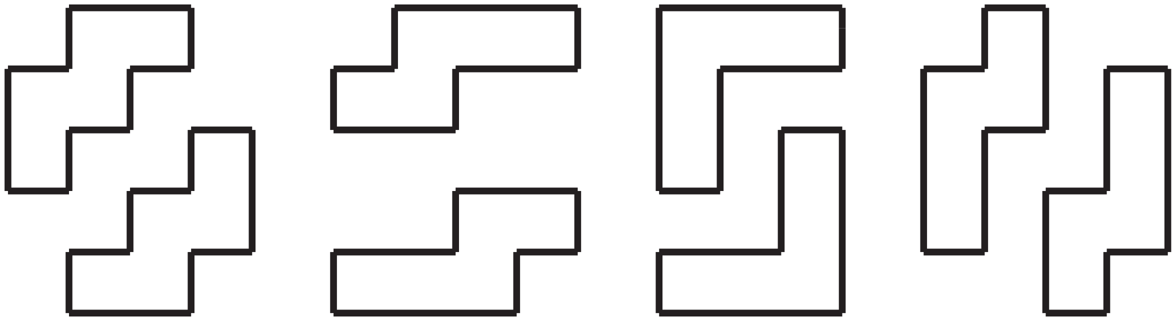


Figure 7.7

Figure 7.8



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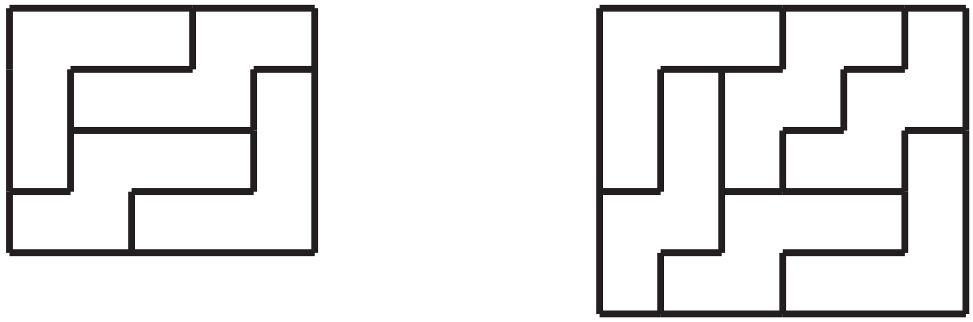

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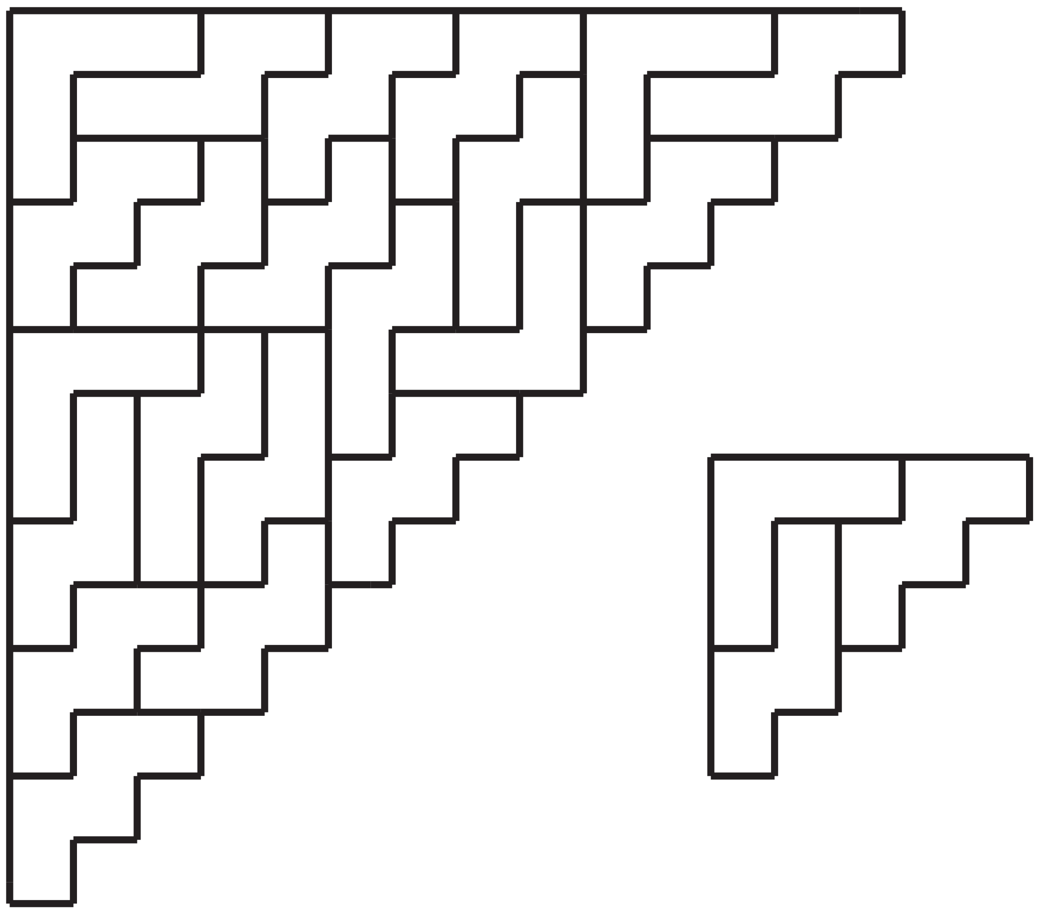

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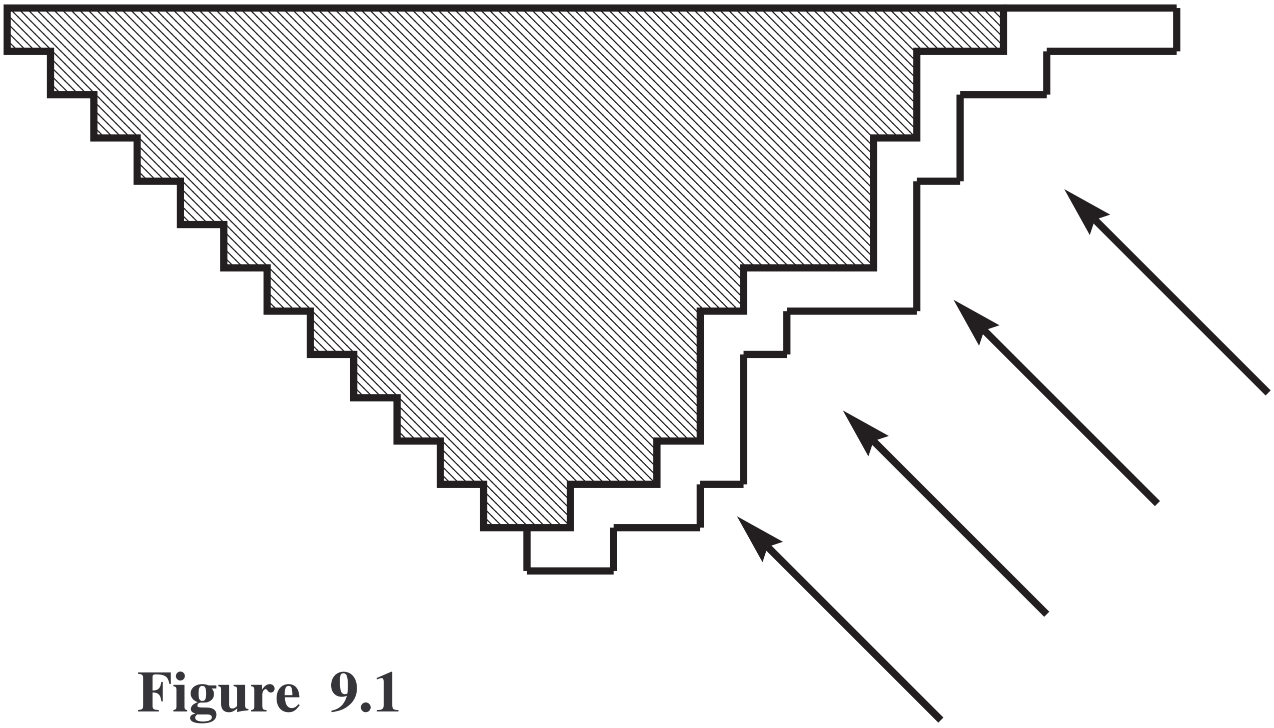

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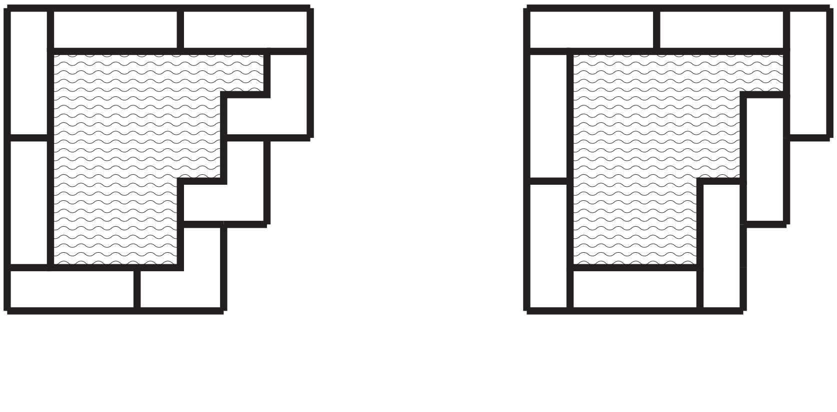

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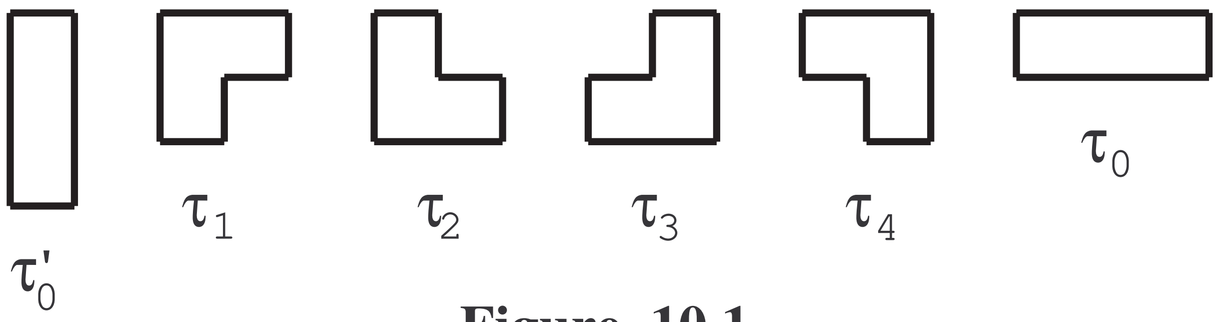

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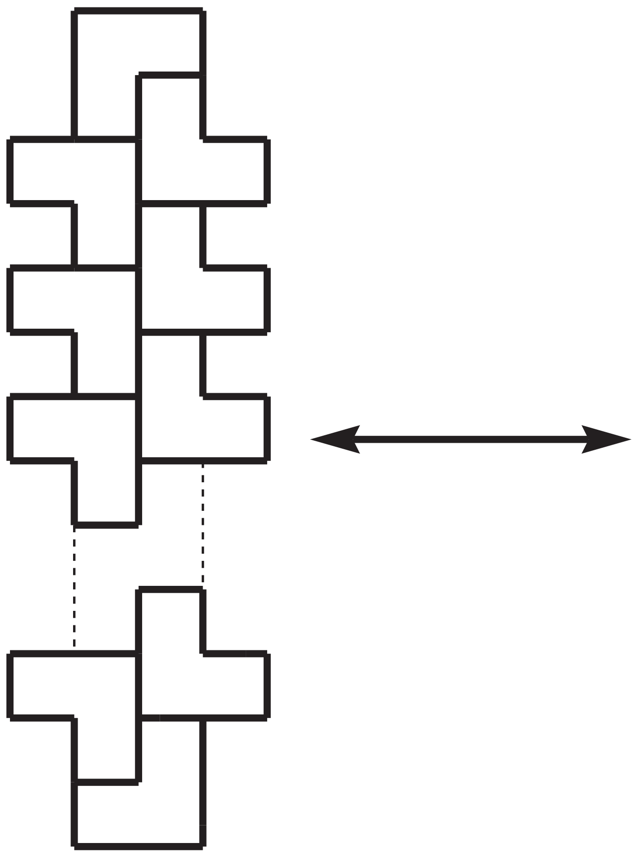

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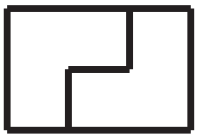

Figure 10.3

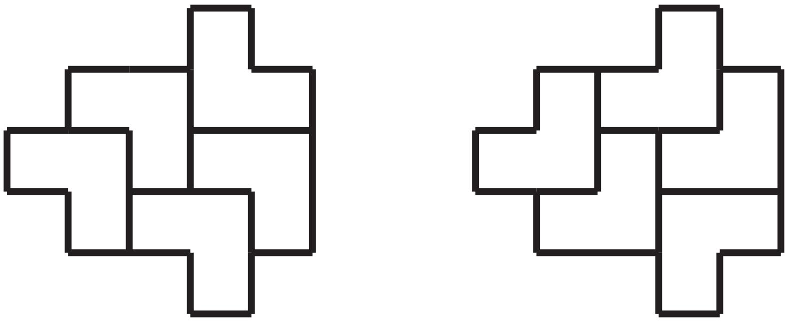

Figure 10.4

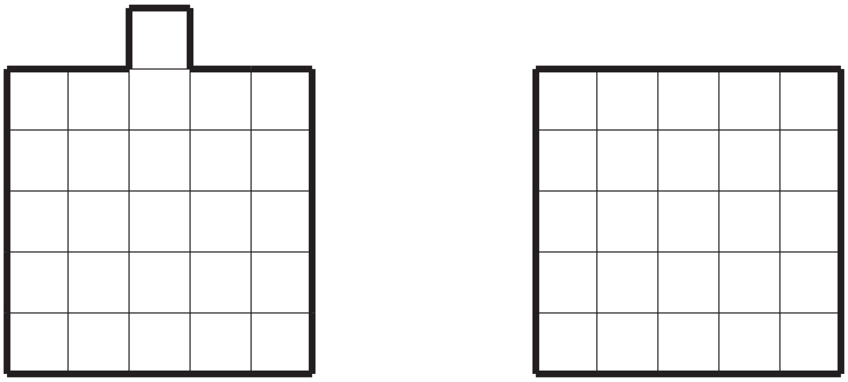

Figure 10.5