A DUALITY OF THE TWISTED GROUP ALGEBRA OF THE SYMMETRIC GROUP AND A LIE SUPERALGEBRA

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Abstract. We construct a duality relation between the twisted group algebra $\mathcal{A}_k$ (cf. (1.5)) of the symmetric group $\mathfrak{S}_k$ and a Lie superalgebra $\mathfrak{q}(n)$ (cf. §2). This is an analogue of Schur-Weyl duality between $\mathfrak{S}_k$ and the general linear group $GL(n)$.

Schur-Weyl duality explains Frobenius' formula for the irreducible characters of $\mathfrak{S}_k$. In a method analogous to Frobenius' description, I. Schur introduces the Q-functions in order to describe the characters of the irreducible representations of $\mathcal{A}_k$. Our duality relation explains Schur's formula.

First, we construct an isomorphism from $\mathcal{C}_k\hat{\otimes}\mathcal{A}_k$ to a twisted group algebra $\mathcal{B}_k$ (cf. (1.6)) of the hyperoctahedral group H_k (cf. §1, **A**), where $\mathcal{C}_k$ denotes the 2^k-dimensional Clifford algebra and $\hat{\otimes}$ denotes the **super tensor product** of **super algebras** (cf. §1, **D**), and give a simple relation between the irreducible representations of $\mathcal{A}_k$ and $\mathcal{B}_k$. Secondly, restricting Sergeev's duality [Se] (cf. §2, Theorem 2.1) between $\mathcal{B}_k$ and $\mathfrak{q}(n)$, we obtain a duality relation between $\mathcal{A}_k$ and $\mathfrak{q}(n)$.

Résumé. Nous construisons une dualité entre l'algèbre de groupe tordue $\mathcal{A}_k$ (cf. (1.5)) du groupe symétrique $\mathfrak{S}_k$ et une super-algèbre de Lie $\mathfrak{q}(n)$ (cf. §2). Ce résultat est un analogue de la dualité de Schur-Weyl entre $\mathfrak{S}_k$ et le groupe linéaire général $GL(n)$.

La dualité de Schur-Weyl explique la formule de F. G. Frobenius [Fr] pour les caractères irréductibles de $\mathfrak{S}_k$. Avec une méthode analogue à la description de Frobenius, I. Schur introduit les Q-fonctions pour décrire les caractères des représentations irréductibles de $\mathcal{A}_k$. Notre dualité explique la formule de Schur.

En premier lieu, nous construisons un isomorphisme de $\mathcal{C}_k\hat{\otimes}\mathcal{A}_k$ à une algèbre de groupe tordue $\mathcal{B}_k$ (cf. (1.6)) du groupe de l'hyperoctahèdre H_k (cf. §1, **A**), où $\mathcal{C}_k$ dénote l'algèbre de Clifford de dimension 2^k, et $\hat{\otimes}$ dénote le **super-produit tensoriel** des **super-algèbres**, (cf. §1, **D**), et donnons une relation simple entre les repésentations irréductibles de $\mathcal{A}_k$ et $\mathcal{B}_k$. Ensuite, en restreignant la dualité de A. N. Sergeev [Se] (cf. §2, Theorem 2.1) entre $\mathcal{B}_k$ et $\mathfrak{q}(n)$, nous obtenons une dualité entre $\mathcal{A}_k$ et $\mathfrak{q}(n)$.

1. Introduction

All vector spaces, and associative algebras, and representations in this paper assume to be finite dimensional and over complex field $\mathbb{C}$.

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A. Preliminaries. The hyperoctahedral group H_k is the group generated by t and the s_i with $1 \leq i \leq k-1$ subject to relations

$$
\begin{aligned}
(1.1) \qquad & t^2 = s_i^2 = 1 \quad (1 \leq i \leq k-1), \\
& (s_i s_{i+1})^3 = 1 \quad (1 \leq i \leq k-2), \quad (s_i s_j)^2 = 1 \quad (|i-j| \geq 2), \\
& (ts_i)^2 = 1 \quad (2 \leq i \leq k-1), \quad (ts_1)^4 = 1.
\end{aligned}
$$

The subgroup of H_k generated by the s_i with $1 \leq i \leq k-1$ is isomorphic to the symmetric group of degree k, which denoted by $\mathfrak{S}_k$.

Let P_k denote the set of all partitions of k. Let DP_k and OP_k denote the distinct and odd partitions of k. Let $DP_k^{\pm}$ be the sets of all $\lambda \in DP_k$ such that $(-1)^{k-l(\lambda)} = \pm 1$, respectively. Note that these notations, DP and OP, are used by J. R. Stembridge in [St1, 2].

B. Projective representations and twisted group algebras of finite groups. Let G be a finite group. A map $R\colon G \to GL(V)$ is called a **projective representation** of G if $R(x)R(y) = \alpha(x,y)R(xy)$ for all x, $y \in G$, where α is a function from $G \times G$ to $\mathbb{C}^\times$. The associative law of multiplication in $GL(V)$ implies

$$
(1.2) \qquad \alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z)
$$

for all x, y, $z \in G$. A function $\alpha\colon G \times G \to \mathbb{C}$ is called **2-cocycle** on G with values in $\mathbb{C}^\times$ if it satisfies the identity (1.2) for all x, y, $z \in G$.

Let $Z^2(G,\mathbb{C}^\times)$ denote the abelian group generated by all 2-cocycles of G with values in $\mathbb{C}^\times$. A 2-cocycle α of G is called **trivial** if there exists a function $c\colon G \to \mathbb{C}^\times$ such that $\alpha(x,y) = \dfrac{c(x)c(y)}{c(xy)}$ for all $x, y \in G$. Let $B^2(G,\mathbb{C}^\times)$ denote the subgroup of $Z^2(G,\mathbb{C}^\times)$ generated by all trivial 2-cocycles of G. Let $H^2(G,\mathbb{C}^\times)$ denote the factor group $Z^2(G,\mathbb{C}^\times)/B^2(G,\mathbb{C}^\times)$, which is called the **second cohomology group** of G (with coefficients in $\mathbb{C}^\times$).

Remark. The structures of $H^2(\mathfrak{S}_k,\mathbb{C}^\times)$ and $H^2(H_k,\mathbb{C}^\times)$ are clarified by I. Schur [Sc] and J. W. Davies, A. O. Morris [DM], respectively as follows:

$$
(1.3) \qquad H^2(\mathfrak{S}_k,\mathbb{C}^\times) \cong \begin{cases} 0 & \text{for } k \leq 3 \\ \mathbb{Z}/2\mathbb{Z} & \text{for } k \geq 4 \end{cases}
$$

$$
(1.4) \qquad H^2(H_k,\mathbb{C}^\times) = \begin{cases} 0 & \text{if } k = 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k = 2 \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } k = 3 \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } k \geq 4. \end{cases}
$$

Let α be a 2-cocycle of G. Let $\mathbb{C}^\alpha G$ be a **twisted group algebra** of G with the 2-cocycle α, namely $\mathbb{C}^\alpha G$ is an associative algebra whose basis is a set $\{a_x\ ;\ x \in G\}$ of the symbols indexed by the elements of G and multiplication is defined by $a_x a_y = \alpha(x,y)a_{xy}$ for all $x, y \in G$.

Let $R\colon G \to GL(V)$ be a projective representation of G with a 2-cocycle α. Define a representation $\rho\colon \mathbb{C}^\alpha G \to \mathrm{End}\,(V)$ of $\mathbb{C}^\alpha G$ by $\rho(a_x) = R(x)$ for all $x \in G$. The map $(R,V) \mapsto (\rho,V)$ establishs a 1-1 correspondence between the equivalence classes of the projective representations of G with a 2-cocycle belonging to the cohomology class of α and the equivalence classes of the representations of $\mathbb{C}^\alpha G$.

An associative algebra is called **semisimple** if it is isomorphic to a direct sum of some total matrix algebras.

Proposition 1.1. *A twisted group algebra of a finite group G for a 2-cocycle of G is a semisimple associative algebra.*

Remark 1. It is well known (cf. [Sc], [St1]) that, for all $k \geq 4$, any twisted group algebra $\mathbb{C}^\alpha \mathfrak{S}_k$ of $\mathfrak{S}_k$ with a non-trivial 2-cocycle α is isomorphic to the associative algebra, which is denoted by $\mathcal{A}_k$, generated by the elements γ_i with $1 \leq i \leq k-1$ subject to relations

$$\begin{aligned} &\gamma_i^2 = -1 \ \ (1 \leq i \leq k-1), \ \ (\gamma_i\gamma_{i+1})^3 = -1 \ \ (1 \leq i \leq k-2), \\ &(\gamma_i\gamma_j)^2 = -1 \ \ (|i-j| \geq 2). \end{aligned} \tag{1.5}$$

For any $k \leq 3$, let $\mathcal{A}_k$ denote the associative algebra generated by the γ_i with $1 \leq i \leq k-1$ subject to relations (1.5). If $k \leq 3$, then $\mathcal{A}_k$ is isomorphic to the ordinary group algebra $\mathbb{C}\mathfrak{S}_k$, via $\gamma_i \mapsto \sqrt{-1}s_i$.

We introduce a $\mathbb{Z}_2$-gradation of $\mathcal{A}_k$ by giving the generator γ_i degree $1 \in \mathbb{Z}_2$ for all $1 \leq i \leq k-1$. Note that we abbreviate $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$ as $\mathbb{Z}_2$.

Remark 2. It is well known (cf. [DM], [St2]) that any twisted group algebra $\mathbb{C}^\alpha H_k$ of H_k is isomorphic to an associative algebra, which is denoted by $\mathcal{B}_k$, generated by τ and the σ_i with $1 \leq i \leq k-1$ subject to relations

$$\begin{aligned} &\tau^2 = \sigma_i^2 = 1 \quad (1 \leq i \leq k-1), \quad (\sigma_i\sigma_{i+1})^3 = 1 \quad (1 \leq i \leq k-2), \\ &(\sigma_i\sigma_j)^2 = \varepsilon_1 \quad (|i-j| \geq 2), \quad (\tau\sigma_i)^2 = \varepsilon_2 \quad (2 \leq i \leq k-1), \\ &(\tau\sigma_1)^4 = \varepsilon_3 \end{aligned} \tag{1.6}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$ depend on α.

For any $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^{\times 3}$, let $\mathbb{C}^\varepsilon H_k$ denote the above associative algebra corresponding ε. Let $\mathcal{B}_k$ denote the associative algebra $\mathbb{C}^{(+1,+1,-1)}H_k$. We introduce a $\mathbb{Z}_2$-gradation of $\mathcal{B}_k$ by giving the generator τ degree 1 and the generator σ_i degree 0 for all $1 \leq i \leq k-1$.

In the following, we use the formulation of the $\mathbb{Z}_2$-graded representations and modules of $\mathbb{Z}_2$-graded algebras, which is also used in [J1] and [J2]. This formulation is a little different from the traditional parametrization of the non-graded representations, but there exists a explicit relation between the graded representations of a graded algebra A and the non-graded representations of the underlying algebra $|A|$ (cf. [J1, Lem. (2.8), Cor. (2.16), Prop. (2.17)], [J2, Prop. 2.5, Cor. 2.6]).

C. semisimple super algebras. A $\mathbb{Z}_2$-graded algebra A, which is called a **super algebra** in this paper, is called **simple** if it has no $\mathbb{Z}_2$-graded two-sided ideals except itself and 0.

Let V (resp. W) be a vector space with a fixed $\mathbb{Z}_2$-gradation: $V = V_0 \oplus V_1$ (resp. $W = W_0 \oplus W_1$). The vector space $\mathrm{Hom}\,(V,W)$ consisting of all linear maps from V to W has a $\mathbb{Z}_2$-gradation as follows:

$$\mathrm{Hom}\,(V,W)_\alpha = \{f \in \mathrm{Hom}\,(V,W)\,;\, f(V_\beta) \subset W_{\alpha+\beta} \text{ for all } \beta \in \mathbb{Z}_2\}$$

for each $\alpha \in \mathbb{Z}_2$. Therefore, the endomorphism algebra $\mathrm{End}\,(V) = \mathrm{Hom}\,(V,V)$, which is isomorphic to the total matrix ring M_{n+m} ($\dim V_0 = n, \dim V_1 = m$), can be regarded as a super algebra. If the algebra M_{n+m} is regarded as a super algebra, it is denoted by $M(n,m)$. It is clear that $M(n,m)$ is simple, since the underlying algebra $|M(n,m)| = M_{n+m}$ is simple algebra.

There exists another examples of simple super algebra. Let $Q(n)$ denote a super subalgebra of $M(n,n)$ which consists of all $2n \times 2n$-matrices of the form $\begin{pmatrix} C & D \\ D & C \end{pmatrix}$, where C and D are $n \times n$-matrices. It is easy to show that $Q(n)$ is a simple super algebra.

Theorem 1.2. (cf. [J1, Th. 2.6], [J2, Th. 2.1], [Wa]) *A simple super algebra is isomorphic to either $M(n,m)$ for some n,m or $Q(n)$ for some n.*

A simple algebra A is said to be of **type** M if A is isomorphic to $M(n,m)$ for some n,m, and say that A is of **type** Q if A is isomorphic to $Q(n)$ for some n.

Let A be a super algebra. Let V be a vector space. Fix a $\mathbb{Z}_2$-gradation of V: $V = V_0 \oplus V_1$. An algebra homomorphism $\rho\colon A \to \mathrm{End}\,(V)$ is called a **super representation** of A if $\rho(A_\alpha) \subset \mathrm{End}\,(V)_\alpha$. Then V is called a **super A-module**. We simply write $\rho(a)v = av$ for all $a \in A$ and $v \in V$. Let V and W be two super A-modules. For each $\alpha \in \mathbb{Z}_2$, let $\mathrm{Hom}_A(V,W)_\alpha$ denote the subspace of $\mathrm{Hom}\,(V,W)_\alpha$ consisting of all elements $f \in \mathrm{Hom}\,(V,W)_\alpha$ such that $f(av)(w) = (-1)^{\alpha\cdot\beta} af(v)(w)$ for all $a \in A_\beta$ ($\beta \in \mathbb{Z}_2$) and $v \in V$ and $w \in W$. Put $\mathrm{Hom}_A(V,W) = \mathrm{Hom}_A(V,W)_0 \oplus \mathrm{Hom}_A(V,W)_1$ and $\mathrm{End}_A(V) = \mathrm{Hom}_A(V,V)$. It is clear that $\mathrm{End}_A(V)$ is a super subalgebra of $\mathrm{End}\,(V)$. We call $\mathrm{End}_A(V)$ the **centralizer** of V.

Two super A-module V and W are called **isomorphic** if there exists an invertible linear map $f \in \mathrm{Hom}_A(V,W)$. Then we write $V \cong_A W$. In particular, if $V \cong_A W$ and there exists no invertible linear map $f \in \mathrm{Hom}_A(V,W)_0$, then V and W are called **shift-isomorphic**.

Let V be a super A-module. Let W be a subspace of V. If $aw \in W$ for all $a \in A$ and $w \in W$, then W is called a super A-submodule of V. We say that V is **irreducible** if it has no super A-submodules except itself and 0.

The following theorem contains an analogue of Schur's Lemma.

Theorem 1.3. (cf. [J1, Prop. 2.17], [J2, Prop. 2.5, Cor. 2.6]) *Let $V = V_0 \oplus V_1$ be an irreducible super A-module.*

(1) *The super centralizer $\mathrm{End}_A(V)$ is isomorphic to $M(1,0) = \mathbb{C}$, if (and only if) V_0 is not isomorphic to V_1 as an A_0-module.*

(2) *The super centralizer $\mathrm{End}_A(V)$ is isomorphic to $Q(1)$, if (and only if) V_0 is isomorphic to V_1 as an A_0-module.*

If an irreducible super A-module V satisfies the equivalent conditions in (1), then we say that it is of **type** M, and if it satisfies the equivalent conditions in (2), then we say that it is of **type** Q.

Whenever we treat the tensor products $A \otimes B$ and $V \otimes W$ of super algebras A and B and super A-module V and super B-module W, respectively, assume that they are the **super tensor products**, which are denoted by $A \hat{\otimes} B$ and $V \hat{\otimes} W$, respectively, defined by $(a \otimes b)(c \otimes d) = (-1)^{\beta \cdot \gamma} ac \otimes bd$ for any $a \in A$, $c \in A_\gamma$ and $b \in B_\beta$, $d \in B$, and $(a \hat{\otimes} b)(v \otimes w) = (-1)^{\beta \cdot \alpha} av \otimes bw$ for any $a \in A$, $b \in B_\beta$ and $v \in V_\alpha$, $w \in W$.

A super A-module V is called **semisimple**, if there exist some irreducible super A-submodules $V_1, V_2, \ldots, V_r$ of V such that V is a direct sum of the V_i with $1 \leq i \leq r$: $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$.

For any super algebra A, the following conditions are equivalent (cf. [J1, Prop. 2.4], [J1, Cor. 2.12], [J2, Th. 2.2, Cor. 2.3]):

(1) A is semisimple as a super (regular) A-module,
(2) every super A-module is semisimple,
(3) A is a direct sum of a finite number of simple super algebras.

We call a super algebra A **semisimple** if it satisfies the above equivalent conditions (1)-(3).

Proposition 1.4. (cf. [J1, Cor. 2.16]) *A super algebra A is semisimple if and only if a underlying algebra $|A|$ is semisimple.*

Here we introduce the double centralizer theorem for super modules of semisimple super algebras.

Theorem 1.5. *Let A be a semisimple super algebra. Let V be a super A-module with a super representation $\rho\colon A \to \mathrm{End}\,(V)$, and*

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s \tag{1.7}$$

be a direct sum of the non-zero A-homogeneous components of V. Put $B = \mathrm{End}_A(V)$. We define a super representation $\tilde{\rho}\colon A \hat{\otimes} B \to \mathrm{End}\,(V)$ of $A \hat{\otimes} B$ by $\tilde{\rho}(a \hat{\otimes} b) = \rho(a) \circ b$ for all $a \in A$ and $b \in B$. Then the V_i with $1 \leq i \leq s$ are irreducible super $A \hat{\otimes} B$-submodules of V of type M.

If V_i is a super A-module of type M, then there exist an irreducible super A-modules (resp. an irreducible super B-modules) U_i (resp. W_i) of type M such that

$$U_i \hat{\otimes} W_i = U_i' \hat{\otimes} W_i' \cong_{A \hat{\otimes} B} V_i$$

where U_i' (resp. W_i') denotes a super A (resp. B)-module shift-isomorphic to V (resp. W).

If V_i is a super A-module of type Q, then there exist an irreducible super A-modules (resp. an irreducible super B-modules) U_i (resp. W_i) of type Q such that

$$U_i \hat{\otimes} W_i \cong_{A \hat{\otimes} B} V_i \oplus V_i'$$

where V_i' denotes a super $A \hat{\otimes} B$-module shift-isomorphic to V.

And, for each $1 \leq i \leq s$, U_i (resp. W_i) is uniquely determined up to (shift-)isomorphic A-modules (resp. B-modules). It follows that V_i is a super $A \hat{\otimes} B$-module of type Q for all $1 \leq i \leq s$. Then we have

$$U_i \ncong_A U_j\,, \quad W_i \ncong_B W_j$$

for all $1 \leq i \neq j \leq s$.

Furthermore we have $\mathrm{End}_B(V) = \rho(A)$.

We explain Theorem 1.5 more explicitly. Let $\{U_r\}_{r\in R}$ be the complete set of the irreducible super A-modules. For each $r \in R$, fix a generator x_r of $\mathrm{End}_A(U_r)_1$ such that $x_r^2 = -1$ if U_r is of type Q. For any $1 \leq i \leq s$, there exists a unique element $r_i \in R$ such that $m(U_{r_i}, V_i) \neq 0$. Then put $U_i = U_{r_i}$ and $W_i = \mathrm{Hom}_A(U_i, V_i)$ for all $1 \leq i \leq s$. There exists the fact that $b(V_i) \subset V_i$ for all $b \in B = \mathrm{End}_A(V)$ and all $1 \leq i \leq s$. Thus we define a super representation $\chi_i : B \to \mathrm{End}(W)$ by $(\chi_i(b)w)(u) = b(w(u))$ for all $b \in B$, $w \in W_i$, and $u \in U_i$. Then it follows that, for any $1 \leq i \leq s$, if U_i is of type M (resp. of type Q), then so is W_i. For any $1 \leq i \leq s$ such that W_i is of type Q, define a generator y_i of $\mathrm{End}_B(W_i)_1$ by

$$(y_i(w))(u) = (-1)^{\alpha} x_{r_i}(u) \tag{1.8}$$

for all $w \in (W_i)_\alpha$ ($\alpha \in \mathbb{Z}_2$) and all $u \in U_i$. It is easy to check that $y_i^2 = -1$. Let $(U_i \hat{\otimes} W_i)^{\pm\sqrt{-1}}$ denote the $\pm\sqrt{-1}$-eigenspaces of $x_{r_i} \hat{\otimes} y_i \in \mathrm{End}_{A\hat{\otimes}B}(U_i \hat{\otimes} W_i)_0$ for all $1 \leq i \leq s$. Then it follows that V_i is isomorphic to $(U_i \hat{\otimes} W_i)^{\pm\sqrt{-1}}$ as a graded $A\hat{\otimes}B$-module. Remark that, in particular, V_i is shift-isomorphic to $(U_i \hat{\otimes} W_i)^{-\sqrt{-1}}$. Consequently, we have

$$V \cong_{A\hat{\boxtimes}B} \bigotimes_{U_i:\ \text{type } M} U_i \hat{\boxtimes} W_i \oplus \bigotimes_{U_i:\ \text{type } Q} (U_i \hat{\boxtimes} W_i)^{+\sqrt{-1}} \quad \Big(\text{or } (U_i \hat{\boxtimes} W_i)^{-\sqrt{-1}} \Big) \tag{1.9}$$

where $A\hat{\boxtimes}B$ and $U_i\hat{\boxtimes}W_i$ denote the super tensor products $A\hat{\otimes}B$ and $U_i\hat{\otimes}W_i$, respectively, and $\hat{\boxtimes}$ is used in order to emphasize that A and B act on V as mutual commutants of each other.

E: Characters of super modules. For a super A-module V, let $\mathrm{Ch}\, V : A \to \mathbb{C}$ denote the **character** of V defined by $\mathrm{Ch}\, V(a) = \mathrm{tr}_V(a)$ for all $a \in A$. Every irreducible super A-module V is uniquely (up to isomorphic A-modules) determined by its characters. We say that V **affords** the character $\mathrm{Ch}\, V$. Let $\{U_r\}_{r\in R}$ be the complete set of the irreducible super A-modules. Then the $\mathrm{Ch}\, U_r$ with $r \in R$ are called **complete set** of the irreducible characters of A.

Let $Z(A_0^*)$ denote the subspace of $\mathrm{Hom}(A_0, \mathbb{C})$ consisting of all elements $f : A_0 \to \mathbb{C}$ such that $f(ab) = f(ba)$ for all $a, b \in A$. It follows that $\{\mathrm{Ch}\, U_r\ ;\ r \in R\}$ is a basis of $Z(A_0^*)$.

Remark 1. Let $\mathfrak{S}_k'$ be the subgroup of $(\mathcal{A}_k)^\times$ generated by $-1, \gamma_1, \ldots, \gamma_{k-1}$. Then $\mathfrak{S}_k'$ is a double cover of $\mathfrak{S}_k$ with a group homomorphism $\pi : \mathfrak{S}_k' \to \mathfrak{S}_k$ defined by $\pi(-1) = 1$ and $\pi(\gamma_j) = s_j$ for all $1 \leq j \leq k-1$. Let γ^μ denote a canonical element whose homomorphic image is a canonical elements of the conjugacy class of $\mathfrak{S}_k$ indexed by μ:

$$\gamma^\mu = \gamma_1 \cdots \gamma_{\mu_1 - 1}\gamma_{\mu_1+1} \cdots \gamma_{\mu_1+\mu_2-1} \cdots \gamma_{\mu_1+\cdots+\mu_{r-1}+1} \cdots \gamma_{\mu_1+\cdots+\mu_r-1} \quad (r = l(\mu)) \ .$$

It is well known (cf.[Sc], [St1]) that the characters of $\mathcal{A}_k$ are completely determined by the values of them at the γ^μ with $\mu \in OP_k$.

Remark 2. Let H'_k be the subgroup of $(\mathcal{B}_k)^\times$ generated by $-1, \tau, \sigma_1, \ldots, \sigma_{k-1}$. Then H'_k is a double cover of H_k with a group homomorphism $\pi : H'_k \to H_k$ defined by $\pi(-1) = 1, \pi(\tau) = t$, and $\pi(\sigma_j) = s_j$ for all $1 \leq j \leq k-1$. Let $\sigma^{(\lambda,\mu)}$ denote a canonical element whose homomorphic image is a canonical elements of the conjugacy class of H_k indexed by a pair (λ, μ):

$$\begin{aligned}
\sigma^{(\lambda,\mu)} &= x_1 x_2 \cdots y_1 y_2 \cdots , \\
x_i &= \sigma_{\lambda_1+\cdots+\lambda_{i-1}+1} \cdots \sigma_{\lambda_1+\cdots+\lambda_i-1}, \\
y_i &= \sigma_{|\lambda|+\mu_1+\cdots+\mu_{i-1}+1} \cdots \sigma_{|\lambda|+\mu_1+\cdots+\mu_i-1} \tau_{|\lambda|+\mu_1+\cdots+\mu_i}, \\
\tau_j &= \sigma_{j-1} \cdots \sigma_1 \tau \sigma_1 \cdots \sigma_{j-1} \quad .
\end{aligned}$$

It is well known (cf. [DM], [St2]) that that the characters of $\mathcal{B}_k$ are completely determined by the values of them at the $\sigma^{(\mu,\phi)}$ with $\mu \in OP_k$.

F. Schur's description of the irreducible characters of $\mathcal{A}_k$. I. Schur introduces Q-functions in order to express the $\mathbb{Z}_2$-graded irreducible characters of $\mathcal{A}_k$. We review Schur's result in the form translated by T. Józefiak into the languages of the super representations of $\mathcal{A}_k$ [J2]. Let ε denote a map from DP_k to $\mathbb{Z}_2$ defined by $\varepsilon(\nu) = 0$ if $\nu \in DP_k^+$ and $= 1$ if $\nu \in DP_k^-$.

Theorem 1.6. [Sc], [J2] *Define the $\psi_\nu \in Z((\mathcal{A}_k)_0^*)$ with $\nu \in DP_k$ by*

$$(\sqrt{2})^{l(\mu)} p_\mu = \sum_{\nu \in DP_k} \psi_\nu(\gamma^\mu)(\sqrt{2})^{-l(\nu)-\varepsilon(\nu)} Q_\nu \tag{1.10}$$

for all $\mu \in OP_k$, where Q_ν denotes a Schur's Q-function (cf. [Sc], [St1]*), z_μ denotes the order of the centralizer in $\mathfrak{S}_k$ of a permutation of cycle type μ and $p_\mu = p_{\mu_1} p_{\mu_2} \cdots$ and p_r is the r-th power sum. Then $\{\psi_\nu \,;\, \nu \in DP_k\}$ is th complete set of the irreducible characters of $\mathcal{A}_k$.*

Note that $\{p_\mu \,;\, \mu \in \cup_k OP_k\}$ and $\{Q_\nu \,;\, \nu \in \cup_k DP_k\}$ are bases of a subalgebra $\mathbb{C}[p_i \,;\, i: \text{ odd}]$ of the algebra of the symmetric functions.

For each $\nu \in DP_k$, fix an irreducible super $\mathcal{A}_k$-module V_ν which affords the irreducible character ψ_ν of $\mathcal{A}_k$.

2. Sergeev's duality relation

Let $\mathfrak{p}(n,m)$ denote the Lie superalgebra whose underlying space is the super algebra $M(n,m)$ and Jacobi bracket $[\ ,\ ]: \mathfrak{p}(n,m) \times \mathfrak{p}(n,m) \to \mathfrak{p}(n,m)$ is defined by $[X,Y] = XY - (-1)^{\alpha\cdot\beta} YX$ for all $X \in \mathfrak{p}(n,m)_\alpha$ and $Y \in \mathfrak{p}_\beta$ $(\alpha, \beta \in \mathbb{Z}_2)$.

Let $\mathfrak{q}(n)$ denote the Lie supersubalgebra of $\mathfrak{p}(n,n)$ consisting of all elements of the form $\begin{pmatrix} E & F \\ F & E \end{pmatrix}$ where E, F are $n \times n$-matrices. Let $\mathcal{U}_n = \mathcal{U}(\mathfrak{q}(n))$ denote the universal enveloping algebra of $\mathfrak{q}(n)$, which can be regarded as a $\mathbb{Z}_2$-graded algebra (cf. [Ka]).

Now $\mathfrak{q}(n)$ naturally acts on a $2n$-dimensional space V with a fixed $\mathbb{Z}_2$-gradation $V = V_0 \oplus V_1$, $\dim V_\alpha = n$ for all $\alpha \in \mathbb{Z}_2$. Therefore $\mathcal{U}_n$ also acts on V. We define a super $\mathcal{U}_n$-module structure of the k-th super tensor product $W = V^{\hat{\otimes} k}$ of V,

using an algebra homomorphism (called a coproduct) $\Delta : \mathcal{U}_n \to \mathcal{U}_n \otimes \mathcal{U}_n$ defined by $\Delta(X) = X\hat{\otimes}1 + 1\hat{\otimes}X$ for all $X \in \mathfrak{q}(n)$:

$$\Theta(X) = \Delta^{(k)}(X)(v_1\hat{\otimes}\cdots\hat{\otimes}v_k) = \sum_{j=1}^{k}(-1)^{\alpha\cdot(\beta_1+\cdots+\beta_{j-1})}v_1\hat{\otimes}\cdots\hat{\otimes}\overset{j}{\smile}{X}v_j\hat{\otimes}v_k \tag{2.1}$$

for all $X \in \mathfrak{q}(n)_\alpha$ and $v_i \in V_{\beta_i}$ $(1 \le i \le k)$ and α, $\beta_i \in \mathbb{Z}_2$ $(1 \le i \le k)$.

We define a super representation $\Psi : \mathcal{B}_k \to \mathrm{End}\,(W)$ by

$$\begin{aligned}\Psi(\tau)(v_1\hat{\otimes}\cdots\hat{\otimes}v_k) &= (Pv_1)\hat{\otimes}v_2\hat{\otimes}\cdots\hat{\otimes}v_k \\ \Psi(\sigma_i)(v_1\hat{\otimes}\cdots\hat{\otimes}v_k) &= (-1)^{\beta_i\cdot\beta_{i+1}}v_1\hat{\otimes}\cdots\hat{\otimes}v_{i+1}\hat{\otimes}v_i\hat{\otimes}\cdots\hat{\otimes}v_k \\ &\qquad (v_i \in V_{\beta_i},\ \beta_i \in \mathbb{Z}_2,\ 1 \le i \le k-1)\end{aligned} \tag{2.2}$$

where $P = \begin{pmatrix} 0 & -\sqrt{-1}I_n \\ \sqrt{-1}I_n & 0 \end{pmatrix} \in M(n,n)_1$.

Define a map $d : P_k \to \mathbb{Z}_2$ by $d(\nu) = 1$ if $l(\nu)$ is odd, and $= 0$ if $l(\nu)$ is even.

Theorem 2.1. [Se] (1) *Define the functions $\psi_\nu \in Z((\mathcal{B}_k)_0^*)$ with $\nu \in DP_k$ by*

$$2^{l(\mu)}p_\mu = \sum_{\nu\in DP_k}\psi^\nu(\sigma^\lambda)(\sqrt{2})^{d(\nu)-l(\nu)}Q_\nu. \tag{2.3}$$

for all $\mu \in OP_k$. Then $\{\psi_\nu \ ; \ \nu \in DP_k\}$ is the complete set of the irreducible characters of $\mathcal{B}_k$. For each $\nu \in DP_k$, fix an irreducible super $\mathcal{B}_k$-module W_ν which affords the character ψ_ν. If $l(\nu)$ is even, then W_ν is of type M. If $l(\nu)$ is odd, then W_ν is of type Q.

(2) *Two super algebras $\mathcal{B}_k$ and $\mathcal{U}_n$ act on W as mutual commutants of each other in the context of super representations of super algebras:*

$$\mathrm{End}_{\mathcal{U}_n}(W) = \mathcal{B}_k, \quad \mathrm{End}_{\mathcal{B}_k}(W) = \mathcal{U}_n.$$

(3) *The irreducible super $\mathcal{B}_k$-module W_ν $(\nu \in DP_k)$ occurs in W if and only if $l(\nu) \le n$.*

If $l(\nu)$ is odd, then fix a generator y_ν of $\mathrm{End}_{\mathcal{B}_k}(W_\nu)_1$ such that $y_\nu^2 = -1$. For any $\nu \in DP_k$, let U_ν denote an irreducible super $\mathcal{U}_n$-module corresponding to W_ν and let u_ν be the generator defined using y_ν as (1.8) in which y_i and x_{r_i} are replaced by u_ν and y_ν, respectively. Then we have (cf. (1.9))

(2.4)

$$W \cong_{\mathcal{B}_k\hat{\boxtimes}\,\mathcal{U}_n} \bigoplus_{l(\nu):\ \mathrm{even}} W_\nu\hat{\boxtimes}U_\nu \oplus \bigoplus_{l(\nu):\ \mathrm{odd}} (W_\nu\hat{\boxtimes}U_\nu)^{+\sqrt{-1}} \quad \Big(\text{ or } (W_\nu\hat{\boxtimes}U_\nu)^{-\sqrt{-1}}\Big).$$

3. A simple relation between $\mathcal{A}_k$ and $\mathcal{B}_k$

Let $\mathcal{C}_k$ denote the 2^k-dimensional Clifford algebra, i. e. $\mathcal{C}_k$ is an associative algebra generated by k elements $\xi_1, \dots, \xi_k$ subject to relations

$$\xi_i^2 = 1, \quad \xi_i \xi_j = -\xi_j \xi_i \quad (i \neq j) \ . \tag{3.1}$$

We regard $\mathcal{C}_k$ as a super algebra by giving the generator ξ_i degree 1 for all $1 \leq i \leq k$. For any subset $I = \{i_1 < i_2 < \cdots < i_r\}$ of $[k] = \{1, 2, \dots, k\}$, we write $\xi_I = \xi_{i_1} \xi_{i_2} \cdots \xi_{i_r}$. Then $\{\xi_I \ ; \ I \subset [k]\}$ is a basis of $\mathcal{C}_k$. Note that $\mathcal{C}_k$ is isomorphic to a twisted group algebra for $\mathbb{Z}_2^k$ and therefore semisimple. Furthermore, it is easy to show that $\mathcal{C}_k$ is a simple super algebra. If k is even (resp. odd), then $\mathcal{C}_k$ is of type M (resp. of type Q). Fix an irreducible super $\mathcal{C}_k$-module X_k. Since $\mathcal{C}_k$ is simple, every irreducible super $\mathcal{C}_k$-module is isomorphic to X_k.

Proposition 3.1. *The character value of the irreducible super $\mathcal{C}_k$-module X_k is given by* $\mathrm{Ch}\, X_k \left(\sum_I c_I \xi_I\right) = 2^{\lfloor k+1/2 \rfloor} c_\phi$ *for any coefficients $c_I \in \mathbb{C}$ with $I \subset [k]$.*

Theorem 3.2. (1) *Define a linear map $\vartheta \colon \mathcal{C}_k \hat{\otimes} \mathcal{A}_k \to \mathcal{B}_k$ by*

$$\begin{aligned} &\vartheta(\xi_1 \hat{\otimes} 1) \mapsto \tau_i \quad (1 \leq i \leq k) \\ &\vartheta(1 \hat{\otimes} \gamma_j) \mapsto \frac{1}{\sqrt{2}} (\tau_j - \tau_{j+1}) \sigma_j \quad (1 \leq j \leq k-1). \end{aligned} \tag{3.2}$$

where $\tau_i = \sigma_{i-1} \cdots \sigma_1 \tau \sigma_1 \dots \sigma_{i-1}$ for all $1 \leq i \leq k$. Then it follows that ϑ is an isomorphism and

$$X_k \hat{\otimes} V_\nu \cong \begin{cases} W_\nu & \text{if } k \text{ is even or } \nu \in DP_k^+ \\ W_\nu \oplus W_\nu' & \text{if } k \text{ is odd and } \nu \in DP_k^- \end{cases} \tag{3.3}$$

where W_ν' denotes an irreducible super $\mathcal{B}_k$-module shift-isomorphic to the irreducible super $\mathcal{B}_k$-module W_ν.

Proof. Put $\tilde{\gamma}_j = \frac{1}{\sqrt{2}} (\tau_j - \tau_{j+1}) \sigma_j$ for all $1 \leq j \leq k-1$. It follows that ϑ is a homomorphism, since the τ_i with $1 \leq i \leq k$ and the $\tilde{\gamma}_j$ with $1 \leq j \leq k-1$ satisfy the relations (2.1) and (1.5) in which ξ_i and γ_j are replaced by τ_i and $\tilde{\gamma}_j$, respectively, and $\tau_i \tilde{\gamma}_j = -\tilde{\gamma}_j \tau_i$ for all $1 \leq i \leq k$, $1 \leq j \leq k-1$.

Since

$$\vartheta(\xi_1 \hat{\otimes} 1) = \tau \quad \text{and} \quad \vartheta\left(\frac{1}{\sqrt{2}} (\xi_j - \xi_{j+1}) \otimes \tilde{\gamma}_j\right) = \sigma_j$$

for all $1 \leq j \leq k-1$, it follows that ϑ is surjective. Furthermore, it follows that ϑ is bijective, since $\dim \mathcal{C}_k \otimes \mathcal{A}_k = \dim \mathcal{B}_k = 2^k k!$. Therefore ϑ is an isomorphism.

And, calculating the character values of $X_k \hat{\otimes} V_\nu$ and W_ν, the isomorphism (3.3) immediately follows.

Proposition 3.3. *The restriction $W_\nu|_{\mathcal{A}_k}$ of the irreducible super $\mathcal{B}_k$-module W_ν to $\mathcal{A}_k$ is A_k-homogeneous for all $\nu \in DP_k$. More precisely, we have*

$$W_\nu|_{\mathcal{A}_k} \cong \begin{cases} (V_\nu \oplus V_\nu')^{\oplus 2^{\lfloor (k-1)/2 \rfloor}} & \text{if } \nu \in DP_k^+ \\ V_\nu^{\oplus 2^{\lfloor (k+1)/2 \rfloor}} & \text{if } \nu \in DP_k^- \ . \end{cases}$$

where V'_ν denotes an irreducible super $\mathcal{A}_k$-module shift-isomorphic to the irreducible super $\mathcal{A}_k$-module V_ν.

Proof. From Theorem 3.2 (2) and the fact that $\dim X_{k\,\alpha} = 2^{\lfloor (k-1)/2 \rfloor}$ for all $\alpha \in \mathbb{Z}_2$, we have

$$V_\nu^{\oplus 2^{\lfloor (k-1)/2 \rfloor}} \oplus V_\nu'^{\oplus 2^{\lfloor (k-1)/2 \rfloor}} \cong \begin{cases} W_\nu|_{\mathcal{A}_k} & \text{if } k \text{ is even or } \nu \in DP_k^+ \\ W_\nu|_{\mathcal{A}_k} \oplus W'_\nu|_{\mathcal{A}_k} & \text{if } k \text{ is odd and } \nu \in DP_k^-. \end{cases}$$

Therefore the statement is clear if k is even or $\nu \in DP_k^+$. Furthermore, if k is odd and $\nu \in DP_k^-$, then the above isomorphism is written as follows:

$$V_\nu^{\oplus 2^{\lfloor (k+1)/2 \rfloor}} \cong W_\nu|_{\mathcal{A}_k} \oplus W'_\nu|_{\mathcal{A}_k}.$$

Since $\dim W_\nu|_{\mathcal{A}_k} = \dim W'_\nu|_{\mathcal{A}_k}$, it follows that $W_\nu|_{\mathcal{A}_k} \cong_{\mathcal{A}_k} W'_\nu|_{\mathcal{A}_k}$, and the result follows.

4. A duality relation of $\mathcal{A}_k$ and $\mathfrak{q}(n)$

From Theorem 2.1 (2) and Theorem 3.2, it follows that if k is even, then

$$W \cong_{(\mathcal{C}_k \hat{\otimes} \mathcal{A}_k)\hat{\boxtimes}\mathcal{U}_n} \bigoplus_{\nu \in DP_k^+} (X_k \hat{\otimes} V_\nu)\hat{\boxtimes} U_\nu \oplus \bigoplus_{\nu \in DP_k^-} ((X_k \hat{\otimes} V_\nu)\hat{\boxtimes} U_\nu)^{+\sqrt{-1}}$$

and if k is odd, then

$$W \cong_{(\mathcal{C}_k \hat{\otimes} \mathcal{A}_k)\hat{\boxtimes}\mathcal{U}_n} \bigoplus_{\nu \in DP_k^-} (X_k \hat{\otimes} V_\nu)^{+\sqrt{-1}}\hat{\boxtimes} U_\nu \oplus \bigoplus_{\nu \in DP_k^+} ((X_k \hat{\otimes} V_\nu)\hat{\boxtimes} U_\nu)^{+\sqrt{-1}}$$

where $(X_k \hat{\boxtimes} V_\nu)^{\pm\sqrt{-1}}$ denote the irreducible $\mathcal{C}_k \hat{\otimes} \mathcal{A}_k$-modules of $X_k \hat{\otimes} V_\nu$ defined to be the $\pm\sqrt{-1}$-eigenspaces of $z_\nu \hat{\otimes} x_\nu \in \mathrm{End}_{\mathcal{C}_k \hat{\otimes} \mathcal{A}_k}(X_k \hat{\otimes} V_\nu)_0$.

Put $r = \lfloor k/2 \rfloor$. The $\zeta_i = \sqrt{-1}\xi_{2i-1}\xi_{2i}$ with $1 \le i \le r$ are commuting involutions of $(\mathcal{C}_k)_0$. Then W is a direct sum of the simultaneous eigenspaces W^ε with $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{Z}_2^r$: $W^\varepsilon = \{w \in W\,;\, \zeta_j(w) = (-1)^{\varepsilon_j} w \text{ for all } 1 \le j \le r\}$. It is clear that W^ε is a super $\mathcal{A}_k$ and $\mathcal{U}_n$-module for each $\varepsilon \in \mathbb{Z}_2^r$.

Theorem 4.1. *For each $\varepsilon \in \mathbb{Z}_2^r$, the super submodule W^ε is decomposed into a direct sum of irreducible $\mathcal{A}_k \otimes \mathcal{U}_n$-modules with multiplicity free as (4.1) and (4.3).*

(1) *If k is even, then we have*

$$W^\varepsilon \cong_{\mathcal{A}_k \hat{\boxtimes} \mathcal{U}_n} \bigoplus_{\nu \in DP_k^+} V_\nu \hat{\boxtimes} U_\nu \oplus \bigoplus_{\nu \in DP_k^-} (V_\nu \hat{\boxtimes} U_\nu)^{+\sqrt{-1}} \tag{4.1}$$

where $(V_\nu \hat{\boxtimes} U_\nu)^{\pm\sqrt{-1}}$ denote the irreducible $\mathcal{A}_k \hat{\otimes} \mathcal{U}_n$-modules of $V_\nu \hat{\otimes} U_\nu$ defined to be the $\pm\sqrt{-1}$-eigenspaces of $x_\nu \hat{\otimes} u_\nu \in \mathrm{End}_{\mathcal{A}_k \hat{\otimes} \mathcal{U}_n}(V_\nu \hat{\otimes} U_\nu)_0$. Furthermore, we have

$$\mathrm{End}_{\Theta(U_n)}(W^\varepsilon) = \Psi(\mathcal{A}_k), \quad \mathrm{End}_{\Psi(\mathcal{A}_k)}(W^\varepsilon) = \Theta(\mathcal{U}_n) \tag{4.2}$$

for all $\varepsilon \in \mathbb{Z}_2^r$.

(2) *If* k *is odd, then we have*

$$W^\varepsilon \cong_{\mathcal{A}_k \hat{\boxtimes} \mathcal{U}_n} \bigoplus_{\nu \in DP_k^+} V_\nu \hat{\boxtimes} U_\nu \oplus \bigoplus_{\nu \in DP_k^-} V_\nu \hat{\boxtimes} U_\nu \tag{4.3}$$

$$\mathrm{End}_{\Theta(U_n)}(W^\varepsilon) \cong \mathcal{C}_1 \otimes \Psi(\mathcal{A}_k), \quad \mathrm{End}_{\Psi(\mathcal{A}_k)}(W^\varepsilon) \cong \mathcal{C}_1 \otimes \Theta(\mathcal{U}_n). \tag{4.4}$$

Proof. Fix a generator x_ν of $\mathrm{End}_{\mathcal{A}_k}(V_\nu)_1$ such that $x_\nu^2 = -1$, if $\nu \in DP_k^-$. Fix a generator z_k of $\mathrm{End}_{\mathcal{C}_k}(X_k)_1$ such that $z_k^2 = -1$, if k is odd. Then define a generator y_ν of of $\mathrm{End}_{\mathcal{B}_k}(W_\nu)_1$ such that $y_\nu^2 = -1$ by

$$y_\nu = \begin{cases} 1 \hat{\otimes} x_\nu & \text{if } k \text{ is even and } \nu \in DP_k^-, \\ z_k \hat{\otimes} 1 & \text{if } k \text{ is odd and } \nu \in DP_k^+. \end{cases}$$

Let u_ν denote a generator of $\mathrm{End}_{\mathcal{U}_n}(U_\nu)_1$ defined using y_ν as (1.8).

(1) Assume that k is even. If $\nu \in DP_k^-$, then we have

$$(X_k \hat{\otimes} V_\nu \hat{\otimes} U_\nu)^{\pm\sqrt{-1}} = (X_k)_0 \hat{\otimes} (V_\nu \hat{\otimes} U_\nu)^{\pm\sqrt{-1}} \oplus (X_k)_1 \hat{\otimes} (V_\nu \hat{\otimes} U_\nu)^{\mp\sqrt{-1}}.$$

For each $\varepsilon \in \mathbb{Z}_2^r$, let $(X_k)^\varepsilon$ denote the simultaneous eigenspaces of X_k of the ζ_j with $1 \le j \le r$. Since the degree of ζ_j is 0 for all $1 \le j \le r$, it follows that the $(X_k)^\varepsilon$ with $\varepsilon \in \mathbb{Z}_2^r$ are $\mathbb{Z}_2$-graded subspaces of X_k. Furthermore, since $\dim X_k = 2^r$, it follows that $\dim (X_k)^\varepsilon = 1$ for each $\varepsilon \in \mathbb{Z}_2^r$. In particular, for each $\varepsilon \in \mathbb{Z}_2^r$, there exists $\alpha(\varepsilon) \in \mathbb{Z}_2$ and $\xi^\varepsilon \in (\mathcal{C}_k)_{\alpha(\varepsilon)}$ such that $(X_k)^\varepsilon = \mathbb{C}\xi^\varepsilon$. Consequently, it follows that W^ε is isomorphic to

$$\bigoplus_{\nu \in DP_k^+} V_\nu \hat{\otimes} U_\nu \oplus \bigoplus_{\nu \in DP_k^-} (V_\nu \hat{\otimes} U_\nu)^{+\sqrt{-1}}$$

for each $\varepsilon \in \mathbb{Z}_2^r$. More precisely, if $\alpha(\varepsilon) = 1$, then W^ε is shift-isomorphic to the above super $\mathcal{A}_k \hat{\otimes} \mathcal{U}_n$-module. Therefore (4.1) follows. Then, from Theorem 1.5, (4.2) is trivial.

(2) Assume that k is odd. If $\nu \in DP_k^+$, then we have

$$(X_k \hat{\otimes} V_\nu \hat{\otimes} U_\nu)^{\pm\sqrt{-1}} = V_\nu \hat{\otimes} (X_k \hat{\otimes} U_\nu)^{\pm\sqrt{-1}} = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} V_\nu \hat{\otimes} ((X_k)^\varepsilon \hat{\otimes} U_\nu)^{\pm\sqrt{-1}}$$

since the ζ_j with $1 \le j \le r$ and $z_k \hat{\otimes} u_\nu$ commute, where $(X_k \hat{\otimes} U_\nu)^{\pm\sqrt{-1}}$ denote the $\pm\sqrt{-1}$-eigenspaces of $z_k \hat{\otimes} u_\nu \in \mathrm{End}_{\mathcal{C}_k \hat{\otimes} \mathcal{U}_n}(X_k \hat{\otimes} U_\nu)_0$. Furthermore, since $\dim X_k = 2^{r+1}$, it follows that $\dim (X_k)^\varepsilon = 2$. In particular, for each $\varepsilon \in \mathbb{Z}_2^r$, there exists $\xi^\varepsilon \in (\mathcal{C}_k)_0$ such that $(X_k)^\varepsilon = \mathbb{C}\xi^\varepsilon \oplus \mathbb{C} z_k(\xi^\varepsilon)$. It is easy to show that

$$((X_k)^\varepsilon \hat{\otimes} U_\nu)^{\pm\sqrt{-1}} \cong U_\nu$$

for each $\varepsilon \in \mathbb{Z}_2^r$. If $\nu \in DP_k^-$, then we have

$$(X_k \hat{\otimes} V_\nu)^{\pm\sqrt{-1}} \hat{\otimes} U_\nu = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} ((X_k)^\varepsilon \hat{\otimes} V_\nu) \hat{\otimes} U_\nu$$

since the ζ_j with $1 \leq j \leq r$ and $z_k \hat{\otimes} x_\nu$ commute, where $(X_k \hat{\otimes} V_\nu)^{\pm\sqrt{-1}}$ denote the $\pm\sqrt{-1}$-eigenspaces of $z_k \hat{\otimes} x_\nu \in \mathrm{End}_{\mathcal{C}_k \hat{\otimes} \mathcal{A}_k}(X_k \hat{\otimes} V_\nu)_0$. It is easy to show that

$$((X_k)^\varepsilon \hat{\otimes} V_\nu)^{\pm\sqrt{-1}} \cong V_\nu$$

for each $\varepsilon \in \mathbb{Z}_2^r$. Consequently, it follows that W^ε is isomorphic to

$$\bigoplus_{\nu \in DP_k^+} V_\nu \hat{\otimes} U_\nu \oplus \bigoplus_{\nu \in DP_k^-} V_\nu \hat{\otimes} U_\nu$$

for each $\varepsilon \in \mathbb{Z}_2^r$. Therefore (4.3) follows. Then, from Theorem 1.5, (4.4) is trivial.

Corollary 4.2. *The duality relation of $\mathcal{A}_k$ and $\mathcal{U}_n$ in Theorem 4.1 gives Schur's identity (1.10).*

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