SCHUR'S Q-FUNCTIONS AND AFFINE LIE ALGEBRAS OF TYPE $D^{(2)}_{\ell+1}$

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ABSTRACT. Weight vectors of the basic representation of $D^{(2)}_{\ell+1}$ are studied. They are expressed in terms of Schur's Q-functions [M]. The up and down motion along the string of the fundamental imaginary root is described as a combinatorial game.

RÉSUMÉ. On considère les vecteurs de poids pour la représentation fondamentale (ou de base) de $D^{(2)}_{l+1}$. Ils s'expriment en terme des Q-fonctions de Schur. On donne une régle combinatoire qui décrit le mouvement de tels vecteurs en haut et en bas au long de la chaîne de la racine imaginaire fondamentale.

1. NEUTRAL FREE FERMIONS AND SCHUR'S Q-FUNCTIONS

We first review some ingredients of operator formalism for Schur's Q-functions. Let $\mathcal{B}$ be the $\mathbb{C}$-algebra generated by ϕ_n $(n \in \mathbb{Z})$ with respect to the relations

$$\phi_n\phi_m + \phi_m\phi_n = (-1)^n\delta_{n+m,0} \quad (n, m \in \mathbb{Z}). \tag{1.1}$$

In the literature (*e.g.* [DJKM]) the generators ϕ_n are referred to as the neutral free fermions. Define the degree on $\mathcal{B}$ by $\deg \phi_n = 1$ $(n \in \mathbb{Z})$. If we let $\mathcal{B}_0$ (resp. $\mathcal{B}_1$) be the subspace consisting of even (resp. odd) degree elements, then $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ has a structure of a superalgebra. Let $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 = \mathcal{B}_0|0\rangle \oplus \mathcal{B}_1|0\rangle$ (resp. $\mathcal{F}^* = \langle 0|\mathcal{B}_0 \oplus \langle 0|\mathcal{B}_1$) be the Fock space (resp. the dual Fock space), where the vacuum $|0\rangle$ (resp. $\langle 0|$) is defined by

$$\phi_n|0\rangle = 0 \quad (n < 0) \qquad (\text{resp. } \langle 0|\phi_n = 0 \quad (n > 0)). \tag{1.2}$$

The vacuum expectation value $\langle 0|a|0\rangle$ $(a \in \mathcal{B})$ is uniquely determined by setting $\langle 0|1|0\rangle = 1$, $\langle 0|\phi_0|0\rangle = 0$.

We construct the realization of $\mathcal{F}$. The normal ordering for the neutral fermions is defined by

$${}^\circ_\circ\phi_n\phi_m{}^\circ_\circ = \phi_n\phi_m - \langle 0|\phi_n\phi_m|0\rangle. \tag{1.3}$$

Define the Hamiltonian by

$$H(t) = \frac{1}{2}\sum_{\substack{j \geqslant 1\\ j:\text{odd}}}\sum_{n \in \mathbb{Z}}(-1)^{n+1}t_j\phi_n\phi_{-j-n}, \tag{1.4}$$

where t_j ($j \geqslant 1$, odd) are indeterminates. Let $V = \mathbb{C}[t_j;\ j \geqslant 1, \text{odd}]$ be the algebra of polynomials with infinitely many variables. There is an isomorphism from $\mathcal{F}_0$ to V defined by

$$a|0\rangle \longmapsto \langle 0|e^{H(t)}a|0\rangle \quad (a \in \mathcal{B}_0), \tag{1.5}$$

which is often called the boson-fermion correspondence. The subspace

$$B_\infty = \left\{ \sum_{n,m\in\mathbb{Z}} c_{nm}{}^{\circ}_{\circ}\phi_n\phi_m{}^{\circ}_{\circ};\, c_{nm} = 0 \text{ if } |n-m| \gg 0 \right\} \tag{1.6}$$

of $\mathcal{B}_0$ admits a structure of a Lie algebra isomorphic to the one dimensional central extension of $\mathfrak{o}(\infty)$. The Fock representation of B_∞ on V is defined by the vertex operator

$$Z(p,q) = \frac{p+q}{2(p-q)} \left\{ \exp\left(\sum_{\substack{j\geqslant 1\\ j:\text{odd}}} (p^j + q^j)t_j \right) \exp\left(-2 \sum_{\substack{j\geqslant 1\\ j:\text{odd}}} \frac{1}{j}(p^{-j} + q^{-j})\frac{\partial}{\partial t_j} \right) - 1 \right\}, \tag{1.7}$$

which corresponds to the action of ${}^{\circ}_{\circ}\phi(p)\phi(q){}^{\circ}_{\circ}$ on $\mathcal{F}_0$, where $\phi(p) = \sum_{n\in\mathbb{Z}} \phi_n p^n$.

For a strict partition $\lambda = (\lambda_1, \ldots, \lambda_{2d})$ ($\lambda_1 > \cdots > \lambda_{2d} \geqslant 0$), put $v_\lambda = \phi_{\lambda_1}\cdots\phi_{\lambda_{2d}}|0\rangle \in \mathcal{F}_0$. It is shown in [DJKM] that $\{v_\lambda\ ;\ \lambda \text{ is a strict partition}\}$ forms a weight basis for the Fock representation of B_∞. The above boson-fermion correspondence sends the vector v_λ to Schur's Q-function $(1/2^d)Q_\lambda(t)$, where the variables $t = (t_1, t_3, t_5, \ldots)$ are related with the power sum symmetric functions by $t_j = 2p_j/j$ ($j \geqslant 1$, odd).

2. Basic representation of $D^{(2)}_{\ell+1}$ ($\ell \geqslant 2$)

Put $r = 2\ell + 2$ and $\omega = \exp(2\pi i/r)$. Consider the reduced vertex operators

$$Z(p, -\omega^j p) = \sum_{i\in\mathbb{Z}} Z_i^{(j)} p^i \qquad (1 \leq j \leq \ell). \tag{2.1}$$

Then the homogeneous components $Z_i^{(j)}$ constitute a Lie algebra acting on $V = \mathbb{C}[t_j;\ j \geqslant 1, \text{odd}\ ]$, which is isomorphic to the basic representation of $D^{(2)}_{\ell+1}$.

The simple coroots of $D^{(2)}_{\ell+1}$ are described in terms of the neutral free fermions as follows:

$$\begin{aligned}
h_0 &= -2\sum_{n\in\mathbb{Z}} {}^{\circ}_{\circ}\phi_{nr-1}\phi_{-nr+1}{}^{\circ}_{\circ} + 1,\\
h_i &= \sum_{n\in\mathbb{Z}} (-1)^i \left({}^{\circ}_{\circ}\phi_{nr+i}\phi_{-nr-i}{}^{\circ}_{\circ} - {}^{\circ}_{\circ}\phi_{nr-i-1}\phi_{-nr+i+1}{}^{\circ}_{\circ} \right) \ (1 \leqslant i \leqslant \ell - 1),\\
h_\ell &= 2\sum_{n\in\mathbb{Z}} (-1)^\ell {}^{\circ}_{\circ}\phi_{nr+\ell}\phi_{-nr-\ell}{}^{\circ}_{\circ}.
\end{aligned} \tag{2.2}$$

Since the Cartan subalgebra of $D_{\ell+1}^{(2)}$ is the intersection of $D_{\ell+1}^{(2)}$ and the Cartan subalgebra of B_∞, any Q-function $Q_\lambda(t)$ for a strict partition λ is a $D_{\ell+1}^{(2)}$-weight vector.

We will describe the weight of a given Q-function. To this end we need an r-abacus. For each parts of a strict partition we put a bead on the assigned position. To give an example, the following figure is an r-abacus representing the strict partition $(5\ell+7, 4\ell+3, 3\ell+5, 3\ell+3, 2\ell+3, \ell+1, 2)$.

$$\begin{array}{ccccccccc}
 & 1 & 2 & \cdots & \ell & \ell+2 & \ell+3 & \cdots & 2\ell+1 \\
\ell+1 & & & & & & & & \\
2\ell+2 & 2\ell+3 & 2\ell+4 & \cdots & 3\ell+2 & 3\ell+4 & 3\ell+5 & \cdots & 4\ell+3 \\
3\ell+3 & & & & & & & & \\
4\ell+4 & 4\ell+5 & 4\ell+6 & \cdots & 5\ell+4 & 5\ell+6 & 5\ell+7 & \cdots & 6\ell+5 \\
\cdots & \cdots & \cdots & & & & & &
\end{array}$$

Theorem 2.1. *Let $Q_\lambda(t)$ be a Q-function of weight $wt(\lambda)$. Denote by δ the fundamental imaginary root of $D_{\ell+1}^{(2)}$, i.e., $\delta = \sum_{i=0}^{\ell} \alpha_i$. Then a strict partition corresponding to the weight $wt(\lambda)+\delta$ is obtained by one of the following:*

(1) *Move a bead one position up along the leftmost runner.*
(2) *Remove the bead at the position $\ell+1$.*

A strict partition corresponding to the weight $wt(\lambda)+2\delta$ is obtained by one of the following:

(3) *Iterate the procedures obtaining $wt(\lambda)+\delta$.*
(4) *Move a bead one position up along a runner except for the leftmost one.*
(5) *Remove two beads at positions s and $r-s$ for $1 \leqslant s \leqslant \ell$.*

For the proof we only have to verify that the fermions corresponding to the moves (1)–(5) commute with h_i $(0 \leqslant i \leqslant \ell)$.

The maximal weight vectors are the Q-functions $Q_\lambda(t)$ with λ corresponding to the *stalemates*. For example, the strict partitions of size up to 12 whose Q-functions are maximal weight vectors for $D_4^{(2)}$ are

$$\{\emptyset, (1), (2), (3), (2,1), (3,1), (5), (3,2), (6), (5,1), (3,2,1), (7), (6,1), (5,2), (5,2,1),$$
$$(7,2), (6,3), (9,1), (7,3), (6,3,1), (6,5), (10,2), (7,5), (9,2,1), (6,5,1), (7,3,2)\}.$$

3. Application to the case of $A_1^{(1)}$

We now apply the results of the preceding section to the basic representation of $A_1^{(1)}$. To this end, we first recall the following theorem which is due to M. Wakimoto.

Theorem 3.1. *Let* e_i *and* f_i $(i = 0,1,2,3)$ *be the Chevalley generators and* h_i $(i = 0,1,2,3)$ *be the simple coroots of* $D_4^{(2)}$. *Put*

$$\begin{aligned} \tilde{e}_0 &= e_0 + e_3, & \tilde{e}_1 &= \sqrt{2}(e_1 + e_2), \\ \tilde{f}_0 &= f_0 + f_3, & \tilde{f}_1 &= \sqrt{2}(f_1 + f_2), \\ \tilde{h}_0 &= h_0 + h_3, & \tilde{h}_1 &= 2(h_1 + h_2). \end{aligned} \tag{3.1}$$

Then $\tilde{e}_i$ *and* $\tilde{f}_i$ $(i = 0,1)$ *generate a Lie subalgebra isomorphic to* $A_1^{(1)}$. *Moreover, the restriction of the basic representation of* $D_4^{(2)}$ *to this subalgebra remains irreducible and turns out to be the basic representation of* $A_1^{(1)}$.

In a similar fashion, one can embed $D_{2^n}^{(2)}$ into $D_{2^{n+1}}^{(2)}$ so that the basic representation is preserved under the restriction. Since the above theorem is thought of as the special case $n = 1$, we denote by $D_2^{(2)}$ the subalgebra isomorphic to $A_1^{(1)}$. Therefore $V = \mathbb{C}[t_j; j \geqslant 1, \text{odd}]$ is viewed as the basic representation of $D_2^{(2)} \cong A_1^{(1)}$. The same argument as in Section 2 shows that Schur's Q-functions for the strict partitions form a weight basis for the basic $D_2^{(2)}$-module V. We need the 4-abacus in order to determine the weight of a given Q-function. Here we give the 4-abacus of the strict partition $\lambda = (9,5,3,2)$.

$$\begin{array}{ccc} & 1 & 3 \\ 2 & & \\ 4 & 5 & 7 \\ 6 & & \\ 8 & 9 & 11 \\ \cdots & \cdots & \cdots \end{array}$$

Suppose we are given a Q-function $Q_\lambda(t)$ of weight $\mathrm{wt}(\lambda)$. A strict partition corresponding to the weight $\mathrm{wt}(\lambda) + \delta$ is obtained by one of the following:

(1) Move a bead one position up along the leftmost runner.

(2) Remove the bead at the position 2.

A strict partition corresponding to the weight $\mathrm{wt}(\lambda)+2\delta$ is obtained by one of the following:

(3) Iterate the procedures obtaining $\mathrm{wt}(\lambda)+\delta$.

(4) Move a bead one position up along one of the right two runners.

(5) Remove two beads at the positions 1 and 3.

The strict partitions corresponding to the maximal weight vectors are thus characterized by the stalemates, which form the following set:

$$\mathrm{HC}^{(4)} := \{\emptyset, (4n+1, 4n-3, \dots, 5, 1), (4n+3, 4n-1, \dots, 7, 3) \quad (n \geqslant 0)\}. \tag{3.2}$$

As is known [DJKM], the basic representation $D^{(2)}_{\ell+1}$ describes the symmetry of the $(2\ell+2)$-reduced BKP hierarchy. We remark here the equivalence of the 4-reduced BKP hierarchy and the KdV hierarchy [DJKM]. The vertex operator of the former looks

$$Z(p, -ip) = \frac{1+i}{2(1-i)}\left\{\exp\left(\sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}} (1-i^j)p^j t_j\right)\exp\left(-2\sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}} \frac{(1+i^j)p^{-j}}{j}\frac{\partial}{\partial t_j}\right)-1\right\}. \tag{3.3}$$

Hence, by changing the variables

$$x_j = \cos\left(\frac{j\pi}{4}\right) t_j \quad (j \geqslant 1, \mathrm{odd}) \tag{3.4}$$

and putting $k = e^{-i\pi/4}p$, it reads

$$\frac{1+i}{2(1-i)}\left\{\exp\left(2\sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}} k^j x_j\right)\exp\left(-2\sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}} \frac{k^{-j}}{j}\frac{\partial}{\partial x_j}\right)-1\right\},$$

which equals, up to constant, the vertex operator for the KdV hierarchy.

The set of all the weighted homogeneous polynomial solutions coincides with the set of the maximal weight vectors. As for the 4-reduced BKP hierarchy this set consists of $Q_\lambda(t)$ with λ in $\mathrm{HC}^{(4)}$. Define the *doubling* [BO] $d(\lambda)$ of a strict partition $\lambda = (\lambda_1, \lambda_2, \dots)$ by

$$d(\lambda) = \left(\left[\frac{\lambda_1+1}{2}\right], \left[\frac{\lambda_1}{2}\right], \left[\frac{\lambda_2+1}{2}\right], \left[\frac{\lambda_2}{2}\right], \dots\right). \tag{3.5}$$

Then we have

$$\left\{d(\lambda); \lambda \in \mathrm{HC}^{(4)}\right\} = \left\{\Delta_r; r \geqslant 0\right\} \tag{3.6}$$

where $\Delta_r = (r, r-1, \dots, 2, 1)$ for $r \geqslant 1$ and $\Delta_0 = \emptyset$.

The Schur functions $S_{\Delta_r}(x)$ $(r \geqslant 0)$ cover the weighted homogeneous polynomial solutions to the KdV hierarchy [DJKM], where the variables $x = (x_1, x_3, x_5, \dots)$ are related with the power sum symmetric functions by $x_j = p_j/j$ $(j \geqslant 1$, odd). They are related with the Q-functions by

$$Q_\lambda(t)|_{t \to x} = \text{const.} S_{d(\lambda)}(x) \tag{3.7}$$

for the strict partitions λ in $HC^{(4)}$, where the variables are changed according to (3.4) in the left hand side. The character theory of the spin symmetric group leads that the constant in the right hand side is equal to $2^{l(\lambda)/2}$, where $l(\lambda)$ is the length of λ.

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