Combinatorial identities for series of quasi-symmetric functions

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Abstract

We construct a hypoplactic analog of the Robinson-Schensted-Knuth correspondence, which leads to a bijective proof of the noncommutative Cauchy identity. Next, we establish quasi-symmetric analogs of Schur's and Littlewood's identities and compute the Möbius function of the hypoplactic monoid.

Résumé

Nous donnons un analogue hypoplaxique de la correspondance de Robinson-Schensted-Knuth. Cette correspondance nous permet de donner une preuve bijective de l'analogue non commutatif de l'identité de Cauchy. Nous établissons ensuite des analogues quasi-symétriques d'identités de Schur et de Littlewood. Enfin, nous calculons la fonction de Möbius du monoïde hypoplaxique.

1 Introduction

In the theory of symmetric functions and in the representation theory of the classical groups, Schur function series identities such as

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{\lambda} s_\lambda(X) s_\lambda(Y)\,, \tag{1}$$

$$\text{or} \prod_{i} \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j} = \sum_{\lambda} s_\lambda(X)\,, \tag{2}$$

are of crucial importance. Combinatorial proofs of (1) (Cauchy identity) and (2) (Schur's identity) can be given by means of Knuth's extension of the Robinson-Schensted correspondence [11]. In the same way, the following identity (due to Littlewood)

$$\prod_{i}(1-x_i) \prod_{i<j}(1-x_i x_j) = 1 + \sum_{\mu} (-1)^{(|\mu|+r)/2} s_\mu\,, \tag{3}$$

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where μ is summed over all self-conjugate partitions and r is the rank of the partition, as well as several related identities, is a combinatorial consequence of the Burge correspondences, also derived from Robinson-Schensted [1].

It has been shown in [8], [13] that Gessel's fundamental quasi-symmetric functions F_I [10] can be regarded as quasi-symmetric analogs of Schur functions. That is to say that they correspond to the irreducible representations of various quantum structures (Hecke algebras, quantum linear groups) at $q = 0$, which for $q = 1$ are the classical objects to which Schur functions are attached.

In section 2, we recall the basic notations of the theory of noncommutative symmetric functions [9] and of the theory of quasi-symmetric functions [10]. In section 3, we construct the hypoplactic analog of the Robinson-Schensted-Knuth correspondence, which leads to a bijective proof of the noncommutative Cauchy identity. Next, we establish a quasi-symmetric analog of Schur's identity (2) and compute the Möbius function of the hypoplactic monoid (section 4). We conclude with some conjectural interpretations of other quasi-symmetric series (section 6).

2 Background

2.1 Noncommutative symmetric functions

Noncommutative symmetric functions provide an algebraic theory related to the combinatorics of compositions, in much the same way as ordinary symmetric functions are related to the combinatorics of partitions.

The so-called "fundamental theorem of the theory of symmetric functions" states that the elementary symmetric polynomials $e_1, \dots, e_n$ in n variables $x_1, \dots, x_n$ are algebraically independent. Thus, the algebra of symmetric functions (polynomials with an infinite set of variables) is just a polynomial algebra $\mathbb{Q}[e_1, e_2, \dots]$, graded by the weight function $w(e_k) = k$. One can as well take as generators the complete homogeneous functions h_k (sum of all monomials of degree k) or the power sums $p_k = \sum x_i^k$.

By analogy, one defines the algebra of formal *noncommutative symmetric functions* [9] as the free associative algebra $\mathbf{Sym} = \mathbb{Q}\langle \Lambda_1, \Lambda_2, \dots \rangle$ generated by an infinite sequence of noncommutative indeterminates $(\Lambda_n)_{n\geq 1}$, called *elementary* symmetric functions. The *complete* symmetric functions S_k are defined as in the commutative case by

$$\sigma(t) := \sum_{k\geq 0} S_k t^k = [\lambda(-t)]^{-1} ,$$

where $\lambda(t) = \sum_{k\geq 0} \Lambda_k t^k$. Although it is convenient to define $\mathbf{Sym}$ as an abstract algebra, we can realize this algebra by taking an infinite alphabet $A = \{a_1, a_2, \dots\}$ and defining its complete homogeneous symmetric functions by

$$\sigma(A, t) = \sum_{k\geq 0} t^k S_k(A) = \overrightarrow{\prod_{i\geq 1}} \frac{1}{1 - ta_i}.$$

These elements are not symmetric for the usual action of permutations on free algebra, but they are invariant under the Lascoux-Schützenberger action of the symmetric group [14]. We recall that a composition of n is a sequence I of positive integers whose sum $|I|$ is equal to n. Then for any composition $I = (i_1, \ldots, i_r)$, one defines $S^I = S_{i_1} \ldots S_{i_r}$. The homogeneous component $\mathbf{Sym}_n$ of degree n of **Sym** is the subspace of **Sym** generated by all S^I with $|I| = n$. The family (S^I) is a homogeneous basis of **Sym**. The set of all compositions of an integer n can be equipped with the *reverse refinement order*, denoted by $\preceq$. One has for example $(5,2,3) \preceq (1,4,2,1,2)$. The noncommutative *ribbon Schur* functions (R_I) are then defined by

$$R_I = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)} \; S^J \;, \tag{4}$$

where $\ell(I)$ denotes the *length* of I. The family (R_I) is also a homogeneous basis of **Sym**.

2.2 Quasi-symmetric functions

Malvenuto and Reutenauer proved that noncommutative symmetric functions are in duality with the algebra of quasi-symmetric functions [16, 9]. This algebra was introduced by Gessel [10]. Let $X = \{x_1, x_2, \ldots\}$ be a totally ordered set of commutative indeterminates. An element $f \in \mathbb{C}[X]$ is said to be a *quasi-symmetric function* if for each composition $I = (i_1, \ldots, i_m)$ all the monomials $x_{k_1}^{i_1} \cdots x_{k_m}^{i_m}$ with $k_1 < k_2 < \cdots < k_m$ have the same coefficient in f. The quasi-symmetric functions form a subalgebra of $\mathbb{C}[X]$, denoted QSym. For a composition $I = (i_1, \ldots, i_m)$, we define the quasi-monomial function

$$M_I = \sum_{k_1 < k_2 < \cdots < k_m} x_{k_1}^{i_1} \cdots x_{k_m}^{i_m} .$$

The family of quasi-monomial functions is a basis of QSym. Another important basis of QSym is formed by *quasi-ribbon functions* which are defined by

$$F_I = \sum_{I \preceq J} M_J .$$

The pairing $\langle \cdot, \cdot \rangle$ between **Sym** and QSym [16] is then defined by $\langle M_I, S^J \rangle = \delta_{IJ}$ or equivalenty by $\langle F_I, R_J \rangle = \delta_{IJ}$. This duality is essentially equivalent to the noncommutative Cauchy identity ([9, 12])

$$\prod_{i \geq 1}^{\rightarrow} \left(\prod_{j \geq 1}^{\rightarrow} (1 - x_i a_j)^{-1} \right) = \sum_I F_I(X) R_I(A) \,, \tag{5}$$

where X is a totally ordered alphabet and A a noncommutative one.

2.3 The hypoplactic algebra

The algebra of symmetric functions can be embedded into a noncommutative algebra (the plactic algebra), and this embedding illuminates the combinatorial aspects of the theory. Let A be a totally ordered alphabet. The plactic algebra on A is the $\mathbb{C}$-algebra $Pl(A)$, the quotient of $\mathbb{C}\langle A \rangle$ by the relations

$$\begin{cases} aba \equiv baa, bba \equiv bab, & \text{for } a < b \,, \\ acb \equiv cab, bca \equiv bac, & \text{for } a < b < c \,. \end{cases}$$

These relations, which were obtained by Knuth [11], generate the equivalence relation identifying two words which have the same P-symbol under the Robinson-Schensted correspondence. Schensted has shown that the construction of the P-symbol is an associative operation on words. The monoid structure on the set of tableaux has been studied by Lascoux and Schützenberger under the name "plactic monoid" [14]. In particular, they showed that the Littlewood-Richardson rule is equivalent to the fact that Schur functions, defined as sums of all tableaux of a given shape, are a basis of a commutative subalgebra of the plactic algebra, *i.e.* the algebra of symmetric functions. In the same way, QSym can be embedded in the hypoplactic algebra [13], a quotient of the plactic algebra which arises as a subalgebra of the quantum linear group of Dipper and Donkin at $q = 0$ [5].

The hypoplactic algebra $HPl(A)$ is the quotient of the plactic algebra $Pl(A)$ by the relations

$$\begin{cases} baba \equiv abab, baca \equiv abac, & \text{for } a < b < c\,, \\ cacb \equiv acbc, cbab \equiv bacb, & \text{for } a < b < c\,, \\ badc \equiv dbca, acbd \equiv cdab, & \text{for } a < b < c < d\,. \end{cases}$$

The combinatorial objects playing the rôle of Young tableaux are called *quasi-ribbons*. Given a composition I, a quasi-ribbon tableau of shape I is a ribbon diagram r of shape I filled up by letters of A such that

- each row of r is nondecreasing from left to right and,
- each column of r is strictly increasing from top to bottom.

A word is said to be a *quasi-ribbon word* of shape I if it can be obtained by reading from left to right and from bottom to top the columns of a quasi-ribbon diagram of shape I, *e.g.* $w = abacbcc$ is a quasi-ribbon word of shape $(2, 2, 3)$. The quasi-ribbon tableau corresponding to w is given below.

$$\begin{array}{|c|c|cccc} \hline \mathrm{a} & \mathrm{a} & \multicolumn{1}{|c}{} & & & \\ \hline & \mathrm{b} & \multicolumn{1}{c|}{\mathrm{b}} & & & \\ \cline{2-5} & & \multicolumn{1}{|c|}{\mathrm{c}} & \multicolumn{1}{c|}{\mathrm{c}} & \multicolumn{1}{c|}{\mathrm{c}} & \\ \cline{3-5} \end{array}\,.$$

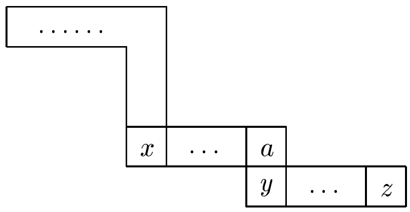
Then, we can realize the algebra of quasi-symmetric functions as a subalgebra of the hypoplactic algebra. For that, we define the quasi-ribbon functions F_I as the sum of all quasi-ribbon words of shape I [17].

3 Hypoplactic Robinson-Schensted-Knuth correspondence

We first recall the hypoplactic version of the Robinson-Schensted algorithm, which gives a correspondence between words $w \in A^*$ and pairs of a quasi-ribbon and a standard ribbon of shape I, denoted by $(Q(w), R(w))$.

Let Q be a quasi-ribbon tableau and let $a \in A$. Let Q' be the diagram obtained from Q by deleting its last row and let x (resp. z) be the first (resp. last) letter of the last row of Q. The result Q_a of the insertion of a in Q is defined by the following rules [13] :

- if $z \leq a$, Q_a is obtained by adding a box containing a at the end of the last row of Q,
- if $x \leq a < z$, let y be the first letter of the last row of Q which is strictly greater than a. The quasi-ribbon diagram Q_a is then



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- if $a < x$, Q_a is obtained by inserting a in Q' and glueing under the quasi-ribbon obtained in this way the last row of Q.

Let $w = a_1 \dots a_n$ be a word of length n. The pair $(Q(w), R(w))$ associated to w can be defined as follows ; the quasi-ribbon diagram $Q(w)$ is obtained by inserting the letters of w, from left to right. The standard ribbon $R(w)$ is iteratively constructed by putting at each step i of the algorithm the number i in the box that contains in Q the letter a_i inserted at this step.
For instance, let $w = bacaba$. The insertion algorithm gives the following sequence

$$\emptyset \longrightarrow \boxed{\mathrm{b}} \longrightarrow \begin{array}{c} \mathrm{a} \\ \mathrm{b} \end{array} \longrightarrow \begin{array}{cc} \mathrm{a} & \\ \mathrm{b} & \mathrm{c} \end{array} \longrightarrow \begin{array}{ccc} \mathrm{a} & \mathrm{a} & \\ & \mathrm{b} & \mathrm{c} \end{array} \longrightarrow \begin{array}{ccc} \mathrm{a} & \mathrm{a} & \\ & \mathrm{b} & \mathrm{b} \\ & & \mathrm{c} \end{array} \longrightarrow \begin{array}{cccc} \mathrm{a} & \mathrm{a} & \mathrm{a} & \\ & & \mathrm{b} & \mathrm{b} \\ & & & \mathrm{c} \end{array}.$$

Thus, the quasi-ribbon diagram $Q(w)$ and the standard ribbon $R(w)$ are respectively

$$Q(w) = \begin{array}{cccc} \mathrm{a} & \mathrm{a} & \mathrm{a} & \\ & & \mathrm{b} & \mathrm{b} \\ & & & \mathrm{c} \end{array}, R(w) = \begin{array}{cccc} 2 & 4 & 6 & \\ & & 1 & 5 \\ & & & 3 \end{array}.$$

The correspondence $w \to (Q(w), R(w))$ is a bijection. In fact, we can directly obtain $R(w)$. For that, it will be convenient to represent w by a *bimonomial* or a commutative *biword*. Generally, a bimonomial is a monomial in pairs of letters $(a, b) \in A \times B$, denoted by $\binom{a}{b}$. In this case, the word $w = a_1 \dots a_n$ is associated with the bimonomial

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}.$$

Let us now recall the notion of standardization introduced by Schensted [18]. For a word w, the standardized $\mathrm{std}(w)$ of w is the permutation obtained in the following way. We read w from left to right, label $1, 2, \dots$ the successive occurences of the smallest letter a of w, then do the same with the next letter b, and so on. For example, if $w = bacaba$, then $\mathrm{std}(w) = 416253$.

Let w be a word of A^* and let $(Q(w), R(w))$ be the pair of quasi-ribbon and standard ribbon obtained with the hypoplactic correspondence. Then $R(w)$ is obtained by reading the shape of the permutation $\mathrm{std}(w)^{-1}$, and $Q(w)$ is the only quasi-ribbon of shape $R(w)$ and with the same evaluation as w.
For instance, if $w = bacaba$ then $\mathrm{std}(w) = 416253$. So $\mathrm{std}(w)^{-1} = 246153$. The ribbon shape of 246153 is

$$\begin{array}{|c|c|c|cc} \hline 2 & 4 & 6 & \multicolumn{1}{c}{} & \\ \cline{1-4} \multicolumn{2}{c|}{} & 1 & 5 & \multicolumn{1}{|c}{} \\ \cline{3-4} \multicolumn{3}{c|}{} & 3 & \multicolumn{1}{|c}{} \\ \cline{4-4} \end{array}.$$

The evaluation of w is $(3, 2, 1)$, then $Q(w)$ is

$$\begin{array}{|c|c|c|cc} \hline \mathrm{a} & \mathrm{a} & \mathrm{a} & \multicolumn{1}{c}{} & \\ \cline{1-4} \multicolumn{2}{c|}{} & \mathrm{b} & \mathrm{b} & \multicolumn{1}{|c}{} \\ \cline{3-4} \multicolumn{3}{c|}{} & \mathrm{c} & \multicolumn{1}{|c}{} \\ \cline{4-4} \end{array}.$$

The map K is the hypoplactic analog of Knuth correspondence between bimonomials and pairs of tableaux of the same shape. A letter of the alphabet $X \times A$ is a pair of letters (x, a) where $x \in X$ and $a \in A$. For convenience, we will use the notation $\begin{bmatrix} x \\ a \end{bmatrix}$ for (x, a). Therefore, a word $w \in (X \times A)^*$ is a noncommutative product of biletters of $X \times A$, *i.e.*

$$w = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \cdots \begin{bmatrix} u_n \\ v_n \end{bmatrix}.$$

The bijection is obtained by means of the following construction. Let $w = \begin{bmatrix} u \\ v \end{bmatrix}$ be a biword of $(X \times A)^*$ and let w' be the biword $\begin{bmatrix} u' \\ v' \end{bmatrix}$ obtained from w by sorting v according to the lexicographic order with priority on the bottom row. For instance, for $w = \begin{bmatrix} 112333 \\ 123112 \end{bmatrix}$, we associate $w' = \begin{bmatrix} 133132 \\ 111223 \end{bmatrix}$. $K(w)$ is the pair $(Q(v), R(u'))$ where $Q(v)$ is the quasi-ribbon of the hypoplactic class of v and $R(u')$ is u', viewed as a ribbon. By construction, $Q(v)$ and $R(u')$ are of same shape.
For example, for $w = \begin{bmatrix} 112333 \\ 123112 \end{bmatrix}$, we associate the pair

$$\begin{array}{|c|c|c|cc} \hline 1 & 1 & 1 & \multicolumn{1}{c}{} & \\ \cline{1-4} \multicolumn{2}{c|}{} & 2 & 2 & \multicolumn{1}{|c}{} \\ \cline{3-4} \multicolumn{3}{c|}{} & 3 & \multicolumn{1}{|c}{} \\ \cline{4-4} \end{array}, \quad \begin{array}{|c|c|c|cc} \hline 1 & 3 & 3 & \multicolumn{1}{c}{} & \\ \cline{1-4} \multicolumn{2}{c|}{} & 1 & 3 & \multicolumn{1}{|c}{} \\ \cline{3-4} \multicolumn{3}{c|}{} & 2 & \multicolumn{1}{|c}{} \\ \cline{4-4} \end{array}.$$

Theorem 3.1 *The hypoplactic Knuth correspondence gives a bijection between nondecreasing biwords and pairs consisting of a quasi-ribbon and a ribbon of the same shape.*

Let X be a totally ordered alphabet and A a noncommutative one. The alphabet XA denotes the ordered alphabet $X \times A$ endowed with the lexicographic ordering.

The noncommutative symmetric functions of the alphabet XA are defined by means of the following generating series

$$\sigma(XA,t) = \sum_{n\geq 0} S_n(XA)t^n = \prod_{x\in X}^{\rightarrow} \sigma(A,xt).$$

Then, from the theorem **3.1**, we deduce a combinatorial proof of the noncommutative analog of the Cauchy formula.

Corollary 3.2

$$S_n(XA) = \sum_{|I|=n} F_I(X)R_I(A).$$

Proof - $S_n(XA)$ is the sum of all nondecreasing words of length n of biletters on the alphabet XA. $F_I(X)$ is the sum of all quasi-ribbons of shape I and $R_I(A)$ is the sum of all ribbons of shape I. □

4 The quasi-symmetric Schur identity

The first result of this section is a quasi-symmetric analog of Schur's identity (2). We recall some definitions. A subset of $\{1,\ldots,n-1\}$ can be encoded by a composition of n. Let $A = \{a_1 < a_2 < \cdots < a_k\}$ be a subset of $\{1,\ldots,n-1\}$, then we define

$$C(A) = \{a_1, a_2 - a_1, \ldots, a_k - a_{k-1}, n - a_k\}.$$

We need the notion of a *recoil set* of a standard tableau [3, 4]. Let us say that i is a *recoil* (or *descent*) of a standard tableau t if $i+1$ occurs in a higher row than i. Denote by $\mathrm{rec}(t)$ the recoil set of t. For example the tableau

$$t = \begin{array}{|c|c|c}\cline{1-2} 3 & 8 & \\ \hline 2 & 5 & \multicolumn{1}{c|}{7} \\ \hline 1 & 4 & \multicolumn{1}{c|}{6} \\ \hline \end{array}$$

has $\mathrm{rec}(t) = \{1,2,4,6,7\}$.

Theorem 4.1 *Let X be a totally ordered alphabet, then*

$$\sum_I F_I = \frac{1}{2}\left[\prod_i \frac{1+x_i}{1-x_i} + 1\right],$$

or equivalently, for $n > 0$

$$\sum_{|I|=n} F_I = \sum_{k=0}^{n-1} s_{n-k,1^k} = P_{(n)}, \tag{6}$$

where $P_{(n)}$ is the Schur P-function indexed by the partition (n).

Proof - We show that identity (6) is valid in the hypoplactic algebra, *i.e.* the sum of hook words is congruent to the sum of quasi-ribbon words. More precisely, let us define the recoil of a hook word as those of its standardized hook tableau. Thus we show that every hook word with D as recoil set is congruent to the quasi-ribbon word of shape $C(D)$.

It is known that the standardization process is compatible with the plactic relations [14]. In the same way, it is compatible with the quartic hypoplactic relations. Thus, it is sufficient to exhibit this correspondence for standard tableaux.

We argue by induction on n. The case $n = 1$ is immediate. We assume the result for n, we will prove it for $n + 1$. Consider a hook word $w = w_1 \dots w_k w_{k+1} \dots w_n$ of shape $(n - k, 1^k)$, *i.e.*

$$\begin{aligned} w_1 > w_2 > \cdots > w_k > w_{k+1}\,, \\ w_{k+1} < w_{k+2} < \cdots < w_n . \end{aligned}$$

Let $D = \{d_1, d_2, \dots, d_i\}$ be the recoil set of this hook tableau. We assume that w is congruent to the quasi-ribbon word of shape $C(D)$.
Consider a hook T obtained by adding to w the letter $n+1$, and let D_T be the recoil set of T. From this, we have two cases :

- T is of shape $(n + 1 - k, 1^k)$, thus $D_T = D$. Then, the hook word is $w_1 \dots w_n(n + 1)$. The insertion of $w_1 \dots w_n$ gives the quasi-ribbon word of shape $C(D)$ (with $|C(D)| = n$). As $n + 1 > w_n$, the insertion of $n + 1$ gives the quasi-ribbon word of shape $C(D)$ (with $|C(D)| = n + 1$),

- T is of shape $(n - k, 1^{k+1})$, thus $D_T = D \cup \{n\}$. Then, the hook word is $(n + 1)w_1 \dots w_n$. The insertion of $(n + 1)w_1 \dots w_{k+1}$ gives the column shape

$$\begin{array}{|c|} \hline w_{k+1} \\ \hline w_k \\ \hline \vdots \\ \hline w_1 \\ \hline n+1 \\ \hline \end{array}$$

  As $n+1$ is the greatest letter, inserting the sequence $w_{k+2} \dots w_n$ proceeds exactly as with the shape

$$\begin{array}{|c|} \hline w_{k+1} \\ \hline w_k \\ \hline \vdots \\ \hline w_2 \\ \hline w_1 \\ \hline \end{array}$$

Thus, the final shape is $(d_1, d_2 - d_1, \ldots, d_i - d_{i-1}, n - d_i, 1)$ which corresponds to $C(D_T)$.

Finally, the hook word T, with D_T as recoil set, is congruent to the quasi-ribbon word of shape $C(D_T)$. $\square$

Example 4.2 Consider the hook tableau

$$\begin{array}{|c|ccc}\hline 5 \\ \hline 2 \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{6} \\ \hline \end{array},$$

with $\{1, 4\}$ as recoil set. The insertion algorithm on 521346 gives

$$\begin{array}{ccc}
\boxed{1} & & \\
\boxed{2} & \boxed{3} & \boxed{4} \\
 & & \boxed{5}\,\boxed{6}
\end{array}.$$

So $521346 \equiv 213546$ (which is a quasi-ribbon word of shape $(1, 3, 2)$).

We can now give a quasi-symmetric analog of Littlewood's identity (3).

Theorem 4.3 *The inverse of the quasi-symmetric Schur series is given by*

$$\left(\sum_I F_I\right)^{-1} = 1 + \sum_{|I|=2n+1} (-1)^{n+1} c_I F_I \,,$$

where c_I is the number of permutations σ such that σ and σ^{-1} respectively have descent compositions I and 12^n.

Proof - We apply an identity proved in [9]. Let $H = \sum_{n \geq 0} H_n$, where $H_n = \sum_{k=0}^{n-1} R_{1^k, n-k}$. Then

$$H^{-1} = 1 - \sum_{n \geq 0} (-1)^n R_{12^n} \,.$$

The commutative image of H_n, denoted $\overline{H_n}$, is $\overline{H_n} = \sum_{k=0}^{n-1} s_{n-k,1^k}$. By the previous theorem, we get

$$\left(\sum_I F_I\right)^{-1} = \overline{H_n^{-1}} = 1 - \sum_{n \geq 0} (-1)^n \overline{R_{12^n}} \,.$$

From the results of [10] and [16] (see [9], theorem **6.1.6**) we know that $R_{12^n} = \sum_I \langle r_{12^n}, r_I \rangle F_I$, where r_I is the commutative image of the ribbon Schur functions R_I and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product of symmetric functions. Moreover, it is easy to see that $\langle r_{12^n}, r_I \rangle$ is the number of permutations σ such that σ fits I and σ^{-1} fits 12^n [10]. $\square$

Arguing as in [4], we deduce the following corollary.

Corollary 4.4 *Let $w \in HPl(A)$ be a quasi-ribbon word of shape I. The Möbius function μ of the hypoplactic monoid is given by*

$$\mu(w) = \begin{cases} (-1)^{n+1} c_I & \textit{if } |w| = 2n+1, \\ 0 & \textit{otherwise.} \end{cases}$$

5 Two constructions on permutations

We exhibit two constructions on permutations which have nice properties related to quasi-symmetric series identities.

Definition 5.1 *Let σ be a permutation of $\mathfrak{S}_n$. The permutation* $\mathrm{Dst}(\sigma)$ *is obtained by reading the ribbon shape of σ from bottom to top and diagonally.*

Example 5.2 let $\sigma = 1764253$ then $\mathrm{Dst}(\sigma) = 2345167$ (see Figure 1).

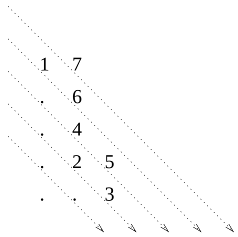


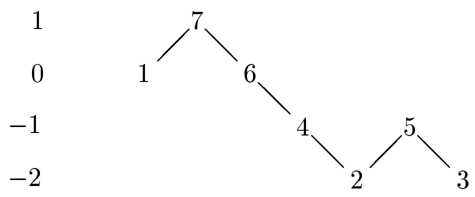
Figure 1: $\mathrm{Dst}(1764253) = 2345167$

There is an easier way to understand this construction. Let σ be a permutation of $\mathfrak{S}_n$. We associate to σ an integer vector, called the *trace vector* and denoted by $\mathrm{Tr}(\sigma) = (l_1, \ldots, l_n)$, of length n built with the following rules :

$$l_i = \begin{cases} l_{i-1} + 1 & \text{if } \sigma_i > \sigma_{i-1}, \\ l_{i-1} - 1 & \text{if } \sigma_i < \sigma_{i-1}, \end{cases}$$

for $i \in [2, n]$ and $l_1 = 0$. The permutation $\mathrm{Dst}(\sigma)$ is the permutation corresponding to the sorted vector obtained from $\mathrm{Tr}(\sigma)$.

Example 5.3 Take again $\sigma = 1764253$. The trace vector $\mathrm{Tr}(1764253) = (0, 1, 0, -1, -2, -1, -2)$ is obtained from the following diagram.



Reading this diagram level by level from bottom to top, we obtained $\mathrm{Dst}(\sigma) = 2345167$.

Definition 5.4 *Let σ be a permutation of $\mathfrak{S}_n$. The permutation* $\mathrm{Bizz}(\sigma)$ *is obtained by reading the ribbon shape of σ from bottom to top and from left to right.*

Example 5.5 Let $\sigma = 1764253$. The ribbon shape of σ is

$$\begin{array}{|c|c|cc}
\hline
1 & 7 & \multicolumn{1}{|c}{} \\
\hline
\multicolumn{1}{c|}{} & 6 & \multicolumn{1}{|c}{} \\
\cline{2-2}
\multicolumn{1}{c|}{} & 4 & \multicolumn{1}{|c}{} \\
\cline{2-3}
\multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c|}{5} \\
\cline{2-3}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{3} \\
\cline{3-3}
\end{array}\;.$$

Then $\mathrm{Bizz}(\sigma) = 1246735$.

We recall some definitions. Given a permutation $\sigma \in \mathfrak{S}_n$, we can associate a subset $\mathrm{des}(\sigma)$ of $\{1, n-1\}$ corresponding to position i of σ such that $\sigma_i > \sigma_{i+1}$. For a composition $I = (i_1, \dots, i_k)$ of n, one associates the subset $\mathrm{des}(I) = \{i_1, i_1 + i_2, \dots, i_1 + \cdots + i_{k-1}\}$. $\mathrm{des}(I)$ is called the *descent set* of I. For a composition I of n, one defines the *descent class* D_I as the set of permutations of $\mathfrak{S}_n$ such that $\mathrm{des}(\sigma) = \mathrm{des}(I)$.

It is known that D_I is an interval of the permutohedron (the weak order). Moreover $D_I = [\alpha(I), \omega(I)]$ where $\alpha(I)$ and $\omega(I)$ are respectively the *minimal* and the *maximum* permutation of D_I. $\alpha(I)$ is the permutation obtained by filling the columns of the shape I from bottom to top and from left to right with the numbers $1, 2, \dots |I|$, and $\omega(I)$ is the permutation obtained by filling the rows of the shape I from left to right and from bottom to top with the numbers $1, 2, \dots, |I|$.

Example 5.6 Given $I = 212$. Then $\alpha(I) = 14325$ and $\omega(I) = 45312$.

$$\alpha(I) = \begin{array}{|c|c|c}
\hline
1 & 4 & \multicolumn{1}{|c}{} \\
\hline
\multicolumn{1}{c|}{} & 3 & \multicolumn{1}{|c}{} \\
\cline{2-3}
\multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c|}{5} \\
\cline{2-3}
\end{array}\,, \quad \omega(I) = \begin{array}{|c|c|c}
\hline
4 & 5 & \multicolumn{1}{|c}{} \\
\hline
\multicolumn{1}{c|}{} & 3 & \multicolumn{1}{|c}{} \\
\cline{2-3}
\multicolumn{1}{c|}{} & 1 & \multicolumn{1}{|c|}{2} \\
\cline{2-3}
\end{array}\,.$$

Proposition 5.7 *Let σ be a permutation of shape I. Then*

$$\mathrm{Bizz}(\sigma) = \sigma \cdot \alpha(I).$$

The following results are immediate consequences of **5.7**.

Proposition 5.8 *Given a permutation σ, the number of inversions of σ is denoted $\ell(\sigma)$. Then,*

$$\ell(\mathrm{Bizz}(\sigma)) = \ell(\sigma) - \ell(\alpha(I))\,,$$

where I is the ribbon shape of σ.

Proposition 5.9 *Given a permutation σ, then*

$$\mathrm{des}(\mathrm{Bizz}(\sigma)) \subseteq \mathrm{des}(\sigma)\,,$$

thus there exists an integer p such that $(\mathrm{Bizz})^p(\sigma) = \mathrm{Id}$.

Theorem 5.10 *Let I be a composition of n, and consider the descent class D_I. The image of D_I by* Bizz *is an interval of the permutohedron.*

6 Some identities

We introduce the algebra of *free quasi-symmetric functions* [6]. The free quasi-ribbon $\mathbb{F}_\sigma$ labelled by a permutation σ of $\mathfrak{S}_n$ is the non commutative polynomial

$$\mathbb{F}_\sigma = \sum_{\mathrm{std}(w)=\sigma^{-1}} w \quad \in \mathbb{Z}\langle A\rangle.$$

Using the hypoplactic version of the Robinson-Schensted correspondence, one shows that the commutative image of $\mathbb{F}_\sigma$ is the quasi-symmetric function F_I where $I = C(\sigma)$.

Moreover, as the standard ribbon which plays the role of the insertion tableau is equal to $\mathrm{std}(w)^{-1}$, it follows that $\mathbb{F}_\sigma$ contains exactly one representative of each hypoplactic class of shape I.

For a word $w = w_1 \dots w_n$ of $\mathbb{N}^*$ and an integer p, $w[p]$ denotes the shifted word $(w_1+p)\dots(w_n+p)$. Given $\alpha \in \mathfrak{S}_p$ and $\beta \in \mathfrak{S}_q$, we have

$$\mathbb{F}_\alpha \cdot \mathbb{F}_\beta = \sum_{\sigma \in \alpha \,\sqcup\!\sqcup\, \beta[p]} \mathbb{F}_\sigma .$$

Thus we have the free quasi-ribbons span a $\mathbb{Z}$-subalgebra of the free associative algebra, and the subalgebra $\mathbf{FQSym} = \bigoplus_{\sigma\in\mathfrak{S}_n} \mathbb{Q}\mathbb{F}_\sigma$ is called the algebra of *free quasi-symmetric functions.*
We consider the function L_n defined by

$$L_n = \sum_{|I|=n} (-1)^{\ell(I)} \mathbb{F}_{\omega(I)} ,$$

and for a composition $I = (i_1, \dots, i_r)$, we set $L_I = L_{i_1} \dots L_{i_r}$.

Lemme 6.1 *Let H be the serie defined by*

$$H = \sum_{n\geq 0} L_n ,$$

then,

$$H^{-1} \quad = \quad \sum_I (-1)^{\ell(I)} L_I .$$

□

It is clear that

$$H = \sum_{n\geq 0} L_n = \sum_I (-1)^{\ell(I)} \mathbb{F}_{\omega(I)} .$$

Using the package `NCSF` [20], we conjecture the following propositions.

Conjecture 6.2 *The inverse serie of H is given by the transformation* Bizz*, i.e.*

$$H^{-1} = \sum_\sigma \mathbb{F}_{\mathrm{Bizz}(\sigma)^{-1}} .$$

Conjecture 6.3 *Let* $H = 1 + \sum_{n\geq 1}(-1)^n \mathbb{F}_{\omega(2^n)}$. *The inverse series is*

$$H^{-1} = \sum_{\sigma} \mathbb{F}_{\mathrm{Bizz}(\sigma)^{-1}} ,$$

where $\sigma \in \mathfrak{S}_{2p}$ *and* $\mathrm{des}(\sigma) = \{2, 4, \ldots, 2p-2\}$.

Conjecture 6.4 *Let* H *defined by*

$$H = 1 + \sum_{I} (-1)^{\ell(I)} \mathbb{F}_{\omega(I)} ,$$

where I *is a composition with even parts. Then*

$$H^{-1} = \sum_{\sigma} \mathbb{F}_{\mathrm{Bizz}(\sigma)^{-1}} ,$$

where $\sigma \in \mathfrak{S}_{2p}$ *and* $\mathrm{des}(\sigma) \subseteq \{2, 4, \ldots, 2p-2\}$.

We conjecture a link between the transformations Dst and Bizz.

Conjecture 6.5 *Given any composition* I *of* n, *then*

$$\sum_{\sigma\in D_I} F_{C(\mathrm{Bizz}(\sigma)^{-1})} = \sum_{\sigma\in D_I} F_{C(\mathrm{Dst}(\sigma)^{-1})}.$$

Note 6.6 One of the referees proposes a proof for the commutative version of conjecture **6.3**, *i.e.*

$$\left[\sum_n (-1)^n F_{2^n}\right]^{-1} = \sum_n \sum_{i_1,\ldots,i_{2n}} x_{i_1} x_{i_2} \ldots x_{i_{2n-1}} x_{i_{2n}} , \qquad (7)$$

where the inner sum on the right is over all sequences $i_1, \ldots i_{2n}$ satisfying

$$i_1 \leq i_2 \geq i_3 \leq i_4 \geq i_5 \leq \cdots \geq i_{2n-1} \leq i_{2n}.$$

This can be proved by observing that in the product

$$\begin{aligned} &\sum_m \sum_{i_1\leq i_2 < i_3 \leq i_4 < \cdots < i_{2m-1}\leq i_{2m}} (-1)^m x_{i_1} x_{i_2} \ldots x_{i_{2m-1}} x_{i_{2m}} \\ &\qquad\qquad \times \sum_n \sum_{j_1\leq j_2 \geq j_3 \leq j_4 \geq \cdots \geq j_{2n-1}\leq j_{2n}} x_{j_1} x_{j_2} \ldots x_{j_{2n-1}} x_{j_{2n}} , \\ = &\sum_{m,n} \sum_{\substack{i_1\leq i_2 < i_3 \leq i_4 < \cdots < i_{2m-1}\leq i_{2m} \\ j_1\leq j_2 \geq j_3 \leq j_4 \geq \cdots \geq j_{2n-1}\leq j_{2n}}} (-1)^m x_{i_1} x_{i_2} \ldots x_{i_{2m-1}} x_{i_{2m}} x_{j_1} x_{j_2} \ldots x_{j_{2n-1}} x_{j_{2n}} \end{aligned}$$

Every term other than 1 appears twice, once with a positive sign and once with a negative sign [2]. Then using the P-partition interpretation, one gets (7).

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