ON THE RING GENERATED BY CHERN 2-FORMS ON $\mathbb{SL}_n/B$

Boris Shapiro[‡], Mikhail Shapiro[*], Alek Vainshtein [†]

[‡] Department of Mathematics, University of Stockholm
S-10691, Sweden, shapiro@matematik.su.se
[*] Department of Mathematics, Royal Institute of Technology
Stockholm, S-10044, Sweden, mshapiro@math.kth.se
[†] Dept. of Mathematics and Dept. of Computer Sci., Haifa University
Mount Carmel 31905, Haifa, Israel, alek@mathcs2.haifa.ac.il

Abstract. In this short note we give an explicit presentation of the ring $\mathcal{A}_n$ generated by the curvature 2-forms of the standard Hermitian linear bundles over $\mathbb{SL}_n/B$ as the quotient of the polynomial ring. The difference between $\mathcal{A}_n$ and $H^*(\mathbb{SL}_n/B)$ reflects the fact that $\mathbb{SL}_n/B$ is not a symmetric space. This question was raised by V. I. Arnold in [Ar].

Résumé: Dans cette note nous donnerons une présentation explicite, en tant que quotient de l'anneau polynomial, de l'anneau A_n engendrée par les 2-formes de courbure des fibrés en droites hermitiens standard sur $\mathbb{S}L_n/B$. La différence entre A_n et $H^*(\mathbb{S}L_n/B)$ reflète le fait que $\mathbb{S}L_n/B$ n'est pas un espace symmétrique. Cette question a été posée par V. I. Arnold dans [Ar].

§1. Introduction

Let $\mathbb{SL}_n/B$ denote the space of complete flags in $\mathbb{C}^n$. One has the obvious sequence of tautological bundles $0 \subset E_1 \subset ... \subset E_n = E$ (where E is the trivial $\mathbb{C}^n$-bundle over $\mathbb{SL}_n/B$) and the corresponding n-tuple of quotient line bundles $L_i = E_i/E_{i-1}$. Fixing some Hermitian metric on the original $\mathbb{C}^n$ one equips every bundle E_i and L_i with the induced Hermitian metric. Let w_i denote the curvature form of the above Hermitian metric on L_i. Each w_i is a $\mathbb{U}_n$-invariant 2-form on $\mathbb{SL}_n/B$ representing $c_1(L_i)$ in $H^2(\mathbb{SL}_n/B)$ (the $\mathbb{U}_n$-action is provided by the chosen metric). Denoting $c_1[L_i]$ by x_i one gets $H^*(\mathbb{SL}_n/B, \mathbb{Z}) = \frac{\mathbb{Z}[x_1,...,x_n]}{(s_1,s_2,...,s_n)}$, where the denominator is the ideal generated by all elementary symmetric functions in variables $x_1, ..., x_n$, see e.g. [Bo]. Since we have the standard representative w_i for each $x_i = c_1(L_i)$, it seems natural to study the $\mathbb{Z}$-ring $\mathcal{A}_n = \mathcal{A}(w_1, ..., w_n)$ generated by all w_i's and compare it to $H^*(\mathbb{SL}_n/B, \mathbb{Z})$. We also discuss the subring $\mathcal{A}_{k,n} = \mathcal{A}(w_{i_1}, ..., w_{i_k})$ generated by any k of w_i's (the structure of $\mathcal{A}_{k,n}$ does not depend on a particular choice of a k-tuple). The main result of this note is as follows.

1.1. Proposition. *$\mathcal{A}_{k,n}$ is a graded ring isomorphic to $\frac{\mathbb{Z}[x_1,\dots,x_k]}{I_{k,n}}$. The ideal $I_{k,n}$ is generated by the set of 2^k-1 polynomials of the form*

$$g^{(n)}_{i_1,\dots,i_j}=(x_{i_1}+\dots+x_{i_j})^{j(n-j)+1}, \tag{1}$$

where $\{i_1,\dots,i_j\}$ runs over the set of all nonempty subsets of the set $\{1,\dots,n\}$.

1.2. Example. The ring $\mathcal{A}_3=\mathcal{A}_{3,3}$ is isomorphic to $\frac{\mathbb{Z}[x_1,x_2,x_3]}{I_{3,3}}$, where $I_{3,3}$ is generated by $x_1^3, x_2^3, x_3^3, (x_1+x_2)^3, (x_1+x_3)^3, (x_2+x_3)^3, x_1+x_2+x_3$. The Hilbert series of A_3 is $(1,2,3,1)$. (For a comparison, the Hilbert series for $H^*(\mathbb{SL}_3/B)$ is $(1,2,2,1)$.)

1.3. Remark One has the standard surjective ring homomorphism $\pi\colon\mathcal{A}_n\to H^*(\mathbb{SL}_n/B,\mathbb{Z})$.

The following conjecture is the result of calculation of the Hilbert series of $\mathcal{A}_n$ for $n\le 7$, see [SS].

1.4. Conjecture. 1) The total dimension of $\mathcal{A}_n$ (as a $\mathbb{Z}$-module) equals the number of forests on n labeled vertices, and there exists a natural monomial basis for $\mathcal{A}_n$ whose monomials are enumerated by the above forests.

2) The total dimension of $\mathcal{A}_{k,n}$ is a monic polynomial in n of degree k.

Besides purely aesthetic reasons, the study of $\mathcal{A}_n$ is motivated by the fact that many known results for $H^*(\mathbb{SL}_n/B,\mathbb{Z})$ (such as the existence of a good monomial basis, S_n-module structure, the action of divided difference operators etc) have natural counterparts for $\mathcal{A}_n$. Finally, the study of $\mathcal{A}_n$ seems to be important in understanding the structure of the ring $\mathfrak{A}(\mathbb{SL}_n/B)$ of all $\mathbb{U}_n$-invariant forms on $\mathbb{SL}_n/B$, which arised resently in the arithmetic intersection theory for flag varieties, comp. [Ta1], [Ta2].

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§2. Proofs

On ring $\mathfrak{A}(\mathbb{SL}_n/B)$ of $\mathbb{U}_n$-invariant forms on $\mathbb{SL}_n/B$. It is known that $\mathbb{SL}_n/B$ has another presentation as a homogeneous space, namely, $\mathbb{S}L_n/B=\mathbb{U}_n/T^n$ where $T^n\subset\mathbb{U}_n$ is the usual torus of diagonal matrices.. Let us recall an old result from the general theory of homogeneous spaces, see e.g. [DNF].

2.1. Proposition. *The ring of all G-invariant differential forms on a homogeneous space G/H is isomorphic to the exterior algebra $\Lambda_{inv}((\mathfrak{G}/\mathfrak{H})^*)$, i.e. to the algebra of skew symmetric polylinear functions on $\mathfrak{G}$ which vanish on $\mathfrak{H}$ and are invariant under the action of internal automorphisms by the elements of H.*

(Here $\mathfrak{G}$ and $\mathfrak{H}$ are the Lie algebras of G and H, resp.)

Take the $\binom{n}{2}$-dimensional vector space V_n of all skew hermitian matrices of the form

$$(2) \qquad V_n = \begin{pmatrix} 0 & a_{1,2} & \dots & \dots & \dots & a_{1,n} \\ -\bar{a}_{1,2} & 0 & a_{2,3} & \dots & \dots & a_{2,n} \\ -\bar{a}_{1,3} & -\bar{a}_{2,3} & 0 & a_{3,4} & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\bar{a}_{1,n} & -\bar{a}_{2,n} & \dots & \dots & \dots & 0 \end{pmatrix}.$$

Its matrix entries form the linear space $(\mathfrak{U}_n/\mathfrak{T}^n)^*$ where $\mathfrak{U}_n$ (resp. $\mathfrak{T}^n$) is the Lie algebra of $\mathbb{U}_n$ (resp. of T^n). Let us denote by $(e^{i\lambda_1}, e^{i\lambda_2}, ..., e^{i\lambda_n})$ the diagonal entries of elements in T^n acting by conjugation on V_n. Under this action each entry $a_{i,j}$ above the main diagonal is multiplied by $e^{i(\lambda_i-\lambda_j)}$, and each entry $-\bar{a}_{i,j}$ below the main diagonal is multiplied by $e^{i(\lambda_j-\lambda_i)}$. Introducing the fundamental weights $\alpha_i = \lambda_i - \lambda_{i+1}$, $i = 1, ..., n-1$, we get that $a_{i,j}$ is multiplied by $e^{i(\alpha_i+\alpha_{i+1}+\dots+\alpha_{j-1})}$. The expression $\alpha_i + \alpha_{i+1} + ... + \alpha_{j-1}$ is called *the multiweight* of the entry $a_{i,j}$. (Under this convention the entry $-\bar{a}_{i,j}$ has the opposite multiweight $-\alpha_i - \alpha_{i+1} - ... - \alpha_{j-1}$.) The multiweight of an exterior monomial $\tilde{a}_{i_1,j_1} \wedge \tilde{a}_{i_2,j_2} \cdots \wedge \tilde{a}_{i_r,j_r}$, where each $\tilde{a}_{i_l,j_l}$ is either a_{i_l,j_l} or $-\bar{a}_{i_l,j_l}$, equals the sum of the multiweights of its factors.

2.2. Corollary. *The ring $\mathfrak{A}(\mathbb{SL}_n/B)$ is the linear span of all exterior monomials $\tilde{a}_{i_1,j_1} \wedge \tilde{a}_{i_2,j_2} \cdots \wedge \tilde{a}_{i_r,j_r}$ whose multiweight vanishes. In particular, $\mathfrak{A}(\mathbb{SL}_n/B)$ has no degree 1 elements.*

Remark. Note that such monomials with the vanishing multiweight can be considered as collections of roots in of $\mathfrak{U}_n$ whose sum vanishes. This notion makes sense for any root system.

2.3. Example. The Hilbert series for $\mathfrak{A}(\mathbb{SL}_3/B)$ is $(1, 0, 3, 2, 3, 0, 1)$. (We assume that $\mathfrak{A}(\mathbb{SL}_n/B)$ contains constants.) Its degree 2 component is spanned by $a_{1,2} \wedge \bar{a}_{1,2}, a_{1,3} \wedge \bar{a}_{1,3}, a_{2,3} \wedge \bar{a}_{2,3}$; degree 3 component is spanned by $a_{1,2} \wedge \alpha_{2,3} \wedge \bar{a}_{1,3}, \bar{a}_{1,2} \wedge \bar{\alpha}_{2,3} \wedge a_{1,3}$; degree 4 component is spanned by $a_{1,2} \wedge \bar{a}_{1,2} \wedge a_{1,3} \wedge \bar{a}_{1,3}, a_{1,2} \wedge \bar{a}_{1,2} \wedge a_{2,3} \wedge \bar{a}_{2,3}, a_{1,3} \wedge \bar{a}_{1,3} \wedge a_{2,3} \wedge \bar{a}_{2,3}$ and, finally, its degree 6 component is spanned by $a_{1,2} \wedge \bar{a}_{1,2} \wedge a_{1,3} \wedge \bar{a}_{1,3} \wedge a_{2,3} \wedge \bar{a}_{2,3}$. The Hilbert series for $\mathfrak{A}(\mathbb{SL}_4/B)$ is $(1, 0, 6, 8, 21, 24, 16, 24, 21, 8, 6, 0, 1)$.

Recall that *an Eulerian digraph* is a directed graph such that the numbers of entering and leaving edges at each vertex are equal (loops are not allowed).

The following proposition is a relatively simple reformulation of 2.2.

2.4. Proposition. *The dimension of $\mathfrak{A}(\mathbb{SL}_n/B)$ as a $\mathbb{Z}$-module equals the number of Eulerian digraphs on n labeled vertices.*

On curvature forms. Using the above desription of $\mathbb{U}_n$-invariant forms on $\mathbb{SL}_n/B$ one can derive easily the following proposition from the results of [GS], see also [Ta2].

2.5. Proposition. *The curvature 2-form w_i, $i = 1, ..., n$, of the tautologic line bundle $L_i = E_i/E_{i-1}$ over $\mathbb{SL}_n/B$ equals up to a constant factor to the* *sum of all entries in the ith row of the following matrix of 2-forms:*

$$
\left(\begin{array}{ccccc}
0 & \gamma_{1,2} & \gamma_{1,3} & \dots & \gamma_{1,n} \\
-\gamma_{1,2} & 0 & \gamma_{2,3} & \dots & \gamma_{2,n} \\
-\gamma_{1,3} & -\gamma_{2,3} & 0 & \dots & \gamma_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\gamma_{1,n} & -\gamma_{2,n} & -\dots & -\gamma_{n-1,n} & 0
\end{array}\right), \tag{3}
$$

where $\gamma_{i,j} = \alpha_{i,j} \wedge \bar{\alpha}_{i,j}$.

Now we are ready to start proving the main Proposition 1.1. Consider the ring $\mathcal{A}(w_{i_1}, ..., w_{i_k})$ generated by $w_{i_1}, ..., w_{i_k}$. Since all w_j are $\mathbb{U}_n$-invariant, one has that $\mathcal{A}(w_{i_1}, ..., w_{i_k})$ is a subring of the exterior algebra $\Lambda(\mathbb{C}^{\binom{n}{2}})$.

2.6. Lemma. *There exists a natural S_n-action on the set of all permutations $(w_{i_1}, ..., w_{i_n})$ of $w_1, ..., w_n$.*

Proof. The elementary transposition τ_i, $i = 1, ..., n-1$, acts on the matrix (3) of 2-forms by the simultaneous interchanging of 1) the ith row with the $(i+1)$st row, 2) the ith column with the $(i+1)$st column and 3) changing sign of $\gamma_{i,i+1}$. One can easily check that this determines the required S_n-action. $\square$

2.7. Corollary. *All rings $\mathcal{A}(w_{i_1}, ..., w_{i_k})$ are isomorphic to each other and, in particular, to $\mathcal{A}(w_1, ..., w_k)$.*

We denote this class of isomorphic algebras by $\mathcal{A}_{k,n}$.

2.8. Lemma. *$\mathcal{A}_{n-1,n}$ is isomorphic to $\mathcal{A}_n = \mathcal{A}_{n,n}$.*

Proof. Indeed, one has $w_1 + w_2 + ... + w_n = 0$. $\square$

The *vanishing ideal* $I_{k,n}$ of $\mathcal{A}_{k,n}$ is the set of all polynomials $p \in \mathbb{Z}[x_1, ..., x_k]$ that vanish if we substitute the the curvature forms $w_1, ..., w_k$ for the variables $x_1, ..., x_k$ resp.

2.9. Lemma. *The vanishing ideal $I_{k,n}$ consists of all $p \in \mathbb{Z}[x_1, ..., x_k]$ such that 2^n derivatives $p, \frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, ..., \frac{\partial p}{\partial x_k}, \frac{\partial^2 p}{\partial x_1 \partial x_2}, ..., \frac{\partial^2 p}{\partial x_{k-1} \partial x_k}, ..., \frac{\partial^k p}{\partial x_1 \partial x_2 ... \partial x_k}$ belong to $I_{k,n-1}$. The derivatives are taken with respect to all subsets $\{i_1, ..., i_l\} \subset \{1, ..., n\}$ of indices (no repetitions of indices are allowed) including the empty set.*

Proof. Notice that $\mathcal{A}_{k,n}$ has a natural module structure over $\mathcal{A}_{k,n-1}$. Indeed, one has $w_i^{(n)} = w_i^{(n-1)} + \gamma_{i,n}$, where the upper index shows the dimension of the initial space, see (3). Therefore, for any polynomial $p \in \mathbb{Z}[x_1, ..., x_k]$ one has, after the substitution of the curvature forms, the following:

$$
\begin{aligned}
(4)\quad & p(w_1^{(n)}, ..., w_k^{(n)}) = p(w_1^{(n-1)} + \gamma_{1,n}, ..., w_k^{(n-1)} + \gamma_{k,n}) = \\
& p(w_1^{(n-1)}, ..., w_k^{(n-1)}) + p_{x_1}(w_1^{(n-1)}, ..., w_k^{(n-1)})\gamma_{1,n} + ... + p_{x_k}(w_1^{(n-1)}, ..., \\
& \qquad w_k^{(n-1)})\gamma_{k,n} + ... + p_{x_1,x_2,...,x_k}(w_1^{(n-1)}, ..., w_k^{(n-1)})\gamma_{1,n}\gamma_{2,n}\cdots\gamma_{k,n}.
\end{aligned}
$$

Since $\gamma_{i,n}^2 = 0$, the monomials in the right hand side are square-free, and the coefficient at the product $\gamma_{i_1,n}...\gamma_{i_l,n}$ equals $\frac{\partial^l p}{\partial x_{i_1}...\partial x_{i_l}}$. Finally, the condition $p \in I_{k,n}$, i.e. $p(w_1^{(n)},...,w_k^{(n)}) = 0$, is equivalent to vanishing of all polynomial coefficients $p_{x_{i_1},...,x_{i_l}}(w_1^{(n-1)},...,w_k^{(n-1)})$ in the right hand side of (4). By definition, this means that $p_{x_{i_1},...,x_{i_l}}(x_1,...,x_k) \in I_{k,n-1}$. □

Denote $\mathcal{D}_{i_1,...,i_l} = \frac{\partial^l}{\partial x_{i_1}...\partial x_{i_l}}$, and let $V_{i_1,...,i_l,r,m,k}$ be the space of all homogeneous polynomials with integer coefficients of degree $m+r$ in k variables of the form $p = (x_{i_1} + ...x_{i_l})^r f$, where $\deg(f) = m$.

2.10. Lemma. *The linear map* $\mathcal{D}_{i_1,...,i_l} : V_{i_1,...,i_l,r+l,m+k-l,k} \to V_{i_1,...,i_l,r,m,k}$ *is a surjection.*

Proof. Simple linear algebra. □

Proof of the main Proposition 1.1. We use a double induction on $k \leq n$, i.e. for a given k we apply induction on n and then cover the first case $\mathcal{A}_{k+1,k+1}$ for $k+1$ using Lemma 2.8.

Base of induction. $\mathcal{A}_{1,n} = \frac{\mathbb{Z}[x_1]}{(x_1^n)}$. Indeed, $w_1 = \gamma_{1,n} + \gamma_{1,3} + ...\gamma_{1,n}$, where $\gamma_{1,i}$ are independent commuting variables satisfying $\gamma_{1,i}^2 = 0$. The statement follows.

Step of induction. Assume that 1.1 is true for all pairs (k', n') with $k' < k$ and for all (k, n') with $n' < n$. Let us show that it holds for the pair (k, n). Notice that all polynomials of the form (1) are contained in $I_{k,n}$. This can be either checked directly or by induction using 2.9. Let us temporarily denote by $\tilde{I}_{k,n}$ the ideal generated by all polynomials in (1). We have to show that $I_{k,n} = \tilde{I}_{k,n}$. By the above remark, $\tilde{I}_{k,n} \subset I_{k,n}$. Take any $p \in I_{k,n}$. Using Lemma 2.10 we will step by step add to p certain polynomials that lie in $\tilde{I}_{k,n}$ in such a way that all the derivatives of the resulting polynomial p_{fin} (that enter the formulation of Lemma 2.9) vanish. Consider $\mathcal{D}_{1,...,k}(p)$. By 2.9 it belongs to $I_{k,n-1}$. By the inductive assumption $I_{k,n-1}$ is generated by $g_{i_1,...,i_l}^{(n-1)}$, see (1). Therefore, one has $\mathcal{D}_{1,...,k}(p) = \sum_{i_1,...,i_l}(x_{i_1} + ... + x_{i_l})^{j(n-j-1)+1} h_{i_1,...,i_l}$. By Lemma 2.10 for each $\phi = (x_{i_1} + ... + x_{i_l})^{j(n-j-1)+1} h_{i_1,...,i_l}$ there exists (but nonunique!) $\psi = (x_{i_1} + ... + x_{i_l})^{j(n-j)+1} H_{i_1,...,i_l}$ such that $\mathcal{D}_{i_1,...,i_l}(\psi) = \phi$. Notice that ψ belongs to $\tilde{I}_{k,n}$ by defintion. Therefore, subtracting from p an appropriate polynomial belonging to $\tilde{I}_{k,n}$ we get $\tilde{p} \in I_{k,n}$ such that $\mathcal{D}_{1,...,k}(\tilde{p}) = 0$. Consider now any operator $\mathcal{D}_{1,...,\hat{i},...,k} = \frac{\partial^k}{\partial x_1...\widehat{\partial x_i}...\partial x_k}$ and apply it to $\tilde{p}$. One sees that $\mathcal{D}_{1,...,\hat{i},...,k}(\tilde{p})$ does not depend on x_i since $\mathcal{D}_{1,...,k}(\tilde{p}) = 0$. Therefore, using the same argument as above, we can subtract from $\tilde{p}$ some polynomial belonging to $\tilde{I}_{k,n}$ which does not depend on x_i and such that $\mathcal{D}_{1,...,\hat{i},...,k}$ applied to the resulting difference vanishes. Let us apply this procedure for all $i = k, k-1, \ldots, 1$. Notice that each step does not affect the vanishing of all $\mathcal{D}_{1,...,\hat{j},...,k}$, $j < i$, applied to the polynomial obtained on the current step, since each time we add a function which does not depend on x_i. Having obtained in this way the modified p such that all $\mathcal{D}_{1,...,\hat{i},...,k}$, $i \leq k$ applied to it vanish, we can proceed with all $\mathcal{D}_{1,\dots,\hat{i},\dots,\hat{j},\dots,k}$ (ordering them e.g. lexicographically). Our assumptions imply that $\mathcal{D}_{1,\dots,\hat{i},\dots,\hat{j},\dots,k}$ applied to our polynomial does not depend on x_i and x_j. Therefore we can subtract a polynomial lying in $\tilde{I}_{k,n}$ that does not depend on both x_i and x_j and such that $\mathcal{D}_{1,\dots,\hat{i},\dots,\hat{j},\dots,k}$ applied to the resulting difference vanish. Using this procedure consecutively we do not change all the derivatives considered so far, since the polynomials we subtract depend on different groups of variables. Continuing in the same manner we get some polynomial p_{fin} whose all derivatives mentioned in 2.9 vanish. This means that p_{fin} is a constant. But since constants distinct from zero do not lie in $I_{k,n}$, we have that p_{fin} equals 0. The statement follows. $\square$

§3. Some related problems

Problem 1. Give a presentation of the obvious analog of $\mathcal{A}_n$ for any $\mathbb{SL}_n/P$ generated by the standard $\mathbb{U}_n$-invariant forms representing the Chern classes of the tautological (quotient) bundles.

Remark. Notice that in the case of Grassmanian $G_{k,n}$ the analogous algebra coincides with $H^*(G_{k,n})$, since $G_{k,n}$ is a symmetric space and therefore has no nontrivial left-invariant forms homologous to 0, see e.g. [St].

Problem 2. Determine the S_n-module structure for $\mathcal{A}_n$ and its $\mathbb{SL}_n/P$-analogs.

Problem 3. Calculate the Poincare series for the ring $\mathfrak{A}(\mathbb{SL}_n/P)$ of $\mathbb{U}_n$-invariant forms on $\mathbb{SL}_n/P$.

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