A NEW MULTIDIMENSIONAL MATRIX INVERSE WITH APPLICATIONS TO MULTIPLE q-SERIES

(Extended abstract)

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Abstract. We compute the inverse of a specific infinite r-dimensional matrix, extending a matrix inverse of Krattenthaler. Our inversion is different from the r-dimensional matrix inversion recently found by Schlosser but generalizes a multidimensional matrix inversion previously found by Chu. As applications of our matrix inversion we derive some multidimensional q-series identities. Among these are q-analogues of Carlitz's multidimensional Abel-type expansion formulas. Furthermore, we derive a q-analogue of MacMahon's Master Theorem.

Resumé. Nous calculons l'inverse d'une certaine matrice de dimension r par généraliser une matrice d'inverse de Krattenthaler. Notre inversion est différent de l'inversion d'une matrice de dimension r récemment trouvée par Schlosser mais elle généralise une inversion d'une matrice multidimensionelle précédemment trouvée par Chu. Comme application de notre inversion d'une matrice nous dérivons quelques q-séries identités multidimensionelles. Parmi lesquelles il y a les q-analogues de Carlitz's Abel-type formules multidimensionelles. De plus, nous dérivons un q-analogue de MacMahon's Master Theorem.

1. Introduction

Matrix inversions are very important tools in combinatorics and special functions theory. In particular, it is a widely spread and often used method to derive and prove identities for (basic) hypergeometric series with the help of so-called "inverse relations" (see Section 3), which are immediate consequences of matrix inversions.

In a recent paper [33], a great deal of unification for "one-dimensional" matrix inversions was achieved by the first author. There it was proved that the matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ ($\mathbb{Z}$ denotes the set of integers), are inverses of each other, where

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j + b_j c_k)}{\prod_{j=k+1}^{n}(c_j - c_k)}, \tag{1.1}$$

and

$$g_{kl} = \frac{(a_l + b_l c_l)}{(a_k + b_k c_k)} \frac{\prod_{j=l+1}^{k}(a_j + b_j c_k)}{\prod_{j=l}^{k-1}(c_j - c_k)}. \tag{1.2}$$

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This matrix inversion contains a vast number of previously known matrix inversions [1, 4, 5, 17, 18, 20, 25, 42] as special cases.

A similar unification for "multidimensional" matrix inversions (according to our terminology these are matrix inversions that arise in the theory of multiple series) that arise in the theory of multiple basic hypergeometric series *associated to root systems* was achieved by the second author in [43]. Theorems 3.1 and 4.1 of [43] cover all previously discovered matrix inversions [3, 36, 39] in that area as special cases. In particular, these matrix inversions also contain the inversion (1.1)/(1.2) as one-dimensional special case.

The main result of this extended abstract is another multidimensional extension of the matrix inverse (1.1)/(1.2) (see Theorem 2.1). This matrix inversion finds its applications in the theory of "ordinary" multiple series. It does not "belong", as far as we can tell, to the theory of multiple series associated to root systems. Also here, special cases of this matrix inversion appeared earlier in the literature. Aside from reducing to (1.1)/(1.2), the matrix inversion from [33], in the one-dimensional case, it also contains Chu's multidimensional matrix inversion [9] and a two-dimensional matrix inversion [32] by the first author. We demonstrate the usefulness of our new multidimensional matrix inverse by deriving several multidimensional q-series identities, among them some new basic hypergeometric double summations, and a multidimensional extension of a very-well-poised ${}_{10}\phi_9$-summation. Further, we derive some q-analogues of Carlitz's multidimensional Abel-type expansion formulas [6, 7, 8] (these q-analogues are new even in the one-dimensional case), related multiple q-Abel and q-Rothe summations, and, for the first time, a (noncommutative) q-analogue of MacMahon's Master Theorem (see Theorem 7.2).

The latter may be of relevance to a question raised by Clarke and Foata [12]. Namely, these authors ask for a suitable q-analogue of MacMahon's Master Theorem to be able to show algebraically that the joint distribution of "exc" and "den" (excedance and Denert statistics, see [11, 13, 15, 26] for more information) on words is Euler-Mahonian, i.e. equidistributed with the joint distribution of "des" and "maj" (descent and major statistics). The question whether Theorem 7.2 is appropriate for the above task is currently being investigated. It is also possible that an approach similar to the one which lead to Theorem 7.2 may yield a different q-analogue of MacMahon's Master Theorem which then would answer Clarke and Foata's question.

For the proofs of our theorems and more details we refer the reader to our preprint [34].

2. A multidimensional matrix inversion

Let $F=(f_{\mathbf{n}\mathbf{k}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ ($\mathbb{Z}$ denotes the set of integers) be an infinite lower-triangular r-dimensional matrix; i.e., $f_{\mathbf{n}\mathbf{k}}=0$ unless $\mathbf{n}\ge\mathbf{k}$, by which we mean $n_i\ge k_i$ for all $i=1,\dots,r$. The matrix $G=(g_{\mathbf{k}\mathbf{l}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ is said to be the *inverse matrix* of F if and only if

$$\sum_{\mathbf{n}\ge\mathbf{k}\ge\mathbf{l}} f_{\mathbf{n}\mathbf{k}}g_{\mathbf{k}\mathbf{l}}=\delta_{\mathbf{n}\mathbf{l}}$$

for all $\mathbf{n},\mathbf{l}\in\mathbb{Z}^r$, where $\delta_{\mathbf{n}\mathbf{l}}$ is the usual Kronecker delta.

The main results of this extended abstract are the following two theorems.

Theorem 2.1. *Let $(a_i(t))_{t\in\mathbb{Z}}$, $(b_{ij}(t))_{t\in\mathbb{Z}}$, and $(c_i(t))_{t\in\mathbb{Z}}$, $i,j=1,\dots,r$, be arbitrary sequences such that $c_i(s)\ne c_i(t)$ for $s\ne t$. Then $(f_{\mathbf{n}\mathbf{k}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{k}\mathbf{l}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{n}\mathbf{k}}=\prod_{i=1}^r\frac{\prod_{t_i=k_i}^{n_i-1}\left(a_i(t_i)+\sum_{j=1}^r b_{ij}(t_i)c_j(k_j)\right)}{\prod_{t_i=k_i+1}^{n_i}\left(c_i(t_i)-c_i(k_i)\right)} \tag{2.1}$$

and

$$g_{\mathbf{kl}} = \frac{\det\limits_{1\le i,j\le r}\Big((a_i(l_i)+\sum_{s=1}^r b_{is}(l_i)c_s(k_s))\delta_{ij}+b_{ij}(l_i)(c_i(l_i)-c_i(k_i))\Big)}{\prod\limits_{i=1}^r (a_i(k_i)+\sum_{j=1}^r b_{ij}(k_i)c_j(k_j))} \times \prod_{i=1}^r \frac{\prod\limits_{t_i=l_i+1}^{k_i}(a_i(t_i)+\sum_{j=1}^r b_{ij}(t_i)c_j(k_j))}{\prod\limits_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_i(k_i))}. \tag{2.2}$$

Remark 2.2. For the proofs of this Theorem we use the operator method of [31] extended with a multidimensional corollary [43, Corollary 2.4] (see [34] for the details).

Remark 2.3. For $r=1$, Theorem 2.1 reduces to the first author's matrix inverse (1.1)/(1.2). The special case $c_i(t)=t$, $i=1,\dots,r$, is equivalent to Chu's [9, Eqs. (2.3)/(2.4)] matrix inversion result. Setting $r=2$, $c_i(t)=q^t$, $a_i(t)=0$, $i=1,2$,

$$\Big(b_{ij}(t_i)\Big)_{1\le i,j\le 2} = \begin{pmatrix} -1 & q^{\mu+t_1} \\ q^{\lambda+t_2} & -1 \end{pmatrix},$$

and simplifying a bit, we recover the first author's two-dimensional inversion [32, (4.15)/(4.16)], which was used there to derive many two-dimensional expansion formulas.

The following Theorem is a "companion" result to Theorem 2.1.

Theorem 2.4. *Assume the conditions of Theorem 2.1. Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}} = \frac{\det\limits_{1\le i,j\le r}\Big((a_i(n_i)+\sum_{s=1}^r b_{is}(n_i)c_s(k_s))\delta_{ij}+b_{ij}(n_i)(c_i(n_i)-c_i(k_i))\Big)}{\prod\limits_{i=1}^r (a_i(k_i)+\sum_{j=1}^r b_{ij}(k_i)c_j(k_j))} \times \prod_{i=1}^r \frac{\prod\limits_{t_i=k_i}^{n_i-1}(a_i(t_i)+\sum_{j=1}^r b_{ij}(t_i)c_j(k_j))}{\prod\limits_{t_i=k_i+1}^{n_i}(c_i(t_i)-c_i(k_i))} \tag{2.3}$$

and

$$g_{\mathbf{kl}} = \prod_{i=1}^r \frac{\prod\limits_{t_i=l_i+1}^{k_i}(a_i(t_i)+\sum_{j=1}^r b_{ij}(t_i)c_j(k_j))}{\prod\limits_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_i(k_i))}. \tag{2.4}$$

Remark 2.5. The special case $c_i(t)=t$, $i=1,\dots,r$, is equivalent to Chu's [9, Eq. (2.9)/(2.10)] companion matrix inversion result.

3. Preliminaries on inverse relations and basic hypergeometric notation

Here we introduce the basic concept of "inverse relations" and introduce some standard q-series notation.

There is a standard technique for deriving new summation formulas from known ones by using inverse matrices (cf. [1], [3], [10], [17], [18], [19, Sec. 3.8], [20], [21], [33], [36], [38], [39],

[40], [41], [42], [43]). If $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are lower triangular matrices being inverses of each other, then of course the following is true:

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} f_{\mathbf{nk}}a_{\mathbf{k}} = b_{\mathbf{n}} \qquad \textit{if and only if} \qquad \sum_{\mathbf{0}\le\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}}b_{\mathbf{l}} = a_{\mathbf{k}}. \tag{3.1}$$

If one side of (3.1) is known, then the other produces another summation formula. The less used dual version, the so-called "rotated inversion", can be used to derive nonterminating summations. It reads

$$\sum_{\mathbf{k}\le\mathbf{n}\le\boldsymbol{\infty}} f_{\mathbf{nk}}a_{\mathbf{n}} = b_{\mathbf{k}} \qquad \textit{if and only if} \qquad \sum_{\mathbf{l}\le\mathbf{k}\le\boldsymbol{\infty}} g_{\mathbf{kl}}b_{\mathbf{k}} = a_{\mathbf{l}}, \tag{3.2}$$

subject to suitable convergence conditions. Again, if one side of (3.2) is known, the other produces a possibly new identity.

In the subsequent sections we use special cases of our Theorems 2.1 and 2.4 to derive a couple of higher dimensional summations for q-series.

Before we start to present some applications of our Theorems, we need to recall the standard basic hypergeometric notation (cf. [19]). Let q be a complex number such that $|q|<1$. Define

$$(a;q)_\infty := \prod_{j\ge 0}(1-aq^j), \tag{3.3}$$

and,

$$(a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}. \tag{3.4}$$

For multidimensional series, we also employ the notation $|\mathbf{k}|$ for $(k_1+\dots+k_r)$ where $\mathbf{k} = (k_1,\dots,k_r)$. Concerning the nonterminating multiple series given in this paper, we have stated their regions of convergence explicitly. The convergence of these series can be checked by application of the multiple power series ratio test [27, 30].

4. A "twisted" multidimensional extension of a very-well-poised ${}_{10}\phi_9$-summation

Special cases of our matrix inversions in Theorems 2.1 and 2.4 can be used to obtain multidimensional extensions of various summations for (one-dimensional) basic hypergeometric series. For illustration, we give one such example below.

Theorem 4.1. *Let a_i and A_i be indeterminates, $i=1,2,\dots,r$. Then*

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \frac{\left(\prod_{i=1}^r A_i(1-a_iq^{n_i+k_{i+1}})(1-a_iq^{n_i-k_{i+1}}/A_{i+1}) - \prod_{i=1}^r a_i(1-A_iq^{n_i+k_i})(1-q^{n_i-k_i})\right)}{\prod_{i=1}^r (1-a_iq^{k_i+k_{i+1}})(A_{i+1}q^{k_{i+1}}-a_iq^{k_i})}$$
$$\cdot\prod_{i=1}^r q^{2k_i}\left(\frac{a_i^2q^{n_i}}{A_iA_{i+1}}\right)^{k_i-k_{i+1}} \frac{(1-q^{2k_i}A_i)}{(1-A_i)}\frac{(A_i;q)_{k_i}\,(q^{-n_i};q)_{k_i}}{(q;q)_{k_i}\,(q^{1+n_i}A_i;q)_{k_i}}$$
$$\cdot\frac{(q^{n_i}a_i;q)_{k_{i+1}}\,(qa_i/A_i;q)_{k_{i+1}}\,(A_iA_{i+1}/a_i;q)_{k_i+k_{i+1}}\,(A_i/a_i;q)_{k_i-k_{i+1}}}{(q^{1-n_i}A_{i+1}/a_i;q)_{k_{i+1}}\,(A_iA_{i+1}/a_i;q)_{k_{i+1}}\,(a_i;q)_{k_i+k_{i+1}}\,(a_i/A_{i+1};q)_{k_i-k_{i+1}}}$$
$$= \prod_{i=1}^r \frac{(qa_i^2/A_iA_{i+1};q)_{n_i}\,(qA_i;q)_{n_i}}{(a_i;q)_{n_i}\,(a_i/A_{i+1};q)_{n_i}}, \tag{4.1}$$

where, by convention, $A_{r+1}=A_1$ and $k_{r+1}=k_1$.

Remark 4.2. The special case $r=1$ of (4.1) can be rewritten as (when writing a for a_1, A for A_1, and n for n_1)

$$ {}_{10}\phi_9\left[\begin{matrix} A, \sqrt{A}q, -\sqrt{A}q, A/\sqrt{a}, -A/\sqrt{a}, A\sqrt{q}/\sqrt{a}, -A\sqrt{q}/\sqrt{a}, aq/A, aq^n, q^{-n} \\ \sqrt{A}, -\sqrt{A}, \sqrt{a}q, -\sqrt{a}q, \sqrt{aq}, -\sqrt{aq}, A^2/a, Aq^{1-n}/a, Aq^{1+n} \end{matrix}; q, q\right] \\ = \frac{(1-a)}{(1-aq^{2n})} \frac{(qa^2/A^2;q)_n\,(qA;q)_n}{(a;q)_n\,(a/A;q)_n}. \quad (4.2) $$

This ${}_{10}\phi_9$-summation (for the standard notation of an ${}_s\phi_t$-series, see [19]) can be obtained by specializing Bailey's transformation formula [2] (see [19, (2.8.5); Appendix (III.27)]; the specializations that have to be performed there are $d=1$, $b\to a^2q/cA$, $a\to A^2/aq$, in that order).

5. q-Analogues of Carlitz's Abel-type expansion formulas

In [7], Carlitz gave multidimensional extensions of Euler's formulas

$$ e^{AZ} = \sum_{k=0}^{\infty} \frac{A(A+Bk)^{k-1}}{k!} Z^k e^{-BZk}, \quad (5.1) $$

where $\left|BZe^{1-BZ}\right| < 1$ [14, p. 354] (cf. [42, Sec. 4.5]), and

$$ (1+Z)^A = \sum_{k=0}^{\infty} \frac{A}{A+Bk}\binom{A+Bk}{k} Z^k (1+Z)^{-Bk}, \quad (5.2) $$

where $\left|\frac{(B-1)Z}{(1+Z)^B}\right| < 1$ [14, p. 350] (cf. [42, Sec. 4.5]). The purpose of this section is to present q-analogues of (5.1) and (5.2) and to derive q-analogues of Carlitz's multidimensional extensions thereof.

First we derive a simple multidimensional extension of the expansion formula

$$ 1 = \sum_{k=0}^{\infty} \frac{(a+b)(a+bq^k)^{k-1}}{(q;q)_k} (z(a+bq^k);q)_\infty\, z^k, \quad (5.3) $$

valid for $|az|<1$, which is a q-analogue of Euler's formula (5.1).

To see that (5.3) is a q-analogue of (5.1), do the replacements $a\to 1-q^A+\frac{1-q^B}{1-q}$, $b\to -\frac{1-q^B}{1-q}$, $z\to Z$ and then let $q\to 1$. In this case, $\lim_{q\to 1}\frac{a+bq^k}{1-q} = A+Bk$. Also, recall that $\lim_{q\to 1}((1-q)Z;q)_\infty = e^{-Z}$.

The second formula that we extend to several dimensions is

$$ (z;q)_\infty = \sum_{k=0}^{\infty} \frac{1-(a+b)}{1-(aq^{-k}+b)} \frac{(aq^{-k}+b;q)_k}{(q;q)_k} (-1)^k q^{\binom{k}{2}} (z(a+bq^k);q)_\infty\, z^k, \quad (5.4) $$

valid for $|az|<1$, which in turn is a q-analogue of Euler's formula (5.2).

To see that (5.4) is a q-analogue of (5.2), do the replacements $a\to q^A-\frac{1-q^B}{1-q}$, $b\to \frac{1-q^B}{1-q}$, $z\to -Z$ and then let $q\to 1$. In this case, $\lim_{q\to 1}\frac{1-(aq^{-k}+b)q^j}{1-q} = A+Bk+j-k$. Furthermore, we use $\lim_{q\to 1}\frac{(z(a+bq^k);q)_\infty}{(z;q)_\infty} = (1+Z)^{-A-Bk}$. Similar limiting processes apply for the multidimensional identities given in the following sections.

Now we state our multiple extension of (5.3):

Theorem 5.1. *Let a_i, b_{ij}, z_i, $i,j=1,\dots,r$, be indeterminate. Then there holds*

$$1=\sum_{k_1,\dots,k_r=0}^{\infty}\Bigg(\det_{1\le i,j\le r}\left(\left(a_i+\textstyle\sum_{s=1}^r b_{is}q_s^{k_s}\right)\delta_{ij}+b_{ij}(1-q_i^{k_i})\right)$$
$$\times\prod_{i=1}^r\frac{\left(a_i+\sum_{j=1}^r b_{ij}q_j^{k_j}\right)^{k_i-1}}{(q_i;q_i)_{k_i}}\left(z_i(a_i+\textstyle\sum_{j=1}^r b_{ij}q_j^{k_j});q_i\right)_\infty z_i^{k_i}\Bigg),\qquad(5.5)$$

provided $|a_iz_i|<1$ for $i=1,\dots,r$.

Remark 5.2. For the proof of this Theorem we apply the "rotated" inverse relations (3.2) and Theorem 2.1 in conjunction with an r-fold q-exponential summation.

Remark 5.3. The expansion (5.5) is a q-analogue of Carlitz's formula [7, Eq. (3.5)] which he derived via MacMahon's Master Theorem. To obtain his result we would have to do the replacements $a_i\to 1-q_i^{A_i}+\sum_{j=1}^r\frac{1-q_i^{B_{ij}}}{1-q_i}$, $b_{ij}\to-\frac{1-q_i^{B_{ij}}}{1-q_i}$, and then let $q_i\to 1$ for $i=1,\dots,r$ (compare with our observation concerning equation (5.3)).

Theorem 5.4. *Let a_i, b_{ij}, z_i, $i,j=1,\dots,r$, be indeterminate. Then there holds*

$$\prod_{i=1}^r(z_i;q_i)_\infty=\sum_{k_1,\dots,k_r=0}^{\infty}\Bigg(\frac{\det_{1\le i,j\le r}\left(\left(a_i-1+\sum_{s=1}^r b_{is}q_s^{k_s}\right)\delta_{ij}+b_{ij}(1-q_i^{k_i})\right)}{\prod_{i=1}^r\left(a_i-q_i^{k_i}+\sum_{j=1}^r b_{ij}q_j^{k_j}\right)}$$
$$\times\prod_{i=1}^r\frac{\left((a_i+\sum_{j=1}^r b_{ij}q_j^{k_j})q_i^{-k_i};q_i\right)_{k_i}}{(q_i;q_i)_{k_i}}q_i^{\binom{k_i+1}{2}}\cdot(-1)^{|\mathbf{k}|}$$
$$\times\prod_{i=1}^r\left(z_i(a_i+\textstyle\sum_{j=1}^r b_{ij}q_j^{k_j});q_i\right)_\infty z_i^{k_i}\Bigg),\qquad(5.6)$$

provided $|a_iz_i|<1$ for $i=1,\dots,r$.

Remark 5.5. For the proof of this Theorem we apply the "rotated" inverse relations (3.2) and Theorem 2.1 in conjunction with an r-fold ${}_1\phi_1$ summation.

Remark 5.6. The expansion (5.6) is a q-analogue of Carlitz's formula [7, Eq. (6.5)] which he also derived via MacMahon's Master Theorem. To obtain his result we would have to do the replacements $a_i\to q_i^{A_i}-\sum_{j=1}^r\frac{1-q_i^{B_{ij}}}{1-q_i}$, $b_{ij}\to\frac{1-q_i^{B_{ij}}}{1-q_i}$, $z_i\to -Z_i$, and then let $q_i\to 1$ for $i=1,\dots,r$ (compare with our observation concerning equation (5.4)).

Theorem 5.7. *Let b_i, x_i, $i=1,\dots,r$, a and z be indeterminate. Then there holds*

$$\prod_{i=1}^r(zx_i;q)_\infty=\sum_{k_1,\dots,k_r=0}^{\infty}\Bigg(\prod_{1\le i<j\le r}\left(\frac{q^{-k_i}/x_i-q^{-k_j}/x_j}{1/x_i-1/x_j}\right)(-1)^{|\mathbf{k}|}q^{\sum_{i=1}^r\binom{k_i}{2}}$$
$$\times\left(1-\sum_{i=1}^r\frac{b_i(1-q^{k_i})}{(x_i-a-\sum_{j=1}^r b_jq^{k_j})}\right)\prod_{i=1}^r\frac{\left((a+\sum_{j=1}^r b_jq^{k_j})q^{1-k_i}/x_i;q\right)_{k_i}}{(q;q)_{k_i}}$$
$$\times\prod_{i=1}^r\left(zq^{1-i}(a+\textstyle\sum_{j=1}^r b_jq^{k_j});q\right)_\infty x_i^{k_i}\cdot z^{|\mathbf{k}|}\Bigg),\qquad(5.7)$$

provided $|azq^{1-r}|<1$.

Remark 5.8. For the proof of this Theorem we apply the "rotated" inverse relations (3.2) and Theorem 2.1 in conjunction with a particular A_r ${}_1\phi_1$ summation.

Remark 5.9. To the authors' knowledge, the expansion (5.7) is not a q-analogue of any of the identities which have appeared in literature yet. In particular, it is of different type than Carlitz's formulas in [6, 7].

Of course, by specializing equation (5.7) we may also obtain an interesting formula for ordinary series. If we do the replacements $a \to q^A - \sum_{j=1}^r \frac{1-q^{B_j}}{1-q}$, $b_i \to \frac{1-q^{B_i}}{1-q}$, $x_i \to q^{A-A_i}$, $i = 1, \dots, r$, $z \to -Z$, then let $q \to 1$ and rewrite (compare with Remark 5.6), we obtain the following multidimensional generalization of (5.2):

Theorem 5.10. *Let A_i, B_i, $i = 1, \dots, r$, and Z be indeterminate. Then there holds*

$$(1+Z)^{\binom{r}{2}+\sum_{i=1}^r A_i} = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg(\prod_{1\le i<j\le r} \left(\frac{A_i - k_i - A_j + k_j}{A_i - A_j} \right) \cdot \left(1 - \sum_{i=1}^r \frac{B_i k_i}{(A_i + \sum_{j=1}^r B_j k_j)} \right) \\ \times \prod_{i=1}^r \binom{A_i + \sum_{j=1}^r B_j k_j}{k_i} Z^{k_i} (1+Z)^{-B_i k_i} \Bigg), \quad (5.8)$$

provided $\left| \frac{(B_i - 1)Z}{(1+Z)^{B_i}} \right| < 1$ *for* $i = 1, \dots, r$.

6. Multiple q-Abel and q-Rothe summations

We can use the expansions (5.3) and (5.4) to obtain terminating q-Abel and q-Rothe summations, respectively (for q-summations of this type, also see [3, 28, 29]). Later, we apply the same method to derive multidimensional generalizations of these formulas.

First, we apply the q-binomial theorem in the form

$$(z(a+bq^k);q)_\infty = (z;q)_\infty \sum_{j=0}^{\infty} \frac{(a+bq^k;q)_j}{(q;q)_j} z^j$$

to the right-hand side of (5.3) and move $(z;q)_\infty$ to the left-hand side of (5.3). Next, by equating coefficients of $z^n/(q;q)_n$ in the resulting identity (again making use of the q-binomial theorem), and employing the q-binomial coefficient notation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

for nonnegative integers $k \le n$ (cf. [19, Appendix (I.39)]), we arrive at the following terminating summation:

$$1 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (a+b)(a+bq^k)^{k-1} (a+bq^k;q)_{n-k} \quad (6.1)$$

(see [28]). This is a q-analogue of Abel's theorem

$$(A+C)^n = \sum_{k=0}^{n} \binom{n}{k} A(A+Bk)^{k-1}(C-Bk)^{n-k}. \quad (6.2)$$

For, (6.2) can easily be obtained from (6.1) via the substitutions $a \to \frac{1-q^{A/(A+C)}}{1-q} + \frac{1-q^{B/(A+C)}}{(1-q)^2}$, $b \to -\frac{1-q^{B/(A+C)}}{(1-q)^2}$, and then letting $q \to 1$.

On the other hand, if we iterate (5.4) $r-1$ times we get

$$(z;q)_\infty = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg(\prod_{i=1}^{r} \frac{1-(a_i+b_i)}{1-(a_iq^{-k_i}+b_i)} \frac{(a_iq^{-k_i}+b_i;q)_{k_i}}{(q;q)_{k_i}} (-1)^{k_i} q^{\binom{k_i}{2}} \\ \times \Big(z\prod_{i=1}^{r}(a_i+b_iq^{k_i});q\Big)_\infty z^{|\mathbf{k}|} \prod_{i=1}^{r} \Big(a_i+b_iq^{k_i}\Big)^{\sum_{j=i+1}^{r} k_j} \Bigg). \tag{6.3}$$

Now, after the following application of the q-binomial theorem

$$\Big(z\prod_{i=1}^{r}(a_i+b_iq^{k_i});q\Big)_\infty = (z/c;q)_\infty \sum_{j=0}^{\infty} \frac{\big(c\prod_{i=1}^{r}(a_i+b_iq^{k_i});q\big)_j}{(q;q)_j} \left(\frac{z}{c}\right)^j$$

to the right-hand side of (6.3) we may put $(z/c;q)_\infty$ to the left-hand side of (6.3). Next, by equating coefficients of $(z/c)^N$ (again making use of the q-binomial theorem), we arrive at the following terminating summation:

$$\frac{(c;q)_N}{(q;q)_N} = \sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le|\mathbf{k}|\le N}} \Bigg(\prod_{i=1}^{r} \frac{1-(a_i+b_i)}{1-(a_iq^{-k_i}+b_i)} \frac{(a_iq^{-k_i}+b_i;q)_{k_i}}{(q;q)_{k_i}} (-1)^{k_i} q^{\binom{k_i}{2}} \\ \times \frac{\big(c\prod_{i=1}^{r}(a_i+b_iq^{k_i});q\big)_{N-|\mathbf{k}|}}{(q;q)_{N-|\mathbf{k}|}} \prod_{i=1}^{r} \Big(a_i+b_iq^{k_i}\Big)^{\sum_{j=i+1}^{r} k_j} c^{|\mathbf{k}|} \Bigg). \tag{6.4}$$

Identity (6.4) may be viewed as a Gould-type generalization of the q-multinomial theorem. The case $r=1$,

$$\frac{(c;q)_n}{(q;q)_n} = \sum_{k=0}^{n} \frac{1-(a+b)}{1-(aq^{-k}+b)} \frac{(aq^{-k}+b;q)_k}{(q;q)_k} \frac{(c\,(a+bq^k);q)_{n-k}}{(q;q)_{n-k}} (-1)^k q^{\binom{k}{2}} c^k \tag{6.5}$$

(see [29]), is a q-analogue of the (Hagen-)Rothe summation formula [23]

$$\binom{A+C}{n} = \sum_{k=0}^{n} \frac{A}{A+Bk} \binom{A+Bk}{k} \binom{C-Bk}{n-k}, \tag{6.6}$$

for (6.6) can be obtained from (6.5) via the substitutions $a \to q^A - \frac{1-q^B}{1-q}$, $b \to \frac{1-q^B}{1-q}$, $c \to q^{-A-C}$, and then letting $q \to 1$.

Many more similar convolution formulas, in the $q=1$ case, are listed in [24] (also see [44]), whereas more q-Abel and q-Rothe summations can be found in [29], where these are derived by means of umbral calculus.

We start our multidimensional exposition with a multiple q-Abel summation:

Theorem 6.1. *Let a_i, b_{ij}, $i,j=1,\dots,r$, be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Then there holds*

$$1 = \sum_{0\le\mathbf{k}\le\mathbf{n}} \Bigg(\det_{1\le i,j\le r} \bigg(\Big(a_i+\sum_{s=1}^{r} b_{is}q_s^{k_s}\Big)\delta_{ij} + b_{ij}(1-q_i^{k_i}) \bigg) \\ \times \prod_{i=1}^{r} \begin{bmatrix} n_i\\ k_i \end{bmatrix}_{q_i} \Big(a_i+\sum_{j=1}^{r} b_{ij}q_j^{k_j}\Big)^{k_i-1} \Big(a_i+\sum_{j=1}^{r} b_{ij}q_j^{k_j};q_i\Big)_{n_i-k_i} \Bigg). \tag{6.7}$$

Our multidimensional q-Abel summation theorem is a q-analogue of Carlitz's formula [7, Eq. (3.8)] which basically can be obtained from (6.7) via the substitutions $a_i \to \frac{1-q^{A_i/(A_i+C_i)}}{1-q_i} + \sum_{j=1}^r \frac{1-q^{B_{ij}/(A_i+C_i)}}{(1-q_i)^2}$, $b_{ij} \to -\frac{1-q^{B_{ij}/(A_i+C_i)}}{(1-q_i)^2}$, and then letting $q_i \to 1$, for $i = 1, \dots, r$.

A multidimensional q-Abel summation of a different type is given in Bhatnagar and Milne [3].

Concerning the next two theorems, we could also have given multidimensional generalizations of the Gould-type q-multinomial convolution (6.4), but have decided to restrict ourselves to stating the special cases which are multiple q-Rothe summations:

Theorem 6.2. *Let a_i, b_{ij}, c_i, $i, j = 1, \dots, r$, be indeterminate, and let $n_1, \dots, n_r$ be nonnegative integers. Then there holds*

$$\prod_{i=1}^r \frac{(c_i;q_i)_{n_i}}{(q_i;q_i)_{n_i}} = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left(\frac{\det\limits_{1\le i,j\le r} \left(\left(a_i - 1 + \sum_{s=1}^r b_{is} q_s^{k_s}\right) \delta_{ij} + b_{ij}(1-q_i^{k_i}) \right)}{\prod\limits_{i=1}^r \left(a_i - q_i^{k_i} + \sum_{j=1}^r b_{ij} q_j^{k_j}\right)} \right.$$
$$\times \prod_{i=1}^r \frac{\left((a_i + \sum_{j=1}^r b_{ij} q_j^{k_j}) q_i^{-k_i}; q_i\right)_{k_i}}{(q_i;q_i)_{k_i}} q_i^{\binom{k_i+1}{2}} \cdot (-1)^{|\mathbf{k}|}$$
$$\left. \times \prod_{i=1}^r \frac{\left(c_i(a_i + \sum_{j=1}^r b_{ij} q_j^{k_j}); q_i\right)_{n_i-k_i}}{(q_i;q_i)_{n_i-k_i}} c_i^{k_i} \right). \quad (6.8)$$

Our multidimensional q-Rothe summation theorem is a q-analogue of Carlitz's formula [7, Eq. (6.10)] which basically can be obtained from (6.8) via the substitutions $a_i \to q^{A_i} - \sum_{j=1}^r \frac{1-q^{B_{ij}}}{1-q_i}$, $b_{ij} \to \frac{1-q^{B_{ij}}}{1-q_i}$, $c_i \to q_i^{-A_i-C_i}$, and then letting $q_i \to 1$, for $i = 1, \dots, r$.

Theorem 6.3. *Let b_i, c_i, x_i, $i = 1, \dots, r$, and a be indeterminate, and let $n_1, \dots, n_r$ be nonnegative integers. Then there holds*

$$\prod_{i=1}^r \frac{(c_i;q)_{n_i}}{(q;q)_{n_i}} = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left(\prod_{1\le i<j\le r} \left(\frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i - 1/x_j} \right) (-1)^{|\mathbf{k}|} q^{\sum_{i=1}^r \binom{k_i}{2}} \right.$$
$$\times \left(1 - \sum_{i=1}^r \frac{b_i(1-q^{k_i})}{(x_i - a - \sum_{j=1}^r b_j q^{k_j})}\right) \prod_{i=1}^r \frac{\left((a + \sum_{j=1}^r b_j q^{k_j}) q^{1-k_i}/x_i; q\right)_{k_i}}{(q;q)_{k_i}}$$
$$\left. \times \prod_{i=1}^r \frac{\left((a + \sum_{j=1}^r b_j q^{k_j}) c_i q^{1-i}/x_i; q\right)_{n_i-k_i}}{(q;q)_{n_i-k_i}} c_i^{k_i} \right). \quad (6.9)$$

In identity (6.9), if we make the substitutions $a \to q^A - \sum_{j=1}^r \frac{1-q^{B_j}}{1-q}$, $b_i \to \frac{1-q^{B_i}}{1-q}$, $c_i \to q^{-A_i-C_i}$, $x_i \to q^{A-A_i}$, $i = 1, \dots, r$, and then let $q \to 1$, we obtain the following nice multidimensional Rothe summation:

Theorem 6.4. *Let A_i, B_i, C_i, $i = 1, \dots, r$, be indeterminate, and let $n_1, \dots, n_r$ be nonnegative integers. Then there holds*

$$\prod_{i=1}^r \binom{A_i + C_i}{n_i} = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left(\prod_{1\le i<j\le r} \left(\frac{A_i - k_i - A_j + k_j}{A_i - A_j} \right) \cdot \left(1 - \sum_{i=1}^r \frac{B_i k_i}{(A_i + \sum_{j=1}^r B_j k_j)}\right) \right.$$
$$\left. \times \prod_{i=1}^r \binom{A_i + \sum_{j=1}^r B_j k_j}{k_i} \binom{C_i + i - 1 - \sum_{j=1}^r B_j k_j}{n_i - k_i} \right). \quad (6.10)$$

Remark 6.5. The (multiple) q-Abel and q-Rothe summations in this section were derived by manipulating the series expansions we had obtained by rotated inversion in Section 5 and then extracting coefficients from them.

7. A q-analogue of MacMahon's Master Theorem

Here we present a q-extension of MacMahon's Master Theorem [37]. Chu [9, Sec. 5] observed that inverse relations imply MacMahon's Master Theorem. Basically, he recovered Carlitz's multidimensional extension of (5.1) [7, Eq. (3.5)] (or rather the related formula in [6, Eq. (4.3)]) by inverse relations, which by some further manipulations he showed to be equivalent to MacMahon's celebrated theorem.

Letting $\langle \mathbf{z}^{\mathbf{n}}\rangle f(\mathbf{z})$ denote the coefficient of $\mathbf{z}^{\mathbf{n}}$ in $f(\mathbf{z})$, the classical version of MacMahon's Master Theorem can be stated as follows:

Theorem 7.1. *Let z_i, b_{ij}, $i,j=1,\dots,r$, be indeterminate, and let $n_1,\dots,n_r$ be arbitrary nonnegative integers. Then there holds*

$$\langle \mathbf{z}^{\mathbf{n}}\rangle \prod_{i=1}^{r}\Big(\textstyle\sum_{j=1}^{r} b_{ij}z_j\Big)^{n_i} \;=\; \langle \mathbf{z}^{\mathbf{n}}\rangle \left(\det_{1\le i,j\le r}(\delta_{ij}-z_ib_{ij})\right)^{-1}.$$

In deriving our q-analogue, Theorem 7.2, we basically "q-extend" Chu's analysis. In our derivation we utilize Theorem 5.4, a multiple extension of the q-expansion (5.4) (i.e., a q-analogue of [7, Eq. (6.5)]), and are able to extend the whole analysis with additional bases $q_1,q_2,\dots,q_r$. In our case certain q-operators come into the game.

Theorem 7.2 (A q-analogue of MacMahon's Master Theorem). *Let z_i, b_{ij}, for $i,j=1,\dots,r$, be indeterminate, and let $n_1,\dots,n_r$ be arbitrary nonnegative integers. Then there holds*

$$\langle \mathbf{z}^{\mathbf{0}}\rangle \prod_{i=1}^{r}\Big(\textstyle\sum_{j=1}^{r} b_{ij}z_j/z_i;q_i\Big)_{n_i} \;=\; \langle \mathbf{z}^{\mathbf{n}}\rangle \left(\det_{1\le i,j\le r}\Big((1-z_i)\delta_{ij}+z_ib_{ij}\mathcal{E}_i\Big)\right)^{-1}\mathbf{1}, \tag{7.1}$$

where $\mathcal{E}_i$ denotes the q-shift operator defined by $\mathcal{E}_iz_i=q_iz_i$, and $\mathbf{1}$ denotes the constant polynomial 1.

Theorem 7.2 indeed includes MacMahon's Master Theorem as a special case. Namely, if we write (7.1) in the way

$$\langle \mathbf{z}^{\mathbf{n}}\rangle \prod_{i=1}^{r}\prod_{s=1}^{n_i}\Big(z_i-\textstyle\sum_{j=1}^{r} b_{ij}z_jq_i^{s-1}\Big) \;=\; \langle \mathbf{z}^{\mathbf{n}}\rangle \left(\det_{1\le i,j\le r}\Big((1-z_i)\delta_{ij}+z_ib_{ij}\mathcal{E}_i\Big)\right)^{-1}\mathbf{1},$$

we conveniently can see that the substitutions $q_i\to 1$, $z_i\to az_i$, $b_{ij}\to -b_{ij}/a$, $i,j=1,\dots,r$, then $a\to 0$, specialize to the classical case, Theorem 7.1.

For illustration, we quickly verify the statement of Theorem 7.2 for one dimension ($r=1$). Writing $b_{11}=b$, $z_1=z$, and $\mathcal{E}_1=\mathcal{E}$, we want to check

$$(b;q)_n \;=\; \langle z^n\rangle\,(1-z+zb\mathcal{E})^{-1}\,\mathbf{1}.$$

We have

$$(1-z+zb\mathcal{E})^{-1}\mathbf{1}=\sum_{n=0}^{\infty}(z-zb\mathcal{E})^n\mathbf{1}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q z^k(-zb\mathcal{E})^{n-k}\mathbf{1}=\sum_{n=0}^{\infty}z^n\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^kq^{\binom{k}{2}}b^k.$$

The last inner sum evaluates to the desired quantity by the q-binomial theorem.

We plan to give a more detailed discussion of our q-analogue of MacMahon's Master Theorem including several applications in a forthcoming paper [35].

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