ENUMERATION OF MAPS ON A SURFACE OF GENUS g AND PRODUCTS OF PERMUTATIONS

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Résumé

Nous présentons une formule qui donne le nombre de décompositions d'un $n-$cycle en produit de deux permutations dont les types cycliques, respectivement λ et μ, sont quelconques. La même expression dénombre également les cartes à une face, bicolorées et enracinées sur une surface orientable de genre g dont les sommets appartenant à chaque couleur ont pour distribution de degrés les partages λ et μ. Nos arguments sont développés dans le contexte de la théorie des caractères et du centre de l'algèbre de groupe du groupe symétrique. L'essentiel de notre argumentation consiste en la construction d'une bijection reliant d'une part, la formule des caractères et d'autre part, notre nouvelle expression. Quelques applications à l'énumération de cartes selon le genre et le nombre d'arêtes sont données.

Abstract

We present an explicit expression for the number of decompositions of an $n-$cycle as a product of any two permutations of cycle types given by partitions λ and μ. The same expression is also counting the number of unicellular rooted bicolored maps on an orientable surface of genus g with vertex degree distribution given by λ and μ. We use character theory and the group algebra of the symmetric group to develop our expression. The key argument is the construction of a bijection involving the character formula at one end and our final expression at the other end. Some applications to the enumeration of maps of given genus and number of edges are given.

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1 Introduction.

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a finite non-increasing sequence of positive integers λ_i such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$. The non-zero terms of λ are called the *parts* of λ and the number k of parts is the length of λ, denoted $\ell(\lambda)$. We also write $\lambda = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$ when α_i parts of λ are equal to i $(i = 1 \ldots n)$. When the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$, we call n the *weight* of λ and we write $\lambda \vdash n$ or $|\lambda| = n$.

The conjugacy classes C_λ of the symmetric group S_n are indexed by partitions of n which are called the cycle types of the permutations $\sigma \in C_\lambda$. Let $\mathbb{Q}[S_n]$ be the group algebra of the symmetric group over the rational numbers $\mathbb{Q}$ and let $C[S_n]$ be the center of this group algebra. The formal sum of the permutations in a conjugacy class C_λ belongs to $C[S_n]$ and we denote it K_λ. We abuse notation and also call the elements K_λ conjugacy classes. The set $\{K_\lambda\}_{\lambda \vdash n}$ forms a linear basis for the center $C[S_n]$. Thus a product of two conjugacy classes $K_\lambda * K_\mu$ in $C[S_n]$ can be decomposed in the basis $\{K_\lambda\}_{\lambda \vdash n}$ and the coefficients of the decomposition are non-negative integers called *connection coefficients* or *structure constants* of $C[S_n]$:

$$K_\lambda * K_\mu = \sum_{\gamma \vdash n} c_{\lambda,\mu}^{\gamma} K_\gamma \tag{1}$$

Efforts for computing the connection coefficients $c_{\lambda,\mu}^{\gamma}$ have been made by several authors, beginning in the late 1970's with D. W. Walkup [15], Bertram and Wei [2], G. Boccara [3] and also R. P. Stanley [14], J. Katriel [12], Jones [11], D. M. Jackson et al. [9, 10]. All these authors treat special cases of the decomposition (1) and most of their explicit results concern the case $\gamma = n$, i.e. at least one permutation is an n-cycle, and restricted values of λ and μ. Our main result (Theorem 2.1) is an explicit expression for $c_{\lambda,\mu}^{n}$, valid for any partitions λ and μ of n.

From now on, we shall be interested only with the case $\gamma = n$ and thus we can define the *genus* $g(\lambda, \mu)$ of a pair of partitions of weight n by the equation

$$\ell(\lambda) + \ell(\mu) \quad = \quad n + 1 - 2g(\lambda, \mu) \tag{2}$$

It is immediate from a decomposition into transpositions that $c_{\lambda,\mu}^{n} = 0$ if $g(\lambda, \mu)$ is a negative integer. We shall therefore assume this condition in the sequel.

In a previous result ([7], Theorem 3.1), a simple expression was given for the coefficients $c_{\lambda,\mu}^{n}$ when $g(\lambda, \mu) = 0$. This coefficient was later interpreted combinatorially by Goulden and Jackson [5] as the number of unicellular rooted bicolored maps with n edges on a surface of genus zero, the vertices of each color having degree distribution given by λ and μ respectively. Our main result (Theorem 2.1) is the extension of this result to maps on a surface of arbitrary genus i.e. to pairs of partitions (λ, μ) with arbitrary $g(\lambda, \mu)$. Except for strongly restricted values of (λ, μ, γ), this is, as far as we know, the first general expression for connection coefficients which do not belong to the so-called minimal transitive case $g = 0$.

In Section 2, we state Theorem 2.1 and give a sketch of the proof. Unlike Goulden and Jackson's proof of the special case $g = 0$, our proof follows from the general character theoretic expression and is not constructive. However crucial use is made of a new bijective interpretation during the course of the proof.

In Section 3, we recall the relation between pairs of permutations and maps on oriented surfaces and restate our result in this context. From this point of view, our result generalizes a formula of Walsh and Lehman ([16] Formula 9) for some monochromatic maps to the corresponding bicolored maps.

From their formula, Walsh and Lehman were able to sum over all types of monochromatic unicellular maps with given genus and number of vertices to obtain a formula for the number of maps with given number of vertices. Several equivalent expressions for this coarser enumeration were obtained by Harer and Zagier [8] in connection with the Euler characteristic of the moduli spaces of algebraic curves. Independently, Jackson [9] used the character theoretic approach to obtain the same result. His approach was simplified by Zagier [17]. Following Zagier, Adrianov [1] obtained recursive expressions which are analogue to Harer-Zagier Formulae for bicolored unicellular maps. In Section 4, using Theorem 2.1, we are able to deduce explicit expressions for the number of these maps. We insist on the fact that, although equivalent expressions were or could already be obtained for our Theorems 4.2 and 4.3 (in [1] and along Jackson and Visentin's lines [10]), our expressions are of a different nature: they involve summations of positive contributions, with number of terms bounded by a function of g and independent of n. In particular for fixed genus, asymptotic results are derived when n goes to infinity.

2 A general expression for $c^n_{\lambda,\mu}$

A *composition* p of n is a finite sequence of non-negative integers summing up to n and we denote by $p \models n$. Following [13], let $z_\lambda = \prod_i \alpha_i! i^{\alpha_i}$ for a partition $\lambda = 1^{\alpha_1} \dots n^{\alpha_n}$.

Theorem 2.1 *Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \dots, \mu_m)$ be any two partitions of n with $g(\lambda, \mu) = g$. Then*

$$\begin{aligned} c^n_{\lambda,\mu} &= \frac{n}{z_\lambda z_\mu 2^{2g}} \sum_{g_1+g_2=g} (\ell + 2g_1 - 1)!(m + 2g_2 - 1)! \times \\ & \Big(\sum_{(i_1,\dots,i_\ell)\models g_1} \prod_k \binom{\lambda_k}{2i_k+1} \Big) \Big(\sum_{(j_1,\dots,j_m)\models g_2} \prod_k \binom{\mu_k}{2j_k+1} \Big), \end{aligned}$$

or equivalently, with $\lambda = 1^{\alpha_1} \dots n^{\alpha_n}$ and $\mu = 1^{\beta_1} \dots n^{\beta_n}$,

$$c^n_{\lambda,\mu} = \frac{n}{\prod \alpha_i! \prod \beta_i! 2^{2g}} \sum_{g_1+g_2=g} S_{\ell,g_1}(\lambda) S_{m,g_2}(\mu),$$

where

$$S_{k,g}(x_1, \dots, x_k) = (k + 2g - 1)! \sum_{(p_1,\dots,p_k)\models g} \prod_i \frac{1}{2p_i + 1} \binom{x_i - 1}{2p_i}$$

is a symmetric polynomial of degree $2g$ in the x_i.

This formula has some interesting properties:

- The coefficient $c^n_{\lambda,\mu}$ is expressed as a sum of positive contributions, thus giving easily an idea of its order of magnitude. This is used in section 4 to obtain asymptotic results for maps.
- The number of terms in the summation is

  $$n(\ell, m, g) = \sum_{g_1+g_2=g} \binom{\ell + g_1 - 1}{g_1} \binom{m + g_2 - 1}{g_2}$$

  and hence is a polynomial in ℓ and m of total degree g.

These properties are meant to be compared with those of the expression which is readily obtained from character theoretic point of view:

$$c^n_{\lambda,\mu} = \frac{n}{z_\lambda z_\mu} \sum_{r=0}^{n-1} (-1)^r r!(n - 1 - r)! \chi_\lambda^{1^r(n-r)} \chi_\mu^{1^r(n-r)}, \tag{3}$$

with

$$\chi_\lambda^{1^r(n-r)} = (-1)^r [T^r] \frac{1}{1-T} \prod_{i \geqslant 0} (1-T^i)^{\alpha_i}, \tag{4}$$

where $[T^r]F(T)$ denotes the coefficient of T^r in the power serie F.

With formula (4) (which is a direct translation of Murnaghan-Nakayama rule) the number of terms in the evaluation of $\chi_\lambda^{1^r(n-r)}$ can be exponential with respect to the number of parts of λ. Moreover a direct asymptotic evaluation of $c^n_{\lambda,\mu}$ is made even more difficult by the occurence of alternating signs.

A word on the proof. The proof of Theorem 2.1 is done with a sequence of bijections establishing a correspondence between two sets that are respectively the combinatorial models of formula 3 and of our final expression. We give a sketch of the successive steps.

As already mentioned, formula 3 is obtained by the character theoretic approach which is exposed in [11] or in [6] for instance. Then this expression is rewritten as

$$c^n_{\lambda,\mu} = \frac{n}{z_\lambda z_\mu} \sum_{\phi_{RG} \in \mathcal{QPM}(\lambda,\mu)} (-1)^{\varepsilon(\phi_{RG})}$$

by introducing a proper combinatorial set $\mathcal{QPM}(\lambda,\mu)$ of "quasi-painted diagram matchings" of type (λ,μ). These objects are bijections between cells of diagrams of shape λ and μ that are colored with two colors.

Then the inner summation is transformed along combinatorial considerations: a first bijection is used to turn it into the following

$$\sum_{\psi_{RG} \in \mathcal{PM}(\lambda^\ell,\mu^m)} (-1)^{\varepsilon(\psi_{RG})},$$

where the summation is over a set of properly "painted diagram matchings" of some derived shape, onto which a sign-reversing involution can be defined. As expected the fix-points yield positive contributions and can be described as some specific graphical configurations, a graph $\mathcal{G}$ being weighted by $2^{\mathrm{cyc}(\mathcal{G})}$ where $\mathrm{cyc}(\mathcal{G})$ denote the cyclomatic number of the graph $\mathcal{G}$.

Finally a bijective correspondence is produced which maps these configurations onto a last set of objects, naturally interpreting the right-hand side of Theorem 2.1. □

This approach has been extended to decompositions of an n-cycle into more than two permutations (joined work with Dominique Poulalhon).

3 Pairs of permutations and combinatorial maps

An account of the relations between pairs of permutations and maps on oriented surfaces can be found e.g. in [4]. But in order to have a self contained exposition, we shall briefly sketch the definitions of maps: a *map* $(\mathcal{S},\mathcal{G})$ on a compact oriented surface $\mathcal{S}$ without boundary is a graph $\mathcal{G}$ together with an embedding of $\mathcal{G}$ into $\mathcal{S}$ such that connected components of the complement $\mathcal{S}\backslash\mathcal{G}$ of the embedding of $\mathcal{G}$ in $\mathcal{S}$, called the faces of the map, are homeomorphic to discs. Multiple edges are allowed and our maps are rooted, i.e. one edge of $\mathcal{G}$ is distinguished. Two maps $(\mathcal{S},\mathcal{G})$ and $(\mathcal{S}',\mathcal{G}')$ are isomorphic if there exists an orientation-preserving homeomorphism $f : \mathcal{S} \to \mathcal{S}'$ such that $f(\mathcal{G}) = \mathcal{G}'$. A map is *bicolored* if its vertices are colored in black or white so that each edge is incident to one vertex of each color. A map is *unicellular* if it has one face. The *type* of a bicolored unicellular map $\mathcal{M}$ with n edges is a pair of partitions (λ,μ) whose parts give the respective degrees of black and white vertices of $\mathcal{M}$.

A relation between products of conjugacy classes and enumeration of maps is presented in the following proposition (see [4] for more details):

Proposition 3.1 *Bicolored unicellular maps of type (λ,μ) are maps on a compact orientable surface of genus g which satisfy*

$$g = g(\lambda,\mu)\,.$$

Moreover, the number of bicolored unicellular maps of type (λ,μ) with n edges is the number of pairs $(\sigma,\tau) \in C_\lambda \times C_\mu$ such that $\sigma\tau = (1,2,\dots,n)$, which is also the coefficient $c^n_{\lambda,\mu}$.

In particular when $g(\lambda,\mu) = 0$, we are interested in unicellular maps on the sphere which are plane trees as remarked by Goulden and Jackson [5]. When $\mu = 2^{n/2}$, all white vertices have degree two and can be seen as centers of edges linking black vertices to black vertices so that the white vertices can be forgotten and we obtain the monochromatic unicellular maps counted by Walsh and Lehman [16].

4 Application to a coarser enumeration problem

In this section, we study some consequences of Theorem 2.1 for the enumeration of maps. We call $Bi(\lambda,\mu,n) = c^n_{\lambda,\mu}$ the number of bicolored unicellular rooted maps on a surface of genus $g = g(\lambda,\mu)$ with black degree distribution given by λ, white degree distribution given by μ and n

edges. We observe that if we fix a partition λ of n and the parameter g and if we sum over all conjugacy classes indexed by partitions μ satisfying $\ell(\mu) = m = n + 1 - 2g - \ell(\lambda)$, then the coefficient

$$Bi(\lambda, m, n) = \sum_{\substack{\mu \vdash n \\ \ell(\mu)=m}} Bi(\lambda, \mu, n) = \sum_{\substack{\mu \vdash n \\ \ell(\mu)=m}} c^n_{\lambda,\mu} \tag{5}$$

counts the number of bicolored unicellular rooted maps on a surface of genus g with black degree distribution given by λ and no restriction on white degree distribution other than the number m of white vertices. Similarly, if we sum over all conjugacy classes with partition λ of a fixed length (or rank), then the coefficient

$$Bi(\ell, m, n) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=\ell}} Bi(\lambda, m, n) = \sum_{\substack{\lambda \vdash n,\ \ell(\lambda)=\ell \\ \mu \vdash n,\ \ell(\mu)=m}} c^n_{\lambda,\mu} \tag{6}$$

counts the number $Bi(\ell, m, n)$ of unicellular bicolored rooted maps of genus g with ℓ black vertices and $m = n + 1 - 2g - \ell$ white vertices. These maps were recently considered by Adrianov [1]. Finally, if we sum the product in (6) over all possible lengths of λ, we obtain

$$Bi(n, g) = \sum_{\substack{\ell \geq 1 \\ m \geq 1 \\ \ell+m=n+1-2g}} Bi(\ell, m, n) = \sum_{\substack{\ell \geq 1 \\ m \geq 1 \\ \ell+m=n+1-2g}} \sum_{\substack{\lambda \vdash n,\ \ell(\lambda)=\ell \\ \mu \vdash n,\ \ell(\mu)=m}} c^n_{\lambda,\mu} \tag{7}$$

which is the number $Bi(n, g)$ of unicellular bicolored rooted maps of genus g with n edges. The purpose of this section is to give explicit expressions for the sums of coefficients in (5), (6) and (7). These expressions can be derived from Theorem 2.1 by some algebra.

Theorem 4.1 *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$ and m, g two non-negative integers such that $\ell + m = n + 1 - 2g$. Then*

$$Bi(\lambda, m, n) = \frac{n!}{z_\lambda 2^{2g} m!} \sum_{g_1+g_2=g} \left(\sum_{i_1+\cdots+i_\ell=g_1} \prod_k \binom{\lambda_k}{2i_k+1} \right) P_{g_2}(m)$$

where $P_g(x)$ is a polynomial in x of degree g defined by:

$$P_g(x) = \sum_{\substack{\gamma \vdash g \\ \gamma=1^{c_1}\ldots g^{c_g}}} (x)_{\ell(\gamma)} \frac{1}{\prod_i c_i!(2i+1)^{c_i}}, \tag{8}$$

with $(x)_\ell = x(x-1)\ldots(x-\ell+1)$ the descending factorial.

We use the notation $u_n \sim_{n\to\infty} v_n$ to denote the fact that $\lim_{n\to\infty}(u_n - v_n)/v_n = 0$, that is u_n is asymptoticaly equivalent to v_n.

Let us call $Mono(n,g)$ the number of monochromatic maps of genus g with n edges.

Corollary 4.1 *Let g be a given non-negative integer, then*

$$Mono(n,g) \sim_{n\to\infty} \frac{n^{3(g-\frac{1}{2})}4^n}{\sqrt{\pi}g!12^g}.$$

Proof. As mentioned in section 3, there is a well known bijection between maps with n edges and bicolored maps with n black vertices, all of degree 2, and $2n$ edges: the bicolored map is simply obtained from the monochromatic map by putting a black vertex in the middle of each edge. Then using Theorem 4.1, we have (see also [16])

$$Mono(n,g) = Bi((2^n), n+1-2g, 2n) = \frac{(2n)!}{2^{2g}n!(n+1-2g)!}P_g(n+1-2g).$$

Now for a given g, the summation in $P_g(x)$ is finite and the leading term is obtained when $\gamma = 1^g$. Then using Stirling's formula

$$n! = n^n e^{-n}\sqrt{2\pi n}e^{E(n)} \tag{9}$$

where $E(n)$ is an error that approaches 0 as n gets large, the result becomes straightforward. □

Theorem 4.2 *Let ℓ, m, n, g be non-negative integers such that $\ell + m = n+1-2g$. Then*

$$Bi(\ell,m,n) = \frac{(n+1)_{2g}}{(n+1)2^{2g}}\binom{n+1-2g}{\ell}\sum_{g_1+g_2=g}\binom{n-1}{\ell+2g_1-1}P_{g_1}(\ell)P_{g_2}(m),$$

Theorem 4.3 *Let n, g be non-negative integers. Then*

$$Bi(n,g) = \frac{(n+1)_{2g}}{(n+1)2^{2g}}\sum_{g_1+g_2=g}\sum_{\substack{0\leqslant \ell_1\leqslant g_1\\ 0\leqslant \ell_2\leqslant g_2}} c_{g_1,\ell_1}c_{g_2,\ell_2}(n+1-2g)_{\ell_1+\ell_2}\binom{2n-2g-\ell_1-\ell_2}{n-2g_1-\ell_1}$$

where the $c_{g,\ell}$ are constants given by:

$$c_{g,\ell} = \sum_{\substack{\gamma\vdash g\\ \ell(\gamma)=\ell\\ \gamma=1^{\gamma_1}\dots g^{\gamma_g}}} \frac{1}{\prod_i \gamma_i!(2i+1)^{\gamma_i}}.$$

Corollary 4.2 *Let g be a non-negative integer, then*

$$Bi(n,g) \sim_{n\to\infty} \frac{n^{3(g-\frac{1}{2})}4^n}{\sqrt{\pi}g!48^g}.$$

In particular, we recover Adrianov's result [1] that

$$\frac{Bi(n,g)}{Mono(n,g)} \sim_{n\to\infty} \left(\frac{1}{4}\right)^g.$$

Proof. Since the summation is finite for given g, Stirling formula (9) can be used in each internal summand and leading terms are obtained when $\ell_1 = g_1$ and $\ell_2 = g_2$. Summing their respective contribution yields the result. $\square$

We give in tables 1 and 2, the expressions of low genera for the polynomials $Bi(\lambda,\mu,n)$, $P_g(x)$, $Bi(\ell,m,n)$ and $Bi(n,g)$ where

$$\begin{aligned}
q_2(x,y) &= 13(x^2+y^2)+5xy(x+y)+86(x+y)+47xy+129,\\
q_3(x,y) &= 502(x^4+y^4)+35x^2y^2(x^2+y^2)+273xy(x^3+y^3)+70x^3y^3\\
&\quad +9978(x^3+y^3)+1260x^2y^2(x+y)+6512xy(x^2+y^2)\\
&\quad +71842(x^2+y^2)+13410x^2y^2+54123xy(x+y)\\
&\quad +185554xy+219918(x+y)+238480\,.
\end{aligned}$$

genus	$Bi(\lambda,\mu,n)$	$P_g(x)$
0	$\frac{n(\ell-1)!(m-1)!}{\prod_i \alpha_i!\beta_i!}$	1
1	$\frac{n(\ell-1)!(m-1)!}{3!\prod_i \alpha_i!\beta_i!}\left[\binom{\ell+1}{2}\sum_i\binom{\lambda_i-1}{2}+\binom{m+1}{2}\sum_i\binom{\mu_i-1}{2}\right]$	$\frac{x}{3}$
2	$\frac{n(\ell-1)!(m-1)!}{5!\prod_i \alpha_i!\beta_i!}\Big[\binom{\ell+3}{4}\big[36\sum_i\binom{\lambda_i-1}{4}+20\sum_{i\neq j}\binom{\lambda_i-1}{2}\binom{\lambda_j-1}{2}\big]+$ $\binom{m+3}{4}\big[36\sum_i\binom{\mu_i-1}{4}+20\sum_{i\neq j}\binom{\mu_i-1}{2}\binom{\mu_j-1}{2}\big]+$ $\frac{10}{3}\binom{\ell+1}{2}\binom{m+1}{2}\sum_i\binom{\lambda_i-1}{2}\sum_i\binom{\mu_i-1}{2}\Big]$	$\frac{x(5x+13)}{90}$

Table 1:
The number of rooted bicolored unicellular maps of genus g on n vertices with vertex degree distributions given by λ and μ.

genus	$Bi(\ell,m,n)$	$Bi(n,g)$
0	$\frac{1}{n}\binom{n}{\ell-1}\binom{n}{m-1}$	$\frac{1}{2n+1}\binom{2n+1}{n}$ Catalan numbers
1	$\frac{\binom{n+1}{2}}{3!}\binom{n-1}{\ell-1}\binom{n-1}{m-1}$	$\frac{\binom{2n-1}{n-1}\binom{n}{3}}{(2n-1)2}$
2	$\frac{\binom{n+1}{2}q_2(\ell,m)}{6\cdot 5!}\binom{n-1}{\ell-1}\binom{n-1}{m-1}$	$\frac{\binom{2n-3}{n-2}\binom{n-1}{4}2(5n^2-7n+6)}{(2n-3)5!}$
3	$\frac{\binom{n+1}{2}q_3(\ell,m)}{36\cdot 7!}\binom{n-1}{\ell-1}\binom{n-1}{m-1}$	$\frac{\binom{2n-5}{n-2}\binom{n-2}{5}5(35n^4-182n^3+397n^2-346n+240)}{(2n-5)3(7)!}$

Table 2
The number of rooted bicolored unicellular maps
of genus g on n vertices.

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