A Bijection for Partitions with All Ranks at Least t Extended Abstract

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Summary (English)

It follows from work of Andrews and Bressoud that for $t \leq 1$, the number of partitions of n with all successive ranks at least t is equal to the number of partitions of n with no part of size $2-t$. However, no simple combinatorial explanation of this fact has appeared in the literature. We give a simple bijection for this identity which generalizes a result of Cheema and Gordon for 2-rowed plane partitions. The bijection yields several refinements of the identity when the partition counts are parametrized by the minimum successive rank, the number of parts and/or the size of the Durfee rectangle. In addition, it gives a natural interpretation of the difference of (shifted) successive Gaussian polynomials which we relate to other interpretations of Andrews and Fishel.

Summary (French)

D'après un travail d'Andrews et Bressoud, il suit que pour $t \leq 1$, le nombre de partitions de n dont le rang minimal vaut au moins t est égal au nombre de partitions sans parts de taille $2-t$. Pourtant aucune explication combinatoire simple de ce fait n'est apparue dans la littérature. Nous donnons une bijection simple pour cette identité qui généralise un résultat de Cheema et Gordon pour les partitions planes ayant au plus deux lignes. Cette bijection entraine plusieurs raffinements de l'identité quand les partitions sont comptées selon le rang

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minimal, le nombre de parts et/ou la taille du rectangle de Durfee. De plus, elle donne une interprétation naturelle de la différence de polynômes gaussiens (decalés) consécutifs que nous relions à d'autres interprétations d'Andrews et Fishel.

1 Introduction

A partition π of a non-negative integer n is a nonincreasing sequence $\pi = (\pi_1, \dots, \pi_t)$ of positive integers whose sum is n and the *weight* of π, denoted $|\pi|$, is n. The Ferrers diagram of π is an array of dots, left justified, in which the number of dots in row i is π_i. The largest square subarray of dots in this diagram is the Durfee square and $d(\pi)$ refers to the length of a side. The conjugate of π, denoted π', is the partition whose ith part is the number of dots in the ith column of the Ferrers diagram of π. The *sequence of successive ranks* of π is the sequence $(\pi_1 - \pi'_1, \dots, \pi_d - \pi'_d)$, where $d = d(\pi)$ [Dys44, Atk66].

Let $P(n)$ denote the set of all partitions of n and for integer $b > 0$, let $P_b(n)$ be the set of partitions of n with no part equal to b. Then:

$$|P_b(n)| = |P(n)| - |P(n-b)|. \tag{1}$$

We consider generalizations and refinements of the identity

$$|R(n)| = |P_1(n)|, \tag{2}$$

where $R(n)$ is the set of those partitions of n with all successive ranks positive. As observed in [ER93], (2) follows from Theorem 1 in Bressoud [Bre80] which is an extension of Theorem 5 in Andrews [And71] to even as well as odd moduli. The results of Bressoud and Andrews are actually a generalization of the Rogers-Ramanujan identities and (2) follows as a very special case. Direct proofs of (2) can be found in [And93] and [RA95], but apparently no simple bijective proof of the result has appeared.

The family $R(n)$ has received attention recently in connection with graphical partitions, that is, partitions which are the degree sequences of simple graphs [ER93, RA95, BS95], since the conjugate of any partition in $R(n)$ is graphical.

In Section 2 of this paper, we give a simple bijective proof of (2). In Section 3, we consider a generalization of (2) which also follows from the Andrews-Bressoud theorem, stated below.

Theorem 1 [And71, Bre80] *For integers M, r, satisfying $0 < r < M/2$, the number $B_{M,r}(n)$ of partitions of n whose successive ranks lie in the interval $[-r+2, M-r-2]$ is equal to the number $A_{M,r}(n)$ of partitions of n with no part congruent to 0, r, or $-r$ modulo M.*

(For $r = 1$, $M = 4$ and $r = 2$, $M = 5$, this gives the Rogers-Ramanujan identities.)

Let $R_{\geq t}(n)$ denote the set of partitions of n in which all ranks are at least t. Then for $1 - n \leq t \leq 1$, it follows from Theorem 1 by setting $r = 2 - t$ and $M = n + r + 1$ that

$$|R_{\geq t}(n)| = |P_{2-t}(n)|. \tag{3}$$

In Section 3, we give a bijection for (3) by first showing bijectively that for $t \leq 0$,

$$|R_{=t}(n)| = |R_{\geq 1}(n-1+t)|, \tag{4}$$

where $R_{=t}(n)$ is the set of partitions whose minimum rank is exactly t, that is, $R_{=t}(n) = R_{\geq t}(n) - R_{\geq t+1}(n)$.

We define the *Durfee rectangle* of a partition π to be the largest $d \times (d+1)$ rectangle contained in the Ferrers diagram of π and let $d^*(\pi)$ denote the height of the Durfee rectangle of π.

It turns out that the bijections in Sections 2 and 3 which establish (2) and (4) preserve both the number of parts in a partition and the size of the Durfee rectangle. (The size of the Durfee square need not be preserved!) As a result, we get several refinements of identities (2), (3), and (4) which are highlighted in Section 4 in terms of generating functions. In Section 5, we note the connection with plane partitions and a result of Cheema and Gordon when $t \geq 1$. Our bijection gives an interpretation of the difference of successive Gaussian polynomials which we relate to other interpretations in Section 6.

We show in the full paper, how a result of Burge [Bur81] can be used to get another bijection for (3). However, the second bijection is far from simple and it does not have the natural implications of the one presented here.

2 A Bijection for Partitions With All Ranks Positive

In this section, we give a simple bijection between $P_1(n)$ and $R_{\geq 1}(n)$. Define the *rank vector* of a partition π to be the vector $[r_1(\pi), r_2(\pi), \ldots, r_{d(\pi)}(\pi)]$ whose ith entry is the ith successive rank $\pi_i - \pi'_i$ of π.

Define a partial function F on all partitions π as follows. $F(\pi)$ is that partition obtained from π by the following procedure:

$F(\pi)$:
While some rank of π is less than 1 do the following:
1. Let t be the minimum rank of π.
2. Let i be the largest index such that $r_i(\pi) = t$.
3. Delete a part of size i from π.
4. Add a part of size $i-1$ to π'.
5. Add a part of size 1 to π.

So, for example, $F((6,5,4,4,4,3,2,2)) = ((10,8,6,2,1,1,1,1))$, as illustrated in Figure 1, and $F((6,5,4,1)) = ((6,5,4,1))$. However, $F((4,4,3,2,1))$ is undefined since the procedure does not terminate. F would also be undefined if it happened that at step (3), π contained no part of size i.

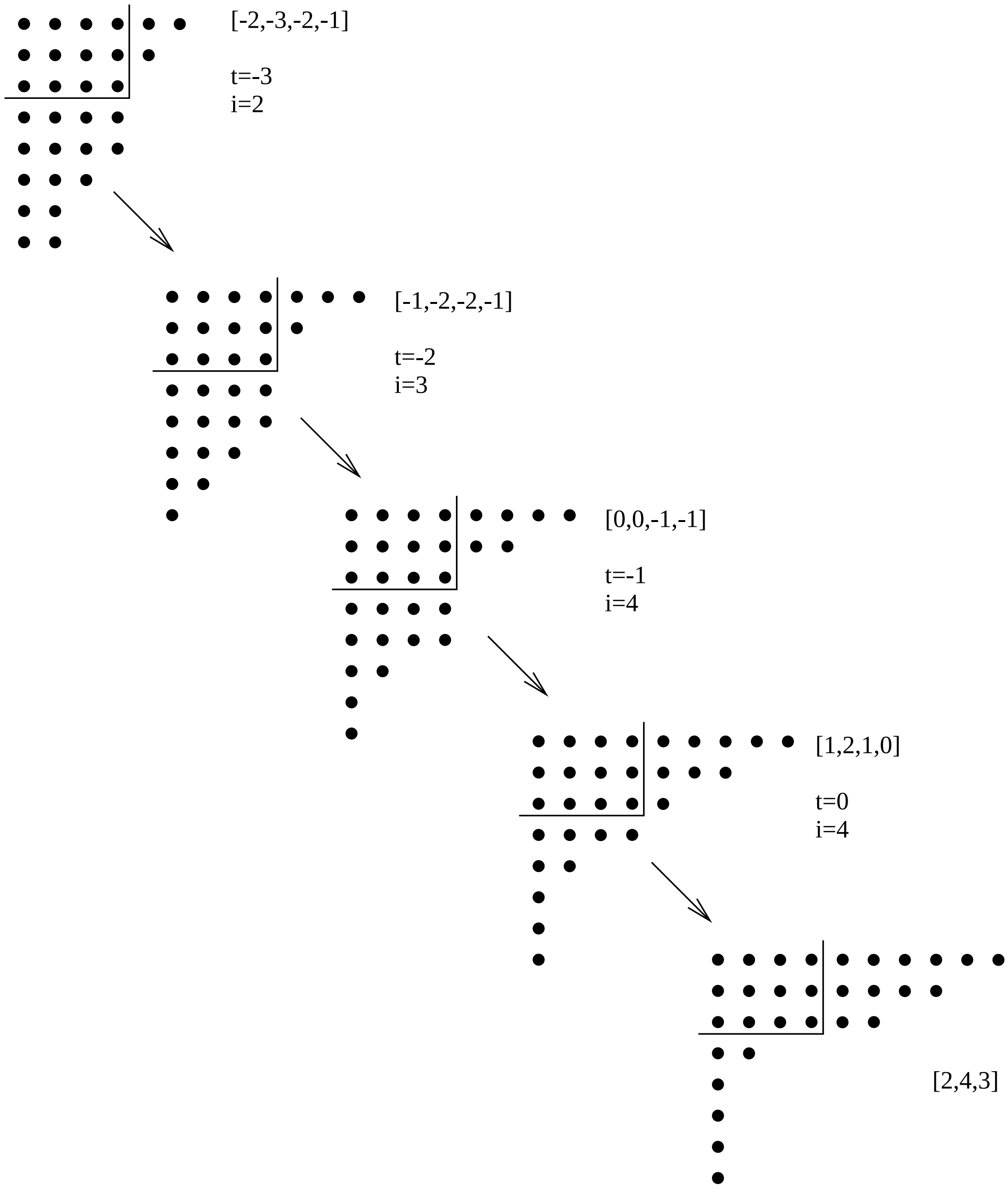

Figure 1: Computation of $F((6, 5, 4, 4, 4, 3, 2, 2))$, with rank vector shown at each iteration and with Durfee rectangle indicated.

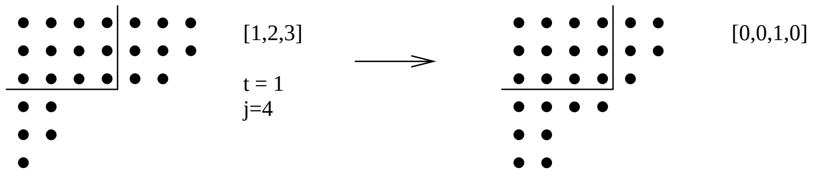

Figure 2: Computation of $G((7,7,6,2,2,1))$, which uses case b(ii) of G.

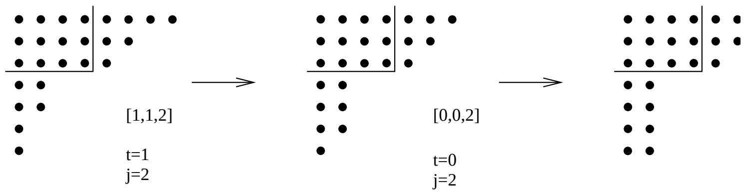

Figure 3: Computation of $G((8,6,5,2,2,1,1))$, which uses case b(iii) of G.

Define another partial function G on partitions π so that $G(\pi)$ is that partition obtained from π by the following procedure.

> $G(\pi)$:
> While π contains a part of size 1 do the following:
> a. Let t be the minimum rank of π.
> b. (i) If $t > 1$, let $j = d(\pi) + 1$.
> (ii) Otherwise, if $r_i(\pi) = t$ only for $i = 1$, let $j = d(\pi) + 1$.
> (iii) Otherwise, let j be the smallest index with $j > 1$ and $r_j(\pi) = t$.
> c. Delete a part of size $j - 1$ from π'.
> d. Add a part of size j to π.
> e. Delete a part of size 1 from π.

For example, then $G((10,8,6,2,1,1,1,1)) = ((6,5,4,4,4,3,2,2))$ (read Figure 1 in reverse.) Note that in the first iteration of G for this example, case b(i) applies. Examples of b(ii) and b(iii) occur in the first iteration of the computations of $G((7,7,6,2,2,1)) = (6,6,5,4,2,2))$ (Figure 2) and $G((8,6,5,2,2,1,1)) = (6,6,5,2,2,2,2))$ (Figure 3), respectively. Partition $(3,2,2)$ is fixed by G. G would be undefined if in step (c), π' contained no part of size $j - 1$.

Our main result is:

Theorem 2 *F gives a bijection, with inverse G, from the set of partitions with no part equal to 1 to the set of partitions with all ranks positive. Furthermore, when applied to a partition π with no '1', F preserves the weight of π, the number of parts of π, and the size of the Durfee rectangle of π.*

Figure 4 shows the one-to-one correspondence $F : P_1(10) \to R_{\geq 1}(10)$.

To prove Theorem 2, we first give conditions which guarantee that in step (3) of proce-

$P_1(10)$		$R_{\geq 1}(10)$
(10)	$\longrightarrow$	(10)
(8,2)	$\longrightarrow$	(9,1)
(7,3)	$\longrightarrow$	(7,3)
(6,4)	$\longrightarrow$	(6,4)
(6,2,2)	$\longrightarrow$	(8,1,1)
(5,5)	$\longrightarrow$	(5,5)
(5,3,2)	$\longrightarrow$	(6,3,1)
(4,4,2)	$\longrightarrow$	(4,4,2)
(4,3,3)	$\longrightarrow$	(5,4,1)
(4,2,2,2)	$\longrightarrow$	(7,1,1,1)
(3,3,2,2)	$\longrightarrow$	(5,3,1,1)
(2,2,2,2,2)	$\longrightarrow$	(6,1,1,1,1)

Figure 4: The bijection $F : P_1(10) \to R_{\geq 1}(10)$.

dure F, π will have a part of size i (Lemma 1 below) and that in step (c) of procedure G, π' will have a part of size $j-1$ (Lemma 2 below).

Lemma 1 *Let t be the minimum rank of π and let i be the largest index with $r_i(\pi) = t$. If $t \leq 0$, then π contains a part of size i.*

Proof. Let $d = d(\pi)$. By definition of i, $i \leq d$, so both π and π' contain parts of size at least i. If $i = d$, then $r_d = \pi_d - \pi'_d = t \leq 0$. Thus $\pi'_d = \pi_d - t \geq d - t$ and so $\pi_{d-t} = d = i$. Otherwise ($i < d$), let j be the largest index with $\pi_j \geq i$. If $\pi_j \geq i+1$, then $\pi'_{i+1} = j = \pi'_i$. But then since $\pi_i \geq \pi_{i+1}$, we would have

$$r_i = \pi_i - \pi'_i \geq \pi_{i+1} - \pi'_{i+1} = r_{i+1},$$

contradicting choice of i. $\Box$

Lemma 2 *Let t be the minimum rank of π and let $d = d(\pi)$.*
(i) *If $t > 1$ then π' contains a part of size d.*
(ii) *If $t = 1$ and if $r_i = t$ only for $i = 1$, then π' contains a part of size d.*
(iii) *If $t \leq 1$ and $r_i = t$ for some $i > 1$, let j be the smallest such index. If π contains a part of size 1, then π' contains a part of size $j-1$.*

Proof. In cases (i) and (ii), $r_d \geq 1$, so $\pi_d > d$ and therefore $\pi'_{d+1} = d$. In case (iii), it suffices to show that $\pi_{j-1} > \pi_j$. If $j = 2$, $\pi_1 > \pi_2$ since π contains a 1. Otherwise, $j \geq 3$ and by definition of t and j, $r_{j-1} > r_j$ and therefore $\pi_{j-1} > \pi'_{j-1} + (\pi_j - \pi'_j) \geq \pi_j$. $\Box$

We now focus on the effect of one iteration of steps (1-5) of the computation of $F(\pi)$. Let f be the partial function which assigns to a partition π the partition $f(\pi)$ derived from π by one application of steps (1-5) of procedure $F(\pi)$. Similarly, let g be the partial function which maps π to the partition resulting from one application of steps (a-e) of procedure $G(\pi)$. Then, e.g., $f((6,5,4,4,4,3,2,2)) = (7,5,4,4,4,3,2,1)$ and $f((7,5,4,4,4,3,2,1)) =$

$(8,6,4,4,4,2,1,1)$ (see Figure 1); f fixes both $(6,5,4,1)$ and $(4,4,3,2,1)$. Also, e.g., $g((7,7,6,2,2,1)) = (6,6,5,4,2,2)$ (Figure 2); $g((8,6,5,2,2,1,1)) = (7,6,5,2,2,2,1)$ (Figure 3); g fixes $(3,2,2)$.

The key to the proof of Theorem 2 is Lemma 3 below. The proof is a tedious examination of cases and is included in the full paper. Because we use these repeatedly, define $i(\pi), j(\pi)$ to be the values assigned to i and j by application of f, g, respectively, to π.

Lemma 3 *For $s \geq 0$, f gives a bijection, with inverse g:*

$$f : \{\pi \in R_{=0}(n) \mid i(\pi) > 1, \quad \pi \textit{ has } s \textit{ ones}\} \longrightarrow \{\pi \in R_{\geq 1}(n) \mid \pi \textit{ has } s+1 \textit{ ones}\},$$

and, if $t < 0$,

$$f : \{\pi \in R_{=t}(n) \mid i(\pi) > 1, \ \ \pi \textit{ has } s \textit{ ones}\} \longrightarrow \{\pi \in R_{=t+1}(n) \mid i(\pi) > 1, \ \ \pi \textit{ has } s+1 \textit{ ones}\}.$$

Furthermore, for π in these domains, f preserves the number of parts, the size of the Durfee rectangle, and, if $t < 0$, the size of the Durfee square; f increases the size of the largest part by 1.

Proof of Theorem 2:

For partition π of n with no part '1', let t be the minimum rank. If $t \geq 1$, then $F(\pi) = \pi$ and $G(\pi) = \pi$. Otherwise, since π has no '1', $\pi'_1 = \pi'_2$ and therefore $r_1 = \pi_1 - \pi'_1 \geq \pi_2 - \pi'_2 = r_2$ so that $i(\pi) > 1$. Thus by repeated application of Lemma 3, $F(\pi) = f^{1-t}(\pi)$ is a partition in $R_{\geq 1}(n)$ with exactly $1-t$ ones and $G(F(\pi)) = g^{1-t}(f^{1-t}(\pi)) = \pi$, showing that F is one-to-one. To show F is onto, if τ is a partition in $R_{\geq 1}(n)$ with exactly s parts of size 1, again by repeated application of Lemma 3, $G(\tau) = g^s(\tau)$ is a partition in $R_{=1-s}(n)$ with no '1' (and therefore with $i(g^s(\tau)) > 1$)and $F(G(\tau)) = f^{1-(1-s))}g^s(\tau) = \tau$. $\square$

3 The Bijection for $R_{\geq t}(n) = P_{2-t}(n)$

Let f^* be the function defined by one application of steps (1-4) of F and let g^* be the following modification of steps (a-d) of G.

$g^*(\pi)$:
a. Let t be the minimum rank of π.
b. (i) If $t > 1$, let $j = d(\pi) + 1$.
(ii) Otherwise, let j be the smallest index with $r_j = t$.
c. Delete a part of size $j - 1$ from π'.
d. Add a part of size j to π.

The proof of Lemma 3 can be modified to show the following.

Lemma 4 *f^* gives a bijection, with inverse g^*:*

$$f^* : R_{=0}(n) \longrightarrow R_{\geq 1}(n-1)$$

and, if $t < 0$,

$$f^* : R_{=t}(n) \longrightarrow R_{=t+1}(n-1)$$

Furthermore, for π in these domains, f^* preserves the size of the Durfee rectangle, and, if $t < 0$, the size of the Durfee square; f^* increases the size of the largest part by 1 if $i(\pi) \neq 1$ and decreases the number of parts by 1.

It follows then by repeated application of Lemma 4 that for $t \leq 1$,

$$|R_{=t}(n)| = |R_{\geq 1}(n+t-1)|.$$

In fact, we have the following theorem.

Theorem 3 *For $t \leq 0$, $f^{*(1-t)}$ is a bijection, with inverse $g^{*(1-t)}$, from the set of partitions with minimum rank t to the set of partitions with all ranks positive. Furthermore, when applied to a partition π, $f^{*(1-t)}$ decreases the weight of π by $-t+1$, decreases the number of parts of π by $-t+1$, and preserves the size of the Durfee rectangle of π.*

Combining Theorems 2 and 3, we have the following result, which establishes (4) of Section 1.

Corollary 1 *For $t \leq 0$, $G \circ f^{*(1-t)}$ is a bijection,*

$$R_{=t}(n) \rightarrow P_1(n-t-1),$$

with inverse $g^{*(1-t)} \circ F$.

Using this we now construct a bijection for identity (3) of Section 1, mapping partitions of n with all ranks at least t to partitions of n with no part '$2-t$'. Define h_t for $\pi \in R_{\geq t}(n)$ as follows.

> $h_t(\pi)$: *(given $\pi \in R_{\geq t}(n)$)*
> Let s be the minimum rank of π; *(Note $s \geq t$.)*
> If $s \geq 1$ then $s \leftarrow 1$.
> i. Let $\sigma = G \circ f^{*(1-s)}(\pi)$ *(Then $\sigma \in P_1(n+s-1)$ by Corollary 1.)*
> ii. Add $1-s$ copies of part '1' to σ. *(Now $\sigma \in P(n)$ with exactly $1-s$ ones.)*
> iii. Replace every occurrence of part '$2-t$' in σ by $2-t$ parts of size 1.
> The result is $h_t(\pi)$.

Note that after step (iii), $h_t(\pi) = \sigma \in P_{2-t}(n)$ and the number of parts '1' in $h_t(\pi)$ is congruent to $(1-s)$ modulo $(2-t)$, since $0 \leq 1-s < 2-t$.)

Theorem 4 *As defined above, h_t is a bijection*

$$h_t : R_{\geq t}(n) \rightarrow P_{2-t}(n).$$

Proof. See full paper for details. □

The bijection $h_0 : R_{\geq 0}(7) \rightarrow P_2(7)$ is illustrated in Figure 5.

$R_{\geq 0}(7)$		$P_2(7)$
(7)	$\longrightarrow$	(7)
(6,1)	$\longrightarrow$	(5,1,1)
(5,2)	$\longrightarrow$	(6,1)
(5,1,1)	$\longrightarrow$	(3,1,1,1,1)
(4,3)	$\longrightarrow$	(4,3)
(4,2,1)	$\longrightarrow$	(4,1,1,1)
(4,1,1,1)	$\longrightarrow$	(1,1,1,1,1,1,1)
(3,3,1)	$\longrightarrow$	(3,3,1)

Figure 5: The bijection $h_0 : R_{\geq 0}(7) \to P_2(7)$.

4 Generating Functions

The results of Sections 2 and 3 can be rephrased in terms of generating function identities. We let $p_1(d,k)$, $r_{\geq t}(d,k)$, $r_{=t}(d,k)$ denote the set of partitions with k parts and with Durfee rectangle size $d \times (d+1)$ in, respectively, P_1, $R_{\geq t}$, and $R_{=t}$. When we fix only the Durfee rectangle, we use $p_1(d)$, $r_{\geq t}(d)$, $r_{=t}(d)$, and when only the number of parts is fixed: $\overline{p}_1(k)$, $\overline{r}_{\geq t}(k)$, $\overline{r}_{=t}(k)$.

Theorem 5 *The following results hold for $t \leq 0$.*
Fixing both the size of the Durfee rectangle, d, and the number of parts, k:

$$\sum_{\pi \in p_1(d,k)} q^{|\pi|} = \frac{q^{d(d+1)} q^{2(k-d)}}{(q)_d} \begin{bmatrix} k-1 \\ d-1 \end{bmatrix}_q = \sum_{\lambda \in r_{\geq 1}(d,k)} q^{|\lambda|} = q^{t-1} \sum_{\sigma \in r_{=t}(d,k-t+1)} q^{|\sigma|}. \quad (5)$$

Summing (5) over k, fixing only the size of the Durfee rectangle:

$$\sum_{\pi \in p_1(d)} q^{|\pi|} = \frac{q^{d(d+1)}(1-q)}{(q)_{d+1}(q)_d} = \sum_{\lambda \in r_{\geq 1}(d)} q^{|\lambda|} = q^{t-1} \sum_{\sigma \in r_{=t}(d)} q^{|\sigma|}. \quad (6)$$

Summing (5) over d, fixing only the number of parts:

$$\sum_{\pi \in \overline{p}_1(k)} q^{|\pi|} = \frac{q^{2k}}{(q)_k} = \sum_{\lambda \in \overline{r}_{\geq 1}(k)} q^{|\lambda|} = q^{t-1} \sum_{\sigma \in \overline{r}_{=t}(k-t+1)} q^{|\sigma|}. \quad (7)$$

Summing (5) over d and k, so that partitions are otherwise unrestricted:

$$\sum_{\pi \in P_1} q^{|\pi|} = \frac{1-q}{(q)_\infty} = \sum_{\lambda \in R_{\geq 1}} q^{|\lambda|} = q^{t-1} \sum_{\sigma \in R_{=t}} q^{|\sigma|}. \quad (8)$$

Proof. In each of the identities (5)-(7), the first equalities follow from well-known partition generating functions together with certain decompositions of the Durfee square. In (8), the first equality follows from (1) of Section 1 with $b = 1$.

The second equalities in (5)-(8) follow from Theorem 2 and the last equalities follow from Theorem 3. □

The first two equalities in (8) appear explicitly in [And93] and [RA95] and, as mentioned earlier, also follow as a special case of Andrews and Bressoud's Theorem 1 in Section 1. Note that for partitions in $R_{\geq 1}(n)$, the Durfee square and the Durfee rectangle are the same size. As a result, the second equality in 6 is the same as the one in [RA95] which uses MacMahon's generating function for plane partitions with bounded part size. Connections with plane partitions are discussed in the next section.

5 Two-rowed Plane Partitions

Our breakthrough in the search for a bijective proof of the identity $R_{\geq 1}(n) = P_1(n)$ came when we found a result of Cheema and Gordon on two-rowed plane partitions.

An *r-rowed plane partition* of n is an array of non-negative integers

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & & & \\ a_{r1} & a_{r2} & a_{r3} & \dots \end{array}$$

where $\sum_{i,j} a_{ij} = n$ and rows and columns are non-increasing.

We can regard a two-rowed plane partition of n as a pair of partitions (σ, τ), where $\sigma = \sigma_1, \sigma_2, \dots$, $\tau = \tau_1, \tau_2, \dots$, $|\sigma| + |\tau| = n$, and $\sigma_i \geq \tau_i$, $i = 1, 2, \dots$.

Let $T(n)$ be the set of 2-rowed plane partitions of n which we will regard as pairs of partitions satisfying the constraints above. Let $S(n)$ be the set of pairs of partitions (α, β) satisfying:

- $|\alpha| + |\beta| = n$ and
- α has no part of size one.

In [CG64], Cheema and Gordon gave a bijection

$$\Theta : S(n) \to T(n).$$

We observed that by "pulling out" the Durfee rectangle of a partition $\pi \in P_1(n)$ and applying the Cheema-Gordon bijection to the pair of partitions remaining to the east and south of the rectangle, we could extend Θ to a bijection $\Theta^* : P_1(n) \to R_{\geq 1}(n)$. Details appear in the full paper.

It can be further shown that $\Theta^* = F$, the mapping described in Section 2. This is not surprising since we devised F as an alternative formulation of Θ^* which decomposes Θ^* into a sequence of basic steps, f, each of whose effect can be analyzed and altered to produce bijective proofs for more general versions of the identity.

6 Difference of Successive Gaussian Polynomials

There are several interpretations of the difference of successive Gaussian polynomials, $\left[\begin{array}{c} n \\ k \end{array}\right]_q - \left[\begin{array}{c} n \\ k-1 \end{array}\right]_q$, including those in [But87], [And93], and [Fis95]. We make just a few remarks here to relate our work to these. For some of the many interesting properties of these polynomials, their differences, and generalizations, see [But87, But90, Fis95].

Let $L[n,k]$ be the set of partitions whose Ferrers diagram lies in a $k \times (n-k)$ box. The number of such partitions is $\left(\begin{array}{c} n \\ k \end{array}\right)$. Let $L[m;n,k]$ be the partitions in $L[n,k]$ of weight m. The generating function for $L[m;n,k]$ is

$$\sum_{n \geq 0} L[m;n,k] q^m = \left[\begin{array}{c} n \\ k \end{array}\right]_q .$$

By Lemma 4, when $n \geq 2k$, g^* applied to a partition in $L[n,k]$ gives a partition in $L[n,k-1]$ and

$$g^* : L[n,k-1] \longrightarrow L[n,k]$$

is an injection. Furthermore, the nonempty partitions in $L[n,k]$ not in the image of g^* are exactly those also in $R_{\geq 1}$. Letting $R_{\geq 1}[n,k]$ denote those partition in $L[n,k]$ with all ranks positive, we have

$$1 + |R_{\geq 1}[n,k]| = |L[n,k]| - |L[n,k-1]| = \left(\begin{array}{c} n \\ k \end{array}\right) - \left(\begin{array}{c} n \\ k-1 \end{array}\right) \tag{9}$$

and, since g^* increases the weight of a partition by 1, we have the following.

Theorem 6

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q - q \left[\begin{array}{c} n \\ k-1 \end{array}\right]_q = 1 + \sum_{\lambda \in R_{\geq 1}[n,k]} q^{|\lambda|}. \tag{10}$$

In [And93], Andrews shows, using a result from [And71], that

$$q^{-k} \left(\left[\begin{array}{c} n \\ k \end{array}\right]_q - \left[\begin{array}{c} n \\ k-1 \end{array}\right]_q \right) = \sum_{\pi \in A[n,k]} q^{|\pi|}. \tag{11}$$

where $A[n,k]$ is the set of partitions in $L[n,k]$ with all ranks smaller than $n-2k$. Thus, $A[n,k]$ is also counted by (9). Let ϵ denote the empty partition. We can establish a bijection between $A[n,k]$ and $R_{\geq 1}[n,k] \cup \{\epsilon\}$ with an idea used by Fishel in [Fis95] as follows.

Let $\pi \in R_{\geq 1}(n)$. For $1 \leq i \leq d$, replace π_i by $n - k - \pi_i + d$; reorder rows into nondecreasing order. For $1 \leq i \leq d$, replace π'_i by $k - \pi'_i + d$; reorder columns into nondecreasing order. The net result is that rank vector $[r_1, \ldots, r_d]$ becomes $[n-2k-r_d, \ldots, n-2k-r_1]$, giving a nonempty partition in $A[n,k]$. It is easy to check that this is a bijection which is, in fact, its own inverse.

We also mention the result of Fishel in [Fis95] that

$$\left(\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right) = \sum_{\pi \in Q[n,k]} q^{|\pi|}. \tag{12}$$

where $Q[n,k]$ is the set of all partitions π in $L[n,k]$ satisfying $\pi_1 \geq k$, $\pi'_d = d$, and for $i = 1, \ldots, d-1$, $\pi_{i+1} \geq \pi'_i$ (where d is the size of the Durfee square of π). Fishel exhibits a bijection between $Q[n,k]$ and $A[n,k]$.

We define $\phi : Q[n,k] \to R_{\geq 1}[n,k] \cup \{\epsilon\}$ by the following procedure.

```
φ(π):
        if π₁ = n − k
                then delete a part 'n − k' from π;
                Otherwise add a part 'k' to π′.
```

We can show that ϕ is a bijection. Also note that neither of our two bijections nor the one of Fishel is weight-preserving.

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