Some Combinatorial Properties of the Coinvariant Algebra

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Abstract. We consider the action of the symmetric group S_n on the coinvariant algebra. This action induces a well known family of representations of S_n, associated with the Bruhat levels. We point out Kazhdan-Lusztig structures which appear in these representations.

First, a combinatorial rule for restricting these representations to Young subgroups is obtained. This rule is an exact analogue of Barbasch-Vogan's rule for restricting Kazhdan-Lusztig representations. Secondly, we describe the action of the simple reflections on these representations. The action of the simple reflections on the Schubert polynomials basis of the coinvariant algebra is similar to their action on the Kazhdan-Lusztig basis of the regular representation. This yields a combinatorial formula for the characters of S_n associated to its action on the graded coinvariant algebra.

The Major index of a permutation plays a crucial role in this work. The restriction rule follows from a remarkable theorem of Garsia and Gessel, which describes the distribution of the major index of shuffles. Our characters' formula is equivalent to Kraskiewicz-Weyman-Stanley rule for decomposing the coinvariant algebra into irreducible representations, which involves Major indices.

Résumé: L'action du groupe symétrique S_n sur l'algèbre co-invariante est considérée. Cette action induit une famille bien connue de représentations associée aux niveaux de Bruhat. Nous soulignons les structures de Kazhdan-Lusztig qui apparaissent dans ces représentations.

En premier lieu, une règle combinatoire pour la restriction de ces représentations aux sous-groupes de Young est obtenue. Cette règle est un analogue exact de la formule de Barbasch-Vogan pour la restriction des représentations de Kahzdan-Lusztig. Ensuite, une description de l'action des réflections simples sur ces représentations est donnée. L'action des réflections simples sur la base des polynômes de Schubert de l'algèbre co-invariante est similaire à leur action sur la base de Kazhdan-Lusztig de la représentation régulère. Cela conduit à une formule combinatoire pour les caractères de S_n associés à son action sur l'algèbre co-invariante graduée.

L'indice majeur d'une permutation joue un role crucial dans ce travail. La règle de restriction est un avatar d'un théorème remarquable de Garcia et Gessel qui décrit la répartition de l'indice majeur des produits battus. Notre formule des caractères est similaire à la règle combinatoire de Kraskiewicz, Weyman et Stanley pour la décomposition de lálgèbre co-invariante en représentations irréductibles, qui utilise la règle indices majeurs.

1. Introduction.

This extended abstract contains two parts. In the first part we study the restriction of the S_n-representations on the graded coinvariant algebra to parabolic (Young) subgroups. We apply a theorem of Garsia and Gessel on the distribution of the major index of shuffles to describe the combinatorial structure of the restricted algebra. In the second part we derive a combinatorial characters' formula for the representations induced by this algebra. This formula is equivalent to a well known combinatorial rule for decomposing the above graded algebra into irreducible representations of S_n.

The *length* of a permutation $\pi \in S_n$, $\ell(\pi)$, is the number of inversions $j < i$ so that $\pi(j) > \pi(i)$. The *major index* of $\pi \in S_n$, $MAJ(\pi)$, is the sum (possibly zero) of all indices $1 \le i \le n-1$ for which $\pi(i) > \pi(i+1)$. For example, let π be the permutation 31748625. Then $MAJ(\pi) = 1+3+5+6 = 15$ and $\ell(\pi) = 12$.

A classical result of MacMahon shows that the major index and the length function have the same generating function [M]. This implies that the number of permutations with a given major index coincides with the number of those having the same length. A bijective proof was given by Foata [F], [FS].

Garsia and Gessel gave what they call "a remarkable extension of MacMahon's result" [GG Theorem 3.1]. They prove that the major index of shuffles and their length are equidistributed. In [Ro4] we give a new bijective proof to Garsia-Gessel's theorem.

Any Coxeter group decomposes into two fundamental families of subsets of elements: the Bruhat levels and the Kazhdan-Lusztig cells. The Bruhat level $L(k)$ in a Coxeter group W consists of all elements of length k in W. The Kazhdan-Lusztig cells have a much more sophisticated description. See e.g. [Hu ch. 7].

These two families have a representation theory significance. Every Kazhdan-Lusztig cell $\mathcal{C}$ has an associated representation $\rho^{\mathcal{C}}$ of W. The regular representation of a Coxeter group W decomposes into a direct sum of these representations. Namely, the regular representation $\mathbb{Q}[W]$ is isomorphic to $\bigoplus \rho^{\mathcal{C}}$, where $W = \bigcup \mathcal{C}$. In the symmetric groups' case the Kazhdan-Lusztig cells are the well known Knuth equivalence classes, and the Kazhdan-Lusztig representations are the irreducible representations.

The situation is similar in the Bruhat levels' case. There is a representation associated with each Bruhat level. Denote the representation associated with $L(k)$ by R^k. Then $\mathbb{Q}[W]$ is isomorphic to $\bigoplus R^k$. This decomposition of the group algebra arises in the study of the coinvariant algebra. The restriction of these representations to S_{n-1} was studied in [GP Theorem 3.3].

The restriction of the Kazhdan-Lusztig representations to parabolic subgroups has a nice compatibility property [BV Prop. 3.11]. In the first part of this paper we apply Garsia-Gessel's theorem to describe the restriction of R^k to all parabolic subgroups. We show that the restriction of these representations has a similar compatibility property.

At last, another similarity between Kazhdan-Lusztig representations and R^k representations is presented. It is shown that the action of the simple reflections of S_n on the Schubert polynomials and their action on the Kazhdan-Lusztig basis are similar. This allow us to obtain similar characters' formulas for these two families of representations.

The rest of this abstract is organized as follows. Section 2 surveys basic properties of the coinvariant algebra, and some needed results from permutation statistics. A simple combinatorial rule for restricting R^k to Young subgroups is given in section 3. In section 4 we describe the action of the simple reflections on the basis of Schubert polynomials. This description implies a combinatorial formula for the characters of R^k (section 5).

This abstract does not include proofs. For proofs and more details see [Ro4] and [Ro5]. For q-analogues see [APR].

2. Preliminaries.

2.1 Schubert polynomials and the coinvariant algebra.

Let $x_1, x_2, \ldots, x_n$ be n independent variables, and let P_n be the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. The symmetric group acts on this ring by permuting the variables. $f \in P_n$ is called a symmetric function if $\pi f = f$ for every $\pi \in S_n$. Let I_n be the ideal generated by the symmetric functions of positive degree. The quotient P_n/I_n is called the *coinvariant algebra* of S_n. S_n acts naturally on this algebra. The resulted representation is isomorphic to the group algebra $\mathbb{Q}[S_n]$. See e.g. [Hu 3.6] and [Hi II.3].

Schubert polynomials form a distinguished basis of the coninvariant algebra P_n/I_n. For definitions and more details see [Md]. Any Schubert polynomial is indexed by a permutation in S_n. Denote the Schubert polynomial indexed by $\pi \in S_n$ by $\mathfrak{S}_\pi$. Then the degree of $\mathfrak{S}_\pi$ is $\ell(\pi)$. The coinvariant algebra P_n/I_n may be decomposed in the following way. The set of Schubert polynomials of degree k forms a basis for a subrepresentation R^k. It follows that each Bruhat level $L(k) = \{\pi \in S_n | \ell(\pi) = k\}$ has an associated representation R^k of dimension $|L(k)|$, so that $P_n/I_n = \bigoplus R^k$.

2.2 Major index of shuffles.

Denote the set of letters $\{1, 2, \ldots, m\}$ by $[m]$. For a sequence of integers $0 = m_0 < m_1 < m_2 < \cdots < m_k = n$ let $\pi_1 \in S_{m_1}, \pi_2 \in S_{m_2-m_1}, \ldots, \pi_k \in S_{m_k-m_{k-1}}$ be permutations on the letters $[m_1], [m_2] \setminus [m_1], \ldots, [m_k] \setminus [m_{k-1}]$ respectively. A permutation $\sigma \in S_n$ on the letters $[n]$ is called a $shuffle$ of $\pi_1, \pi_2, \ldots, \pi_k$ if the letters of $[m_i] \setminus [m_{i-1}]$ appear in σ in the same order as they appear in π_i for every i.

For example, $\sigma = 5136427$ is a shuffle of $\pi_1 = 132$ and $\pi_2 = 5647$.

Garsia and Gessel prove the following theorem.

Theorem 1. [GG Theorem 3.1] Let $A(\pi_1,\ldots,\pi_k)$ be the collection of all shuffles of $\pi_1,\pi_2,\ldots,\pi_k$. Then

$$\sum_{\sigma\in A(\pi_1,\ldots,\pi_k)} q^{MAJ(\sigma)} = \binom{n}{m_1, m_2-m_1,\ldots,m_k-m_{k-1}}_q q^{MAJ(\pi_1)+\ldots+MAJ(\pi_k)}$$

where $m_i - m_{i-1}$ is the cardinality of the set on which π_i acts, and $\binom{n}{m_1,m_2-m_1,\ldots,m_k-m_{k-1}}_q$ denotes the q-multinomial coefficient.

It is easy to see that the length function on shuffles has the same generating function. Namely,

$$\sum_{\sigma\in A(\pi_1,\ldots,\pi_k)} q^{\ell(\sigma)} = \binom{n}{m_1, m_2-m_1,\ldots,m_k-m_{k-1}}_q q^{\ell(\pi_1)+\ldots+\ell(\pi_k)}$$

Garsia-Gessel's theorem generalizes MacMahon's classical result, which shows that the length function and the major index have the same generating function. Foata gave a bijective proof to MacMahon's result [F]. See also [FS] and [Ca]. The proof of Garsia and Gessel suggests an implicit bijection, which uses heavily the theory of P-partitions. In [Ro4] we give an explicit simpler bijective proof to Garsia-Gessel's extended version.

3. Restriction of the R^k representations to Young subgroups.

In the second part of [Ro4] we apply Garsia-Gessel's theorem to obtain a simple combinatorial description of the restriction of representations to Young subgroups.

Barbasch and Vogan pointed out a beautiful combinatorial property of the Kazhdan-Lusztig cells. This property was first proved for Weyl groups in [BV Prop. 3.11]. It is easily generalized to all Coxeter groups [Ro2 Theorem 5.2].

Proposition 2 (Part I). Let H be a parabolic subgroup of a Coxeter group W, and let $\mathcal{C}$ be a Kazhdan-Lusztig left cell of W. Then $\mathcal{C}$ is a disjoint union of sets of the form $\mathcal{D}r$, where $\mathcal{D}$ is a Kazhdan-Lusztig left cell of H and r is a representative of minimal length of a right coset of H in W.

This decomposition has an algebraic interpretation.

Recall that each Kazhdan-Lusztig left cell $\mathcal{C}$ in a Coxeter group W has an associated representation. Denote this representation by $\rho^{\mathcal{C}}$.

Proposition 2 (Part II). Let $\mathcal{C} = \bigcup \mathcal{D}r$, where $\mathcal{C}$ is a Kazhdan-Lusztig left cell of a Coxeter group W, $\mathcal{D}$ are Kazhdan-Lusztig left cells of a parabolic subgroup H, and r are representatives of minimal length of right cosets of H in W. Then the restricted representation $\rho^{\mathcal{C}} \downarrow_H^W$ is isomorphic to the direct sum $\bigoplus\limits_{\mathcal{D}} \rho^{\mathcal{D}}$.

For any partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ denote the Young subgroup $S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_k}$ by S_μ. The Bruhat level $L(k)$ in S_n is the set of all permutations of length k in S_n. Let $L_\mu(j)$ be the set of all permutations of length j in the Young subgroup S_μ. The following fact is obvious.

Fact. For any fixed partition μ of n, $L(k)$ is a disjoint union of sets of the form $L_\mu(j)r$, where r are representatives of minimal length of cosets of S_μ in S_n and $j = k - \ell(r)$.

In [Ro4] we show that this fact has an algebraic interpretation in the sense of Proposition 2 (Part II).

It is well known that there are representations of S_n associated with its Bruhat levels. Any such a representation may be realized by the action of S_n on the Schubert polynomials indexed by the permutations $w \in L(k)$. See section 2.1. Denote the representation of S_n associated with the Bruhat level $L(k)$ by R^k.

Similarly, there exists a representation of the Young subgroups S_μ associated with the Bruhat level $L_\mu(j) = \{\pi \in S_\mu | \ell(\pi) = j\}$. Denote this representation by R^j_μ. R^j_μ may be defined as $\bigoplus_{i_1+\cdots+i_k=j} R^{i_1} \otimes \cdots \otimes R^{i_k}$, where R^{i_t} is a representation of S_{μ_t}.

Combining Proposition 2 with Garsia-Gessel's result (Theorem 1) we prove

Theorem 3. [Ro4 Theorem 3] The representation $R^k \downarrow^{S_n}_{S_\mu}$ is isomorphic to the direct sum of the representations R^j_μ, where the j indices are exactly those which appear in the decomposition $L(k) = \bigcup L_\mu(j)r$.

4. Action on Schubert polynomials.

The following lemma describes the action of the simple reflection $s_r = (r, r+1)$ on the Schubert polynomials basis of the coinvariant algebra.

Lemma 4. [Ro5 Lemma 3.1] For any simple reflection s_r, $(1 \le r < n)$, and any $w \in S_n$

$$s_r \mathfrak{S}_w = \begin{cases} \mathfrak{S}_w & \text{if } ws_r > w \\ -\mathfrak{S}_w - \sum\limits_{k<r} \mathfrak{S}_{w(k,r+1,r)} - \sum\limits_{k>r+1} \mathfrak{S}_{w(k,r,r+1)} + & \\ + \sum\limits_{k<r} \mathfrak{S}_{w(k,r,r+1)} + \sum\limits_{k>r+1} \mathfrak{S}_{w(k,r+1,r)} & \text{if } ws_r < w \end{cases}$$

where the sums run over elements of length $\ell(w)$, and $(k, r, r+1), (k, r+1, r)$ are cycles of length 3.

Corollary 5. For any simple reflection s and any $w \in S_n$

$$s\mathfrak{S}_w = \begin{cases} \mathfrak{S}_w & \text{if } ws > w \\ -\mathfrak{S}_w + \sum\limits_{zs>z} b_{z,w} \mathfrak{S}_z & \text{if } ws < w \end{cases}$$

where $b_{z,w} \in \{-1, 0, 1\}$ and $\ell(z) = \ell(w)$.

It is interesting to compare this corollary with its Kazhdan-Lusztig analogue.

Kazhdan-Lusztig lemma. [KL (2.3.a)-(2.3.c)] Let W be a Coxeter group, s be a Coxeter generator of W, $w \in W$, and C_w be the corresponding Kazhdan-Lusztig basis element. Then

$$sC_w = \begin{cases} -C_w & \text{if } sw < w \\ C_w + \sum\limits_{sz<z} b_{z,w} C_z & \text{if } sw > w \end{cases}$$

For a q-analogue of lemma 4 see [APR].

5. Combinatorial characters' formulas.

A new family of combinatorial characters' formulas was discovered recently. Characters of the symmetric group and the Iwahori-Hecke algebra, as well as characters of other Coxeter groups and algebras have a presentation as weighted sums of combinatorial structures. It should be noted that in most cases the weights are significantly easy to compute. For characters of the symmetric groups and Hecke algebras of type A see [Ro1], [Ra2], and [HLR]. For characters of other Coxeter groups and their Hecke algebras see [Ro1] and [Ro2]. For characters of Brauer and Birman-Wenzl algebras see [Ra3]

In [Ro1] the irreducible characters of the symmetric groups are presented as weighted sums of elements over Kazhdan-Lusztig cells. In [Ro5] we show that the characters of the graded coinvariant algebra may be presented as sums of same weights over Bruhat levels, where the kth Bruhat level of S_n consists of all permutations of length k. Needless to say that the Bruhat levels are not unions of Kazhdan-Lusztig cells.

The similarity between the action of the simple reflections on the Kazhdan-Lusztig basis and their action on the Schubert polynomials basis, presented in section 4, allow us to obtain similar characters' formula for the R^k-representations and the irreducible representations.

Let $\mu = (\mu_1, \ldots, \mu_t)$ and $\lambda = (\lambda_1, \ldots, \lambda_r)$ be partitions of n, and let χ^λ_μ be the character of a the irreducible representation S^λ at a conjugacy class of type μ. Let χ^k_μ be the character of R^k at a conjugacy class of type μ. In [Ro5] we prove

Theorem 6. [Ro5 Theorem 4.2]

$$\chi^k_\mu = \sum_{\ell(\pi)=k} weight_\mu(\pi)$$

where $weight_\mu(\pi)$ is defined as

$$weight_\mu(\pi) = \prod_{i=1}^{t} weight_{\mu,i}(\pi)$$

and

$$weight_{\mu,i}(\pi) = \begin{cases} (-1)^{\mu_i-1-m} & \text{if there exists } 0 \le m \le \mu_i - 1 \text{ so that} \\ & \pi(j) < \pi(j+1) \text{ for } \sum_{r=1}^{i-1} \mu_r < j \le \sum_{r=1}^{i-1} \mu_r + m \\ & \text{and } \pi(j) > \pi(j+1) \text{ for } \sum_{r=1}^{i-1} \mu_r + m < j < \sum_{r=1}^{i} \mu_r \\ 0 & \text{otherwise} \end{cases}$$

An analogous formula is known for Kazhdan-Lusztig characters (which are the irreducible characters in the symmetric groups' case).

Theorem 7. [Ro1 Corollary 4] Let $\mathcal{C}$ be a Knuth equivalence class of shape λ. Then

$$\chi_\mu^\lambda = \sum_{\pi \in \mathcal{C}} weight_\mu(\pi)$$

where $weight_\mu(\pi)$ is defined as in Theorem 4.

It should be noted that Theorem 7 presents a special case of a general formula for Kazhdan-Lusztig characters of Coxeter groups [Ro1 Corollary 3].

In section 5 of [Ro5] we show that Theorem 6 is equivalent to the following decomposition rule of Kraskiewicz and Weyman.

Let T be a standard tableau. Define the major index of T, $MAJ(T)$, to be the sum of all entries i, so that $i+1$ is southwest of i.

Theorem 8. [KW] The representation R^k is isomorphic to the direct sum $\bigoplus m_{k,\lambda} S^\lambda$, where S^λ are the irreducible representations of S_n and $m_{k,\lambda}$ is the number of standard tableaux of shape λ and major index k.

Theorem 8 was first proved by Kraskiewicz and Weyman [KW] in the context of free Lie algebras. R. Stanley suggested a different proof using symmetric functions arguments. This proof was followed by Stembridge [Ste], Garsia [Ga Theorem 8.6] and Reutenauer [Re Theorem 8.8].

Our proof combines arguments from Kazhdan-Lusztig theory with permutation statistics: Foata and Schützenberger constructed a bijection between all permutations with major index k and all permutations of length k, which preserves the descent set of w^{-1} [FS Theorem 1]. The equivalence of Theorems 6 and 8 is proved by combining Theorem 7, Foata-Schützenberger's bijection and basic facts in Kazhdan-Lusztig theory.

Finally, it is worth mentioning that a q-analogue of Theorem 7 is given in [Ro1] and [Ra2]. A q-analogue of Theorem 6 is given in [APR].

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