RIBBON TILE INVARIANTS

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ABSTRACT. Let $\mathbf{T}$ be a finite set of tiles, $\mathcal{B}$ be a set of regions Γ tileable by $\mathbf{T}$. We introduce a *tile counting group* $\mathbb{G}(\mathbf{T}, \mathcal{B})$ as a group of all linear relations for the number of times each tile $\tau \in \mathbf{T}$ can occur in a tiling of a region $\Gamma \in \mathcal{B}$. We compute the tile counting group for a large set of *ribbon tiles* also known as rim hooks in a context of representation theory of the symmetric group.

The tile counting group is presented by its set of generators which consists of certain new *tile invariants*. In a special case these invariants generalize Conway-Lagarias invariant for tromino tilings and a height invariant which is related to computation of characters of the symmetric group and goes back to G. de B. Robinson.

The heart of the proof is the rim hook bijection betwen rim hook tableaux of a given skew Young shape and certain standard skew Young tableaux. We also discuss signed tilings by the ribbon tiles and apply our results to the tileability problem.

Résumé Laissez $\mathbf{T}$ être un ensemble fini de tuiles, et $\mathcal{B}$ un ensemble de régions Γ tuileable par $\mathbf{T}$. Introduirons une groupe comptant de tuiles $\mathbb{G}(\mathbf{T}, \mathcal{B})$ comme un groupe de toutes les relations linéaires pour le nombre de fois chaque tuile $\tau \in \mathbf{T}$ peut se produire dans un carrelage d'une région $\Gamma \in \mathcal{B}$. Nous calculons le groupe comptant de tuiles pour un grand ensemble de tuiles rubané connues aussi sous le nom de crochets du bord dans un contexte de la théorie de représentation de groupes symétriques.

Le groupe comptant de tuiles est présenté par son ensemble de générateurs qui se compose de certains nouveaux invariants de tuile. Dans un cas spécial ces invariants généralisent l'invariant de Conway-Lagarias pour des carrelages à tromino et un invariant de la hauteur associé avec le calcul des caractères du groupe symétrique qui va de nouveau à G. de B. Robinson.

Nous discutons aussi les carrelages signés par les tuiles rubané et nous appliquons nos résultats au problème de tuileabilité.

1. INTRODUCTION

Let $\mathbb{Z}^2$ be a square lattice, $\mathcal{R}$ be a set of all compact simply connected regions in $\mathbb{Z}^2$. We think of these regions as of disjoint unions of (1×1)-squares. Sometimes they are called *polyominoes*. Fix a finite set of *tiles* $\mathbf{T} = \{\tau_1, \ldots, \tau_N\}$, $\tau_i \in \mathcal{R}$,

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$i = 1, \ldots, N$. Let tiles be invariant under translations. We say that a region $\Gamma \in \mathcal{R}$ is *tileable by* $\mathbf{T}$ if it can be presented as a disjoint union of the regions

$$\Gamma = \coprod_{1 \le j \le l} \tau'_j$$

where each region τ'_j, $1 \le j \le l$ is a translation of some τ_{i_j}. Such a disjoint union is called a *tiling* s of Γ. Denote by $\mathcal{S} = \mathcal{S}(\Gamma, \mathbf{T})$ the set of all tilings of Γ by the set of tiles $\mathbf{T}$.

Let us introduce the *tile counting group* and *tile invariants*. Let $\mathbf{T} = \{\tau_1, \ldots, \tau_N\}$ be a set of tiles. Denote by $\mathcal{R}_{\mathbf{T}} \subset \mathcal{R}$ the set of regions tileable by $\mathbf{T}$. Let $\mathcal{B} \subset \mathcal{R}_{\mathbf{T}}$ be a fixed subset of tileable regions. Consider a tileable region $\Gamma \in \mathcal{B}$. We identify each tiling $s \in \mathcal{S}(\Gamma, \mathbf{T})$ with its multiset of tiles $s \simeq \{\tau_{i_1}, \ldots, \tau_{i_l}\}$. Of course, by doing so we lose some information about the structure of the tilings since there could be many tilings of Γ with the same multiset of tile. As before, by $|\Gamma|$ we denote the area of Γ.

Let $\mathbb{Z}\langle \mathbf{T} \rangle$ be a group of formal integer linear combinations of $\mathbf{T}$. With each pair of tilings $s_1, s_2 \in \mathcal{S}(\Gamma, \mathbf{T})$ of a region $\Gamma \in \mathcal{B}$ we associate a relation:

$$(\tau_{i_1} + \cdots + \tau_{i_l} = \tau_{j_1} + \cdots + \tau_{j_r})$$

Let I be linear span of such relations for all regions $\Gamma \in \mathcal{B}$ and for all pairs of tilings $s_1, s_2 \in \mathcal{S}(\Gamma, \mathbf{T})$. Define *tile counting group* to be a quotient group

$$\mathbb{G}(\mathbf{T}; \mathcal{B}) = \mathbb{Z}\langle \mathbf{T} \rangle / I$$

This will be the main object of our study. Since both groups in the quotient are abelian, one can think of a tile counting group $\mathbb{G}(\mathbf{T}; \mathcal{B})$ as of a subgroup of $\mathbb{Z}\langle \mathbf{T} \rangle$. Thus it is reasonable to describe $\mathbb{G}(\mathbf{T}; \mathcal{B})$ by its set of independent generators (or the *basis*) given in $\mathbb{Z}\langle \mathbf{T} \rangle$.

For example, let $\mathbf{T}_2$ be a set of dominoes (see Fig. 1.1), and $\mathcal{B}$ be a set of simply connected regions. Two tilings in Figure 1.2 correspond to a relation $2 \cdot \tau_1 = 2 \cdot \tau_2$. It is known that every domino tiling of a simply connected region can be obtained from every other domino tiling by a sequence of such flips (see [T]). Therefore I in this case is generated by the relation above and we have

$$\mathbb{G}(\mathbf{T}_2; \mathcal{B}) = \mathbb{Z}^2 / I \simeq \mathbb{Z} \times \mathbb{Z}_2$$

The basis can be given as $\tau_1 + \tau_2$, $\tau_1 - \tau_2 \in \mathbb{Z}\langle \mathbf{T}_2 \rangle$. Note that the second generator has an order 2 as an element in $\mathbb{G}(\mathbf{T}_2; \mathcal{B})$, while it has an infinite order as an element of $\mathbb{Z}\langle \mathbf{T}_2 \rangle$.

Here is another way to look at the tile counting group. Let G be an abelian, not necessarily finite group. A map $f : \mathcal{B} \to G$ is called a *tile invariant* (or just an *invariant*) if for any tileable region $\Gamma \in \mathcal{B}$ and for any tiling $s \in \mathcal{S}(\Gamma, \mathbf{T})$, $s \simeq \{\tau_{i_1}, \ldots, \tau_{i_l}\}$ we have

$$f(\Gamma) = f(\tau_{i_1}) + \cdots + f(\tau_{i_l})$$

The problem is to find all the tile invariants for a fixed set of tiles $\mathbf{T}$. Clearly a tile invariant is determined by its values on $\mathbf{T}$, so the problem of finding invariant is equivalent to finding maps $f : \mathbf{T} \to G$ which can be extended to the set of all regions $\mathcal{B}$.

Let $\sum_{\tau\in\mathbf{T}} a(\tau) \in \mathbb{Z}\langle\mathbf{T}\rangle$ be an element of a tile counting group $\mathbb{G} = \mathbb{G}(\mathbf{T};\mathcal{B})$. Suppose m is its order in $\mathbb{G}$ (m could be infinity). Then a map $f : \mathbf{T} \to \mathbb{Z}_m$, $m < \infty$ or $f : \mathbf{T} \to \mathbb{Z}$, $m = \infty$ defined by $f(\tau) = a(\tau) \pmod m$ or $f(\tau) = a(\tau)$, is a tile invariant, where by $\mathbb{Z}_m$ we mean the additive group of integers modulo m. Conversely, every tile invariant can be lifted to an element of the tile counting group. Thus the problem of computing the tile counting group $\mathbb{G}(\mathbf{T})$ is equivalent to description of all invariants. We say that tile invariants $f_1, f_2, \ldots$ form an *independent basis of invariants* if they correspond to an independent generating set in a tile counting group.

Of course, when $\mathcal{B} = \mathbf{T}$ every map $f : \mathcal{B} \to G$ is a tile invariant, i.e $\mathbb{G}(\mathbf{T},\mathbf{T}) \simeq \mathbb{Z}^{|\mathbf{T}|}$. Generally, the bigger is our set of regions $\mathcal{B}$, the more equations on f we have, and the fewer tile invariants we get.

The first nontrivial example of the tile invariant that comes to mind is given by the area of tiles:

$$f_0 : \mathbf{T} \to \mathbb{Z}\,, \quad f_0(\tau_i) = |\tau_i|$$

which can be extended to all tileable regions: $f_0(\Gamma) = |\Gamma|$. This implies that the tile counting group is nontrivial. In the case of domino tiles $\mathbf{T}_2$ we also get another invariant:

$$f_* : \mathbf{T}_2 \to \mathbb{Z}_2\,, \quad f_*(\tau_1) = 0,\ f_*(\tau_2) = 1$$

(see above).

The main result of this paper is a description of a tile counting group for the following set of tiles.

Let the axis on a plane be as shown in Figure 1.3. We say that squares (i,j) and (i',j') lie on the same diagonal if $i - j = i' - j'$. For example, two squares $(2,4)$ and $(5,7)$ lie on the same diagonal. A *ribbon tile* is a simply connected region with no two squares lying on the same diagonal. An example of a ribbon tile is shown in Figure 1.3. Denote by $\mathbf{T}_n$ the set of all ribbon tiles τ with n squares: $|\tau| = n$. Obviously, set $\mathbf{T}_2$ is the set of domino tiles (see Fig 1.1). Sets $\mathbf{T}_3$ and $\mathbf{T}_4$ are shown in Figures 1.4,5.

Note that $|\mathbf{T}_n| = 2^{n-1}$. Indeed, we can code each ribbon tile by a sequence $(\varepsilon_1, \ldots, \varepsilon_{n-1})$ of $n-1$ zeroes and ones as follows. Call the lower left square the *starting square*. Begin with the starting square and move along the tile. Then, write $\mathbf{0}$ when going right, and write $\mathbf{1}$ when going up. See Figures 1.4,5 for sequences of all tiles in $\mathbf{T}_3$ and $\mathbf{T}_4$ respectively.

Definition 1 Consider a sequence of maps $f_1, \ldots, f_m : \mathbf{T}_n \to \mathbb{Z}$, $m = \left\lfloor \frac{n-1}{2} \right\rfloor$ defined as follows

$$f_i(\varepsilon_1, \ldots, \varepsilon_{n-1}) = \varepsilon_i - \varepsilon_{n-i}$$

We call the map f_i the *i-convexity invariant.*

Definition 2 A constant map $f_0 : \mathbf{T}_n \to \mathbb{Z}$ defined as

$$f_0(\varepsilon_1, \dots, \varepsilon_{n-1}) = 1$$

is called the *area invariant.*

Definition 3 If n is even, a map $f_* : \mathbf{T}_n \to \mathbb{Z}_2$ defined as

$$f_*(\varepsilon_1, \dots, \varepsilon_{n-1}) = \varepsilon_{n/2} \pmod 2$$

is called the *parity invariant.*

Before we state our main results, we need to specify the set of regions $\mathcal{B} \in \mathcal{R}_{\mathbf{T}_n}$. A region $\Gamma \in \mathcal{R}$ is called *row-convex* (*column-convex*) if every horizontal (vertical) line intersects Γ by an an interval or does not intersect it at all (see Fig. 1.6). Let $\mathcal{B}_{rc}$ be a set of tileable row-convex simply connected regions.

The main result of this paper is the following Theorem.

Theorem 4 Let $\mathcal{B} = \mathcal{B}_{rc}$ be as above. Then
1) when $n = 2m + 1$, $\mathbb{G}(\mathbf{T}_n, \mathcal{B}) \simeq \mathbb{Z}^{m+1}$ and the maps $f_0, f_1, \dots, f_m$ form an independent basis of invariants.
2) when $n = 2m$, $\mathbb{G}(\mathbf{T}_n, \mathcal{B}) \simeq \mathbb{Z}^m \times \mathbb{Z}_2$ and the maps $f_0, f_1, \dots, f_{m-1}, f_*$ form an independent basis of invariants.

When $n = 2$ Theorem 4 says that the area and parity invariants form an independent basis. Analogously when $n = 3$ the Theorem 1.3 says that beside the area invariant f_0 there exists one other nontrivial tile invariant $f_1 : \mathbf{T} \to \mathbb{Z}$, where

$$f_1(\mathbf{10}) = 1, \quad f_1(\mathbf{01}) = -1, \quad f_1(\mathbf{00}) = f_1(\mathbf{11}) = 0$$

(see Fig. 1.4). In a different form this invariant was discovered by Conway and Lagarias in [CL] (see also [T]). To say that f_1 is an invariant is equivalent to the following statement:

- $\#\,\mathbf{10} - \#\,\mathbf{01} = Const$

This means that the number of times the **10** tromino occurs in a tiling minus the number of times the **01** tromino occurs in the same tiling of a region Γ depends only on a region Γ and not on a tiling.

Here is one nontrivial invariant that exist for all $n > 1$.

Definition 5 Consider a map $f_\bullet : \mathbf{T}_n \to \mathbb{Z}_2$ defined as follows

$$f_\bullet(\varepsilon_1, \dots, \varepsilon_{n-1}) = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{n-1} \pmod 2$$

We call $f_\bullet$ the *height invariant.*

The reason why $f_\bullet$ is called the *height invariant* can be easily seen from the picture. Consider the smallest rectangular box the ribbon tile τ can fit in (see Fig 1.7). Then $f_\bullet(\tau) = a - 1 \pmod 2$, where a is the height of the rectangle. This invariant was considered earlier in connection with the certain characters of the symmetric group S_n (see [R, JK, St]). Observe that

$$f_\bullet = \begin{cases} f_1 + \cdots + f_{m-1} + f_m \mod 2, & n = 2m+1 \\ f_1 + \cdots + f_{m-1} + f_* \mod 2, & n = 2m \end{cases}$$

This proves that $f_\bullet$ is indeed an invariant provided Theorem 4 holds.

Now let us say a few words about how Theorem 4 is proved. We shall present a finite set of moves which preserve the invariants but enable us to get to any tiling starting with any other. Formally, let $\mathcal{B}_{sy}$ be the set *skew Young shapes* which is the set of row and column-convex regions such that when fit into the smallest possible box contain the upper right and the lower left corner of the box (see Fig. 1.8).

Theorem 6 Let $\mathcal{B} = \mathcal{B}_y$ be as above. For every $n > 1$ there is a finite set of at most $n\,4^n$ moves such that any tiling of $\Gamma \in \mathcal{B}$ by $\mathbf{T}_n$ can be transformed by a sequence of moves to any other such a tiling.

When $n = 2$ we need only one move (see Fig 1.2). When $n = 3$ we already need 6 moves (see Fig 1.9). The proof of Theorem 6 is based on a certain rim hook bijection which goes back to Nakayama and Robinson (see [R, JK, St]). Another version of this bijection is also attributed to Stanton and White (see [SW, FS]).

Now, the proof of Theorem 6 is constructive. Then one can simply check that the tile inariants in Theorem 4 preserve on these moves. Another argument shows that it is possible to extend the invariants from the set of skew Young shapes to the set of all row (column) convex shapes. Finally, we show that there are enough relations between the tiles to prove that there are no other invariant. This finishes the outline of the proof.

To finish this extended abstract, let us note that the nontrivial tile counting group is a phenomenon which exist for many interesting classes of tiles. Here is another example.

Theorem 7 Let $\mathbf{T}$ be a set of four tiles obtained by rotations of the tromino **01**. Let $\mathcal{B} = \mathcal{R}_{\mathbf{T}}$ be the set of all regions tileable by $\mathbf{T}$. Then the tile counting group $\mathbb{G}(\mathbf{T}, \mathcal{B}) \simeq \mathbb{Z} \times \mathbb{Z}_3^2$.

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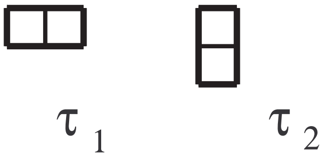


Figure 1.1

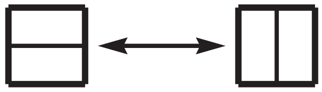


Figure 1.2

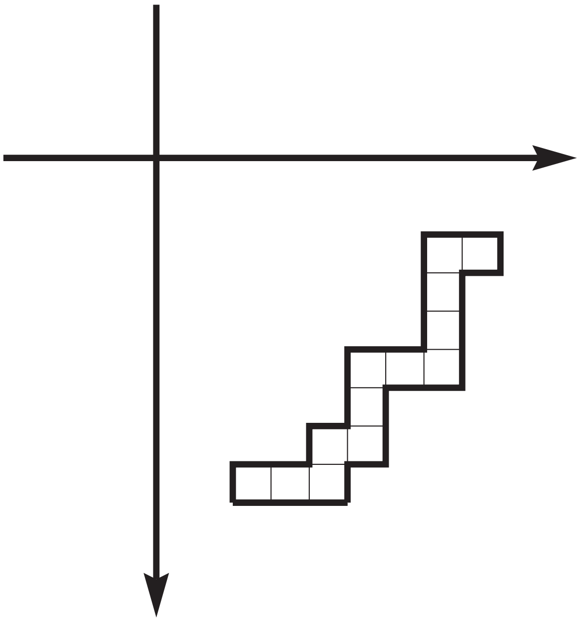


Figure 1.3

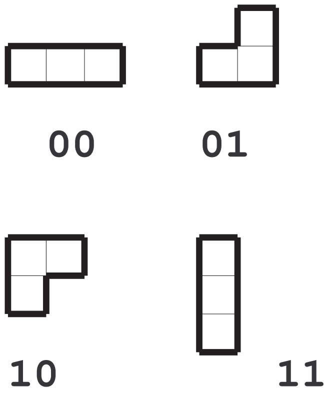


Figure 1.4

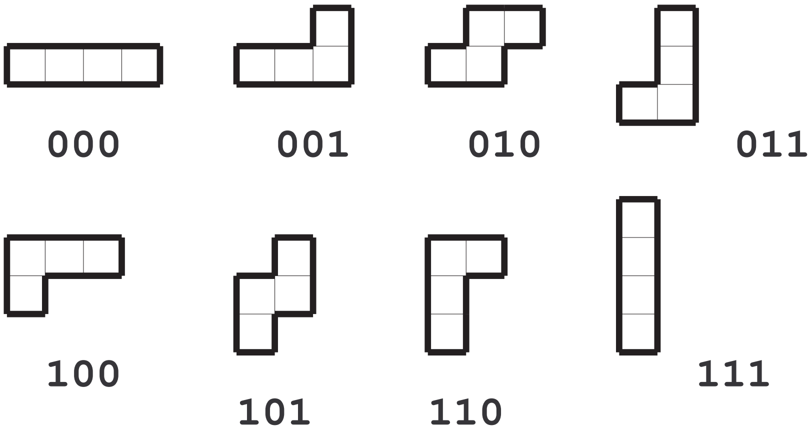


Figure 1.5

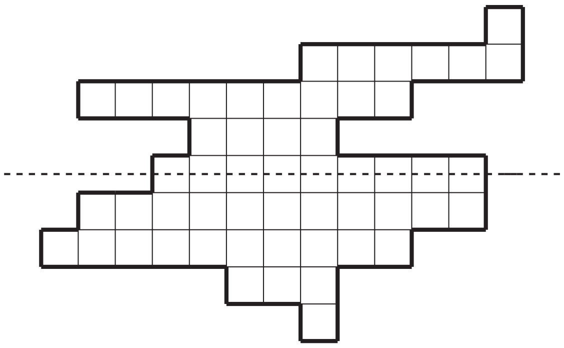


Figure 1.6

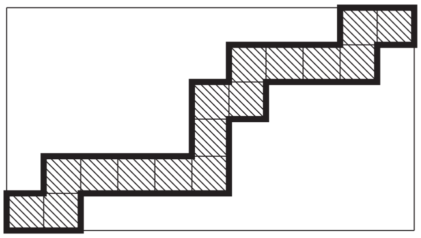


Figure 1.7

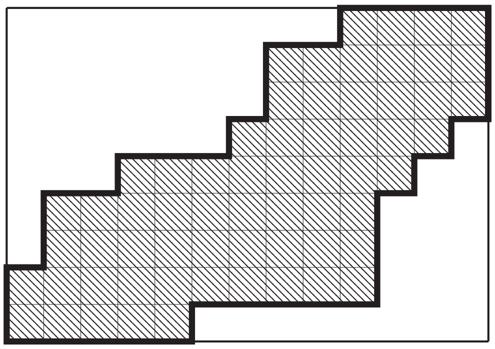
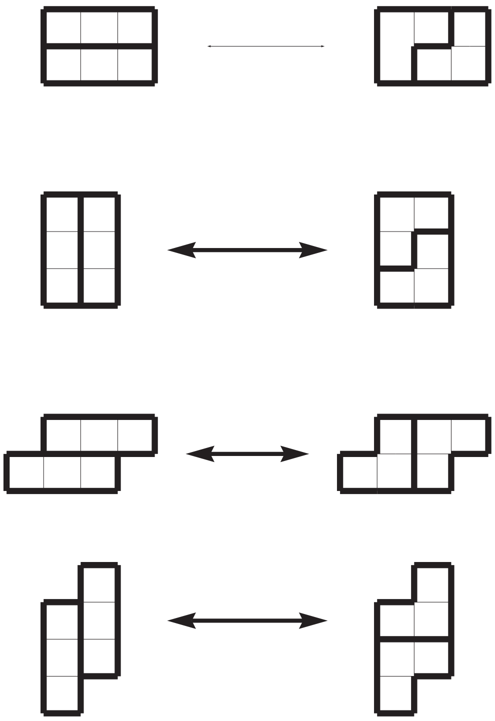


Figure 1.8



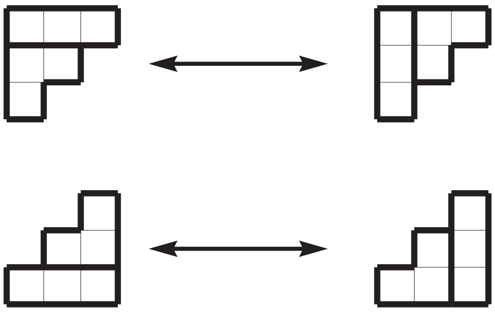


Figure 1.9