RECURSION AND EXPLICIT FORMULAS FOR MULTIVARIATE SAHI-KNOP AND MACDONALD POLYNOMIALS

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Abstract

Sahi and Knop introduced a family of non-homogeneous, non-symmetric polynomials. The top homogeneous components of these polynomials are the non-symmetric Macdonald polynomials. An appropriate Hecke algebra symmetrization of the non-symmetric Macdonald polynomials will then yield the symmetric Macdonald polynomials. The main results here are explicit fomulas for multivariate cases of the Sahi-Knop polynomials. A remarkable by-product of our proofs is the discovery that these polynomials satisfy a recursion on the number of variables.

Résumé

Sahi et Knop ont introduit une famille de polynômes non-homogènes et non-symétriques. Les parties homogènes de degrés supérieurs de ces polynômes donnent les polynômes de Macdonald non-symétriques. Les polynômes de Macdonald peuvent ensuite être obtenus en symétrisant ces polynômes de Macdonald non-symétriques à l'aide de l'algèbre de Hecke. Nous avons comme résultats principaux des formules explicites dans des cas particuliers à plusieurs variables des polynômes de Sahi-Knop. Une conséquence remarquable de nos preuves est l'existence d'une récursion sur le nombre de variables pour ces polynômes.

1. Introduction

The Macdonald polynomial basis, $\{P_\lambda(x;q,t)\}_\lambda$, is an object that has been given a great deal of attention in several areas of mathematics and physics as the result of the many difficult conjectures surrounding this basis. Important developments in the theory of symmetric functions rely on the use of the Macdonald basis which beautifully generalizes several fundamental bases including the Schur, Hall-Littlewood, Zonal, and Jack. The Macdonald polynomials are eigenfunctions of a family of commuting difference operators with significance in many-body physics [6] and they are known to be connected to the theory of basic hypergeometric series [10]. It has also been conjectured that these polynomials occur naturally in representation theory of the symmetric group [2]. A great constraint encountered in the study of the Macdonald basis stems from the absence of explicit formulas for these polynomials in terms of a familiar basis. The determination of such formulas presents a considerable challenge. Even ascertaining that the Macdonald polynomials have two parameter polynomial coefficients when they are expanded in terms of a modified Schur basis was a triumphant breakthrough [2],[4],[5],[7],[12]. Among the proofs that these Macdonald coefficients are polynomials is one which required the introduction of another family of polynomials. Sahi and Knop simultaneously introduced a family of non-symmetric and non-homogenous polynomials, $G_\alpha(x_1,x_2,\ldots,x_n;q,t)$ indexed by compositions. The top component of these polynomials yields the non-symmetric version of the Macdonald polynomials, $E_\alpha(x;q,t)$, which was introduced by Macdonald himself [9]. This non-symmetric version can then be connected to the Macdonald polynomials through a Hecke algebra symmetrization [4]. More precisely it is shown in [9] that if α is any composition that rearranges to λ then

$$P_\lambda \;=\; \sum_{\sigma\in S_n} t^{-length(\sigma)}\, T_\sigma\, E_\alpha$$

where T_σ is an appropriately defined Hecke algebra operator.

The Sahi-Knop polynomials are remarkable in that they are defined by very simple and elementary vanishing properties. This characterization yields a recursive algorithm for constructing the polynomials $G_\alpha(x_1,x_2,\ldots,x_n;q,t)$ and allows the derivation of several nice properties of the Macdonald polynomials. The result presented here is a natural expansion for multivariate cases of the Sahi-Knop polynomials in terms of a q-shifted monomial basis.

Our efforts have not only been motivated by the importance of the Macdonald polynomials, but also by the belief that the Sahi-Knop polynomials are a fundamental basis and a more intimate knowledge of this basis should be significant in any study of general polynomials in several variables.

It will be convenient here and after to let X_n denote the alphabet $\{x_1,x_2,\ldots,x_n\}$.

We shall also use throughout, the symbol $(a;q)_n$ customarily defined by setting

$$(a;q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a)\ .$$

With this notation, our main results can be stated as follows;

Theorem 1.

$$G_{(k,0^{n-1})}(X_n;q,t) = \sum_{b=0}^{k} d_b(k,n)\,(q^{b+1}\,t^{n-1}\,x_n;q)_{k-b}\,G_{(b,0^{n-2})}(X_{n-1};q,t) \tag{1.1}$$

where $(k,0^{n-1})$, $(b,0^{n-2})$ represent n and $n-1$ tuples, respectively, with non negative integral components, and

$$d_b(k,n) = \frac{(-1)^{k+b}\,(q;q)_k\,(t;q)_{k-b}\,(t;q)_{b+1}}{q^{\binom{k}{2}-\binom{b}{2}}\,t^{(n-1)(k-b)}\,(t;q)_{k+1}\,(q;q)_b\,(q;q)_{k-b}} \tag{1.2}$$

Theorem 2. *With the convention $x_{n+1}\equiv x_1$, we have:*

$$G_{(k,r,\ldots,r)}(X_n;q,t) = \sum_{b_1+\cdots+b_n=k-r}\left(C_{b_1,\ldots,b_n}\prod_{i=1}^{n}(x_i;q)_r\,(q^{b_1+\cdots+b_{i-1}+r}t^{i-1}x_{i+1};q)_{b_i}\right) \tag{1.3}$$

where

$$C_{b_1,\ldots,b_n} = \frac{q^{k-\binom{k-r}{2}-r-b_n}\,(q;q)_{k-r}\,(t;q)_{b_1}\cdots(t;q)_{b_{n-1}}\,(qt;q)_{b_n}}{(-1)^{k-r}\,t^{\sum_{j=1}^{n}(j-1)b_j}\,(qt;q)_{k-r}\,(q;q)_{b_1}\cdots(q;q)_{b_n}}\ . \tag{1.4}$$

We have mentioned that the non-symmetric Macdonald polynomials are obtained by taking the top component of the Sahi-Knop polynomials. Theorem 2 thus yields as a corollary an explicit formula for $E_{(k,r,\ldots,r)}(x;q,t)$. Namely we have

Corollary 3.

$$E_{(k,r,\ldots,r)}(X_n;q,t) = C_{k,r}(x)\sum_{b_1+\cdots+b_n=k-r}\left(\frac{(t;q)_{b_1}\cdots(t;q)_{b_{n-1}}\,(qt;q)_{b_n}\,x_1^{b_n}x_2^{b_1}\cdots x_n^{b_{n-1}}}{q^{b_n-\sum_{j=1}^{n}(b_1+\cdots+b_{j-1}+r)b_j-\binom{b_j}{2}}\,(q;q)_{b_1}\cdots(q;q)_{b_n}}\right)$$

with

$$C_{k,r}(x) = \frac{(-1)^{rn}\,(q;q)_{k-r}\,(x_1\cdots x_n)^r}{q^{\binom{k-r}{2}-n\binom{r}{2}+r-k}\,(qt;q)_{k-r}}\ . \tag{1.5}$$

Remarkably, it will be shown that the recursion in (1.1) can be explicitly solved to yield the special case $r = 0$ of (1.3). This given, the general formula in (1.3) readily follows from a characterizing property of Sahi-Knop polynomials.

To see how all this comes about and to understand the results we need to review the definitions and some of the basic properties of Sahi-Knop polynomials.

2. Preliminary information

We recall that a composition is a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with non negative integral components. The parameter n will be referred to as the *length* of α. We shall also set

$$|\alpha| \;=\; \alpha_1 + \alpha_2 + \cdots + \alpha_n \;,$$

and refer to it as the *order* of α. The expression "*α is a composition of m*" will simply mean $|\alpha| = m$.

Denote by α^* the partition obtained by rearranging the components of α in weakly decreasing order. The 'position vector', $k(\alpha) = \big(k_1(\alpha),\, k_2(\alpha), \ldots,\, k_n(\alpha)\big)$, is a crucial ingredient defined as follows: if α has distinct parts then each α_i occupies a well defined position $k_i = k_i(\alpha)$ in α^*. By this we mean that $\alpha_i = \alpha^*_{k_i}$. We may then extend the definition of $k(\alpha)$ to the case in which α has equal components, breaking ties by considering equal parts as decreasing from left to right. In other words, if we label the parts of α by decreasing size and from left to right then $k_i(\alpha)$ is taken to be the label of α_i.

This given, Sahi and Knop associate a vector of monomials $\overline{\alpha}$ to each composition α. The vector contains parts defined by

$$(\overline{\alpha})_i \;=\; q^{-\alpha_i} t^{-n+k_i(\alpha)} \;. \tag{2.1}$$

This notation allows us to present the Sahi-Knop results. To begin with, adhering to Sahi's notation, it is shown in [12] that if α is a composition of m then in the linear span of the monomials $\{x^\beta\}_{|\beta|\le m}$ there exists a unique polynomial $G_\alpha(x;q,t)$ which satisfies the following two conditions:

$$\begin{array}{llll} (a) & G_\alpha(\overline{\beta};q,t) & = & 0 \qquad \text{for all } |\beta| \le |\alpha| \text{ and } \beta \ne \alpha \;, \\ (b) & G_\alpha(x;q,t)\,|_{x^\alpha} & = & 1. \end{array} \tag{2.2}$$

The uniqueness part of the Sahi-Knop result is relatively easy to show, yet uniqueness permits the immediate derivation of a number of surprising identities and recursions. Some of these are given by Sahi in [12] and others only in Knop [5]. Their basic identities which we shall use are expressed by the following three properties:

Property 4. *If $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $\alpha = (\gamma_1 + r, \ldots, \gamma_n + r)$, then*

$$G_\alpha(x;q,t) \;=\; \prod_{i=1}^{n} (x_i;q)_r \;\; G_\gamma(q^r x;q,t) \tag{2.3}$$

Property 5. *If $\alpha_n > 0$, then*

$$G_\alpha(x;q,t) \quad = \quad (1-x_n)\, G_{(\alpha_n-1,\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(qx_n, x_1, x_2, \ldots, x_{n-1};q,t) \;\; . \tag{2.4}$$

For $1 \leq i \leq n-1$, let $s_i = (i, i+1)$ denote the transposition that interchanges x_i and x_{i+1} and set [8]

$$T_{s_i} \;=\; s_i \;+\; \frac{(1-t)}{x_i - x_{i+1}}\, x_i\, (1 - s_i) \;\; . \tag{2.5}$$

It is well known that the operators T_{s_i} generate a faithful representation of the Hecke algebra of S_n in the space of polynomials in $x_1, x_2, \ldots, x_n$. It can be verified, now using symbolic manipulation software, that we have

$$\begin{array}{lrcll} a) & T_{s_i}T_{s_j} & = & T_{s_j}T_{s_i} & \text{for} \quad |i-j| > 1 \;\; , \\ b) & T_{s_i}T_{s_{i+1}}T_{s_i} & = & T_{s_{i+1}}T_{s_i}T_{s_{i+1}} & \text{for} \quad i = 1 \ldots n-1 \;\; , \\ c) & t\,T_{s_i}^{-1} & = & s_i \;+\; \dfrac{(1-t)}{x_i - x_{i+1}}\,(1-s_i)\, x_{i+1} \;\; . & \end{array} \tag{2.6}$$

This permits us to extend the definition of T to all permutations $\sigma \in S_n$ by setting for any reduced expression $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$,

$$T_\sigma \;=\; T_{s_{i_1}} T_{s_{i_2}} \cdots T_{s_{i_k}} \;\; . \tag{2.7}$$

Property 6.

$$(a) \quad G_\alpha(x;q,t) \;=\; T_{s_i}\, G_\alpha(x;q,t) \;=\; s_i\, G_\alpha(x;q,t) \qquad \text{if } \alpha_i = \alpha_{i+1}$$

$$(b) \quad G_{s_i\alpha}(x;q,t) \;=\; (1 - \frac{\overline{\alpha}_{i+1}}{\overline{\alpha}_i})\, T_{s_i} G_\alpha(x;q,t) + (t-1)\, G_\alpha(x;q,t) \qquad \text{if } \alpha_i \neq \alpha_{i+1} \, . \tag{2.8}$$

The properties 4, 5, and 6 combine nicely into a recursive algorithm for computing the polynomials G_α given the initial condition $G_{(0,0,\ldots,0)}(x;q,t) = 1$. This enables rapid computation of extensive tables. Our strategy has been to induct general identities from examination of special cases. The results suggest that further investigation of this kind may lead to explicit formulas in a variety of more general cases.

3. Recursion and induction

It is easy to see that property 4 yields theorem 2 as an immediate corollary of the following special case

Theorem 7. *With the convention $x_{n+1} \equiv x_1$, we have*

$$G_{(k,0,\ldots,0)}(X_n;q,t) \;=\; \sum_{b_1+\cdots+b_n=k} \left(C_{b_1,\ldots,b_n} \prod_{i=1}^{n} (q^{b_1+\cdots+b_{i-1}} t^{i-1} x_{i+1};q)_{b_i} \right) \tag{3.1}$$

where

$$C_{b_1,\ldots,b_n} \;=\; \frac{(-1)^k\, q^{k-\binom{k}{2}-b_n}\, (q;q)_k\, (t;q)_{b_1} \cdots (t;q)_{b_{n-1}}\, (qt;q)_{b_n}}{t^{\sum_{i=1}^{n}(i-1)b_i}\, (qt;q)_k\, (q;q)_{b_1} \cdots (q;q)_{b_n}}\,. \tag{3.2}$$

The proof of this identity is the ultimate consequence of a number of auxiliary results which should be of intrinsic interest. We begin by showing that the family of polynomials defined by setting

$$W_k(X_m) \;=\; \sum_{b_1+\cdots+b_m=k} \left(C_{b_1,\ldots,b_m} \prod_{i=1}^{m} (q^{b_1+\cdots+b_{i-1}} t^{i-1} x_{i+1};q)_{b_i} \right)\,, \tag{3.3}$$

where $x_{m+1} \equiv x_1$ and $C_{b_1,\ldots,b_m}$ is as given in (3.2), satisfies the recursion stated by theorem 1. To be precise we show that

Proposition 8. *For all $k \geq 0$ and $m \geq 1$ we have*

$$W_k(X_m) \;=\; \sum_{b_m=0}^{k} \frac{(q;q)_k\, (t;q)_{k-b_m}\, (t;q)_{b_m+1}\, (q^{b_m+1}\, t^{m-1}\, x_m;q)_{k-b_m}}{(-1)^{k+b_m}\, q^{\binom{k}{2}-\binom{b_m}{2}}\, t^{(m-1)(k-b_m)}\, (t;q)_{k+1}\, (q;q)_{b_m}\, (q;q)_{k-b_m}}\, W_{b_m}(X_{m-1})\,. \tag{3.4}$$

Given this recursion, the proof that W_k are in fact Sahi-Knop polynomials (theorem 7) relies on an inductive argument on the number of variables.

The induction hypothesis may be stated as

$$G_{(k,0,\ldots,0)}(X_{n-1};q,t) \;=\; W_k(X_{n-1})\,, \tag{3.5}$$

and used in conjunction with the recursive nature of W_k will help verify our theorem. Specifically, if we let $m = n$ and $b_m = b$ in (3.4), we obtain the relation

$$W_k(X_n) \;=\; \sum_{b=0}^{k} \frac{(q;q)_k\,(t;q)_{k-b}\,(t;q)_{b+1}\,(q^{b+1}\,t^{n-1}\,x_n;q)_{k-b}}{(-1)^{k+b}\,q^{\binom{k}{2}-\binom{b}{2}}\,t^{(n-1)(k-b)}\,(t;q)_{k+1}\,(q;q)_b\,(q;q)_{k-b}}\,W_b(X_{n-1})\,.$$

The induction hypothesis (3.5) for $k = b$ allows us to rewrite this relation as a useful identity:

$$W_k(X_n) \;=\; \sum_{b=0}^{k} \frac{(q;q)_k\,(t;q)_{k-b}\,(t;q)_{b+1}\,(q^{b+1}\,t^{n-1}\,x_n;q)_{k-b}}{(-1)^{k+b}\,q^{\binom{k}{2}-\binom{b}{2}}\,t^{(n-1)(k-b)}\,(t;q)_{k+1}\,(q;q)_b\,(q;q)_{k-b}}\,G_{(b,0,\ldots,0)}(X_{n-1};q,t)\,. \tag{3.6}$$

The following implication of this identity is shown as a preliminary to carring out the induction.

Lemma 9. *$W_k(X_n)$ is symmetric in the variables $x_2,\ldots,x_n$.*

4. The vanishing

To complete the induction argument, we are requried to show that the Sahi-Knop polynomials $G_{(k,0,\ldots,0)}(X_n;q,t)$ can be expressed as $W_k(X_n)$. This requires that we verify $W_k(X_n)$ satisfies the characterizing properties, (2.2a) and (2.2b), of the Sahi-Knop polynomials. More precisely, we must show that $W_k(X_n)$ vanishes on all $\overline{\beta}$ where $|\beta| \le k$ and $\beta \ne (k,0,\ldots,0)$. To do this, we must separate these β into two groups: those in which there exists some component of β which is stricly larger than β_1, and the β of which β_1 is the weakly largest component. The first case, the vanishing on β in which there exist a component larger than β_1, is the result of two lemmas.

The first lemma is a simple consequence of definition (2.1) for $\overline{\beta}$.

Lemma 10. *Let $\beta = (\beta_1,\ldots,\beta_n)$ where $\beta_j \ge \beta_i$ for $1 \le i \le n$ and $j < i$.* (4.1)
Then for $\gamma = (\beta_1,\ldots,\beta_{j-1},\beta_{j+1},\ldots,\beta_n)$, we have $\overline{\gamma} = (\overline{\beta}_1,\ldots,\overline{\beta}_{j-1},\overline{\beta}_{j+1},\ldots,\overline{\beta}_n)$.

The second lemma contributing to our claim depends on an extra vanishing property of the Sahi-Knop polynomials that is given by Knop. The following ordering,

Let α, γ be compositions with length n. Then $\alpha \le \gamma$ if there is a permutation $\pi \in S_n$ such that $\alpha_i < \gamma_{\pi(i)}$ for $i < \pi(i)$ and $\alpha_i \le \gamma_{\pi(i)}$ for $i \ge \pi(i)$,

is necessary to introduce his important result [6].

Property 11.

$$G_\alpha(\overline{\gamma};q,t) \;=\; 0 \;\text{ for all } \alpha \not\le \gamma.$$

Lemma 12. *Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\gamma = (\gamma_1, \ldots, \gamma_m)$.*
If $\gamma_1 < \alpha_1$ and $\gamma_i \leq \alpha_1$ for all $2 \leq i \leq m$, then $G_\alpha(\overline{\gamma}; q, t) = 0$.

The symmetry held by $W_k(X_n)$ as shown with lemma 9 allows us to obtain the following expression for $W_k(\overline{\beta})$.

$$W_k(\overline{\beta}) \;=\; \sum_{b=0}^{k} \frac{(q;q)_k\,(t;q)_{k-b}\,(t;q)_{b+1}\,(q^{b+1-\beta_j};q)_{k-b}\,G_{(b,0,\ldots,0)}(\overline{\gamma};q,t)}{(-1)^{k+b}\,q^{\binom{k}{2}-\binom{b}{2}}\,t^{(n-1)(k-b)}\,(t;q)_{k+1}\,(q;q)_b\,(q;q)_{k-b}}\;. \tag{4.2}$$

The summand in (4.2) only vanishes if either $(q^{b+1-\beta_j};q)_{k-b} = 0$ or $G_{(b,0,\ldots,0)}(\overline{\gamma};q,t) = 0$. It develops using the preceeding lemmas, that one of these cases occurs for **each** $0 \leq b \leq k$ thus giving the vanishing indicated in our first case.

To complete the verification that $W_k(X_n)$ satisfies property $(2.2a)$ we must prove:

For $\beta = (\beta_1, \ldots, \beta_n)$ where $|\beta| \leq k$ and $\beta \neq (k, 0, \ldots, 0)$,
If $\beta_1 \geq \beta_i$ for all $2 \leq i \leq n$, then $W_k(\overline{\beta}) = 0$.

We first show the vanishing of $W_k(\overline{\beta})$ for all β a partition. This may be accomplished with an iterative argument using the explicit formula 3.3 and again the symmetry of $W_k(X_n)$. The extension of this result to all β as specified in our claim can then be made, proving the vanishing conditions hold.

5. Normalization and conclusion

The uniqueness of the Sahi-Knop result requires only that we ensure $W_k(X_n)$ has the proper normalization having shown that these polynomials satisify the vanishing properties that characterize $G_{(k,0,\ldots,0)}(X_n; q, t)$. The desired normalization is given with condition $(2.2b)$, and it is easy to see that the coefficient of x_1^k in our formula for $W_k(X_n)$ is one.

The extension of these results to entirely general formulas for the Sahi-Knop polynomials has yet to be achieved. Preliminary results suggest that these general formulas could provide a direct relation between non-symmetric Macdonald polynomials and the symmetric Macdonald polynomials using the Kostka-Foulkes coefficients.

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