On the dynamics of homomorphisms of free algebras

Alexander A. Mikhalev, Vladimir Shpilrain, and Jie-Tai Yu

Abstract. We consider several combinatorial problems about polynomial algebras, free associative and Lie algebras. Those problems are mainly concerned with various dynamical properties of homomorphisms.

Our main idea is to study automorphisms and, more generally, homomorphisms of various algebraic systems by means of their action on "very small" sets of elements, as opposed to a traditional approach of studying their action on subsystems (like subgroups, subalgebras, ideals, etc.) We will show that there is a lot that can be said about a homomorphism, given its action on just a single element, if this element is "good enough". Then, we consider somewhat bigger sets of elements, like, for example, automorphic orbits, and study a variety of interesting problems arising in that framework.

Résumé. Nous considerons quelques problèmes combinatoires sur les algèbres polynomiales, associatives libres et l'algèbres de Lie libres. Ces problèmes concernent, en principe, les propriétés dynamiques différentes d'homomorphismes.

Notre idée principale est de traiter des automorphismes et, plus généralement, les homomorphismes des systèmes algébriques différentes, par le moyen de leur actions sur des ensembles "très petits" des éléments, à l'inverse de l'approche traditionnelle de traiter leur actions sur des sous-systémes (comme sous-groupes, sous-algèbres, ideals, etc). Nous montrerons qu'on pourra dire beaucoup de choses de l'homomorphisme, qui est donné par son action sur un élément, si cet élément est "assez bon". Ainsi, nous considerons en quelque point plus grands ensembles des éléments comme, par exemple, l'orbites automorphes et étudions une série des problèmes interessants, qui apparaissent dans ce sujet.

1 Preliminaries

Let K be a field, X a nonempty set, $\Gamma(X)$ *the free groupoid of nonassociative monomials* (free Magma) without unity element on the alphabet X, i.e. $X \subseteq \Gamma(X)$; if $u, v \in \Gamma(X)$, then $u \circ v \in \Gamma(X)$, where $u \circ v$ is the formal multiplication on nonassociative words. We consider the linear space $F(X)$ over K with the basis consisting of the elements of $\Gamma(X)$ with the multiplication given by

$$(\alpha \cdot a) \cdot (\beta \cdot b) = (\alpha\beta) \cdot (a \circ b)$$

for all $\alpha, \beta \in K$, $a, b \in \Gamma(X)$. The algebra $F(X)$ is *the free nonassociative K-algebra without the unity element on the set X*. The algebra $F(X)$ is the free algebra with the set X of free generators in the variety of all K-algebras. Main combinatorial properties of the free nonassociative algebra are different from the properties of free associative algebras and polynomial algebras. For instance, the free nonassociative algebra with a single free generator is not a polynomial algebra in one variable, moreover, each subalgebra of this algebra is free, and it contains free subalgebras of any finite or countable rank.

A K-subspace I of $F(X)$ is said to be a (two sided) *ideal* of $F(X)$ if $ab, ba \in I$ for all $a \in F(X)$ and $b \in I$. Any K-algebra A is a homomorphic image of some free nonassociative K-algebra. For a subset Z of $F(X)$ by $I(Z)$ we denote the ideal of $F(X)$ generated by Z, i.e. the least ideal of $F(X)$ such that $Z \subseteq I(Z)$. Let

$$Z_1 = \{ab - ba \mid a, b \in F(X)\}, \; Z_2 = \{ab + ba \mid a, b \in F(X)\}.$$

The algebra $F(X)/I(Z_1)$ is *the free commutative* (nonassociative) *algebra* on the set X of free generators, and the algebra $F(X)/I(Z_2)$ is *the free anti-commutative* (nonassociative) *algebra*. These algebras are the free algebras in the varieties of all commutative K-algebras and all anti-commutative K-algebras, respectively. For more information about varieties and free algebras see, for example, the monographs [3] by P. M. Cohn and [13] by A. I. Malcev.

Let G be an Abelian semigroup, K a field, $\operatorname{char} K \neq 2$, $\varepsilon\colon G \times G \to K^*$ a skew symmetric bilinear form (a commutation factor, a bicharacter), that is

$$\varepsilon(g_1 + g_2, h) = \varepsilon(g_1, h)\,\varepsilon(g_2, h),\; \varepsilon(g, h_1 + h_2) = \varepsilon(g, h_1)\,\varepsilon(g, h_2),\; \varepsilon(g, h)\,\varepsilon(h, g) = 1$$

for all $g, g_1, g_2, h, h_1, h_2 \in G$,

$$G_- = \{g \in G \mid \varepsilon(g, g) = -1\}, \quad G_+ = \{g \in G \mid \varepsilon(g, g) = +1\}.$$

A G-graded K-algebra $R = \bigoplus_{g \in G} R_g$ is *a color Lie superalgebra* if

$$[x, y] = -\varepsilon(d(x), d(y))[y, x], \quad [v, [v, v]] = 0,$$
$$[x, [y, z]] = [[x, y], z] + \varepsilon(d(x), d(y))[y, [x, z]]$$

with $d(v) \in G_-$ for G-homogeneous elements $x, y, z, v \in R$, where $d(a) = g$ if $a \in R_g$.

If $G = \mathbf{Z}_2$ and $\varepsilon(f, g) = (-1)^{fg}$, then color Lie superalgebra ia a Lie superalgebra. If $\varepsilon \equiv 1$, then we have a G-graded Lie algebra (if $G = \{e\}$, then we have a Lie algebra).

We denote $(\mathrm{ad}\, a)(b) = [a, b]$ for all $a, b \in R$. Let $\mathrm{char}\, K = p > 2$. A color Lie superalgebra R over K is *a color Lie p-superalgebra* if on G-homogeneous components R_g, $g \in G_+$, we have a mapping $x \to x^{[p]}$, $d(x^{[p]}) = pd(x)$, such that for all $\alpha \in K$ and all G-homogeneous elements $x, y, z \in R$ with $d(x) = d(y) \in G_+$, the following conditions are fulfilled:

$$(\alpha x)^{[p]} = \alpha^p x^{[p]}, \quad (\mathrm{ad}\,(x^{[p]}))(z) = [x^{[p]}, z] = (\mathrm{ad}\, x)^p(z),$$
$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum s_j(x, y),$$

where $js_j(x, y)$ is the coefficient on t^{j-1} in the polynomial $(\mathrm{ad}\,(tx + y))^{p-1}(x)$.

If Q is a G-graded associative algebra over K then $[Q]$ denotes the color Lie superalgebra with the operation $[\,,\,]$ where $[a, b] = ab - \varepsilon(d(a), d(b))\, ba$ for G-homogeneous elements $a, b \in Q$. If $\mathrm{char}\, K = p > 2$, and $x^{[p]} = x^p$ for all G-homogeneous $x \in Q$ with $d(x) \in G_+$, then $[Q]$ with the operation $[p]$ is a color Lie p-superalgebra denoted by $[Q]^p$. In what follows we consider only G-homogeneous elements, homomorphisms preserving the G-graded structure, etc.

Let $X = \{x_1, \ldots, x_n\} = \bigcup_{g \in G} X_g$ be a G-graded set, that is $X_g \cap X_f = \emptyset$ for $g \neq f$, $d(x) = g$ for $x \in X_g$, and let $A(X)$ be the free G-graded associative K-algebra, $L(X)$ the subalgebra of $[A(X)]$ generated by X. Then $L(X)$ is *the free color Lie superalgebra with the set X of free generators*. In the case when $\mathrm{char}\, K = p > 2$ let $L^p(X)$ be the subalgebra of $[A(X)]^p$ generated by X. Then $L^p(X)$ is *the free color Lie p-superalgebra* on X. A. A. Mikhalev in [14]–[16] proved that subalgebras of free color Lie superalgebras and p-superalgebras are free (see also monographs [1, 25] and the review paper [17]). For free Lie superalgebras this result was obtained also by A. S. Shtern in [33].

In what follows $F{=}F(X)$ denotes the free algebra without the unity element on the set X of free generators of one of the following varieties of algebras over a field K: the variety of all algebras; the variety of Lie algebras; varieties of color Lie superalgebras; the variety of Lie p-algebras; varieties of color Lie p-superalgebras; varieties of commutative and anti-commutative algebras.

2 Test elements and retracts of free algebras

Fixed points of endomorphisms and automorphisms of algebraic systems are under intensive consideration by many algebraists. An element u is called a *test element* if for any endomorphism φ it follows from $\varphi(u) = u$ that φ is an automorphism.

This definition was explicitly given by V. Shpilrain in [30]. But the history of test elements goes back to J. Nielsen. A classical result of Nielsen [27] states that an endomorphism $x \to f$; $y \to g$ of the free group F_2 with two generators x, y is an automorphism if and only if $[f, g]$ is conjugate to $[x, y]$. Hence the commutator $[x, y] = xyx^{-1}y^{-1}$ is a test element of F_2. W. Dicks [4, 5] proved a similar result for the free associative algebra $K\langle x, y\rangle$ of rank two. In [6, 8], test elements of polynomial algebras and of free associative algebras are under consideration.

In [34] E. C. Turner proved that test elements for monomorphisms of free groups of finite rank are exactly the elements with maximal rank (an element of the free group has maximal rank if it does not belong to any free factor of the group), and that test elements for endomorphisms of free groups are elements not contained in proper retracts. The analogs of these results for free Lie algebras and free color Lie superalgebras and p-superalgebras were proved by A. A. Mikhalev and A. A. Zolotykh in [26], and by A. A. Mikhalev and J.-T. Yu in [18], respectively.

The rank of $a \in F = F(X)$ ($\mathrm{rank}\,(a)$) is the smallest number of generators from X on which an element $\varphi(a)$ depends on, where φ runs on the automorphism group of F (in other words, $\mathrm{rank}\,(a)$ is the smallest rank of a free factor of F containing a). Matrix criteria for a system of elements to have given rank was obtained by A. A. Mikhalev and A. A. Zolotykh in [22, 23] for free Lie algebras, free color Lie superalgebras and p-superalgebras, and by U. U. Umirbaev in [37] for free groups.

A subalgebra H of F is *a retract* if there is a homomorphism ρ: $F \to H$ such that ρ is the identity mapping on H (ρ is *the retraction homomorphism*). Obviously a subalgebra H of F is a retract if and only if there exists an ideal I of F such that $F = H \bigoplus I$.

Let F_n be the free group of rank n, h a non-identity element of F_n, and $\mathrm{Orb}(h)$ the orbit of h under the action of the automorphism group of F_n. The following problem was raised by V. Shpilrain in [30].

Problem 1 *Suppose that φ is an endomorphism of F_n such that*

$$\varphi(\mathrm{Orb}(h)) \subseteq \mathrm{Orb}(h).$$

for some non-identity element h of F_n. Is then φ an automorphism of F_n?

For $n = 2$ the positive solution was given by S. V. Ivanov in [9] and by V. Shpilrain in [31]. It is interesting to consider same problem for free algebras (Lie, associative, relatively free algebras). V. Drensky and J.-T. Yu in [7] solved this problem for $n = 2$ for free Lie algebras and for free metabelian associative and Lie algebras. A. A. Mikhalev and J.-T. Yu solved the problem in [19] for $n = 2$ for free Lie p-algebras over a perfect field (in general case they considered elements of rank one). One may consider a special case when h is a primitive element, i.e., h belongs to a free generating set of the free group (of the free algebra, respectively). In [21, 24], A. A. Mikhalev and A. A. Zolotykh solved the problem for free Lie algebras and for free color Lie (p-) superalgebras assuming that h is a primitive element (matrix criteria for an element and for a system of elements of these algebras to be primitive were obtained by A. A. Mikhalev and A. A. Zolotykh in [22, 23], similar result for free groups was proved by U. U. Umirbaev in [35]). A. van den Essen and V. Shpilrain in [8] obtained analogous result for a primitive element h of the polynomial algebra in two variables (of the free associative algebra of rank two, respectively), and they proved that if *the Jacobian conjecture* (that is that any endomorphism of the polynomial algebra over a field of characteristic zero with the invertible Jacobian matrix is an automorphism) is true, then the problem has an affirmative answer for a primitive element of the finitely generated polynomial algebra over an algebraically closed field of characteristic zero. Finally, recently Z. Jelonek [10] proved this statement for polynomial algebras over the field of complex numbers (his proof used methods of algebraic geometry). In [20, 24] A. A. Mikhalev, J.-T. Yu, and A. A. Zolotykh showed that if K is a non-algebraically closed field, then it is not enough to consider only images of non-degenerate linear combinations of free generators. S. V. Ivanov in [9] solved Problem 1 (under some additional conditions) for a primitive element h of F_n.

Theorem 1 *Let K be a field, $X = \{x_1, \ldots, x_n\}$, $u \in F = F(X)$, $\mathrm{rank}\,(u) = n$, φ a monomorphism of F such that $\varphi(u) = u$. Then φ is an automorphism of F.*

As a corollary, we have

Theorem 2 *Let $X = \{x_1, \ldots, x_n\}$, $u \in F(X)$, $\mathrm{rank}\,(u) = n$, and let φ be a monomorphism of the free algebra $F{=}F(X)$. Then φ is an automorphism of F if and only if the element $\varphi(u)$ belongs to the orbit of u under the action of the automorphism group of F ($\varphi(u) \in \mathrm{Orb}(u)$).*

Note that the statement of Theorem 1 is not true in general for elements u with $\mathrm{rank}\,(u) < n$. Indeed, if u does not depend on x_1, then one can consider the endomorphism φ given by $\varphi(x_1) = x_1 \cdot x_2$ and $\varphi(x_i) = x_i$ for any $i > 1$. At the same time it is impossible to omit the condition that φ is a monomorphism. Indeed, let $X = \{x_1, x_2\}$, $u = x_1 + x_1 \cdot x_2$, $\varphi(x_1) = u$ and $\varphi(x_2) = 0$. Then $\mathrm{rank}\,(u) = 2 = |X|$, $\varphi(u) = u$, but φ is not an automorphism of $F(X)$. The reason of this fact is that if U is the subalgebra of $F(X)$ generated by the element u, J the ideal of $F(X)$ generated by x_2, then $F(X) = U \bigoplus J$, that is the element u belongs to the proper retract U of $F(X)$.

Theorem 3 *Let* $X = \{x_1, \ldots, x_n\}$. *Test elements of the free algebra* $F{=}F(X)$ *are precisely those elements not contained in any proper retract of* F.

Theorem 4 *Let* K *be a field,* $X = \{x_1, \ldots, x_n\}$, H *be a nonzero subalgebra of the free algebra* $F{=}F(X)$.

Then H *is a proper retract of* F *if and only if there exist a set* $Y = \{y_1, \ldots, y_n\}$ *of free generators of the algebra* F, *an integer* r, $1 \leq r < n$, *and a set* $U = \{u_1, \ldots, u_r\}$ *of free generators of the algebra* H *such that*

$$u_i = y_i + u_i^*,\ 1 \leq i \leq r, \tag{1}$$

where the elements u_i^* *belongs to the ideal of the algebra* F *generated by the free generators* $y_{r+1}, \ldots, y_n$.

Theorem 5 *Let* K *be a field,* $X = \{x_1, \ldots, x_n\}$, u *be a nonzero element of the algebra* $F = F(X)$, $\mathrm{Orb}(u)$ *the automorphic orbit of the element* u, φ *an endomorphism of* F *such that* $\varphi(\mathrm{Orb}(u)) \subseteq \mathrm{Orb}(u)$. *Then* φ *is a monomorphism of* F.

Let K be a field, $\mathrm{char}\, K \neq 2$, $X = \{x_1, \ldots, x_n\}$ be a G-graded set. By L_X we denote the free color Lie superalgebra $L(X)$ in the case when $\mathrm{char}\, K = 0$, and the free color Lie p-superalgebra $L^p(X)$ in the case when $\mathrm{char}\, K = p > 2$.

Theorem 6 *Let* K *be a field,* $\mathrm{char}\, K \neq 2$, $X = \{x_1, \ldots, x_n\}$, *and let* u *be a nonzero element of the algebra* $L = L_X$, $\mathrm{Orb}(u)$ *the automorphic orbit of the element* u, φ *an endomorphism of* L *such that* $\varphi(\mathrm{Orb}(u)) \subseteq \mathrm{Orb}(u)$. *Then* φ *is an automorphism of* L.

3 Inverse images

Let $A_n = K\langle X\rangle = K\langle x_1, \ldots, x_n\rangle$ be the free associative algebra over a field K, and L_n the free Lie algebra with the same set of free generators. In this section, we study the dynamics of endomorphisms of those algebras by considering *inverse images* of specific elements as opposed to a more traditional approach of considering *images*. For example, we consider elements u with the following property: there is an endomorphism φ such that $\varphi(u) = x_1$. Another class of elements arises if we only consider *injective* endomorphisms with the same property.

Here we are able to characterize both those classes in L_n and in some other free non-associative algebras. In particular, it may come as a surprise that $\varphi(u) = x_1$ for an injective φ implies u is primitive in L_n (i.e., is part of a free basis). The same result holds for A_2, the free associative algebra of rank 2.

Let K be a field, $X = \{x_1, \dots, x_n\}$ a nonempty finite set. In what folows, $F = F(X)$ is a free K-algebra (without a unit element) on a finite set X of free generators in one of the following varieties of algebras over a field K: the variety of all algebras; the variety of Lie algebras; the variety of Lie p-algebras; varieties of color Lie superalgebras; varieties of color Lie p-superalgebras; varieties of non-associative commutative and anti-commutative algebras (for the information on free Lie superalgebras see monographs [1] or [25]). These varieties are the main types of Schreier varieties of algebras (a variety of algebras is called Schreier if any subalgebra of a free algebra of this variety is itself free in the same variety). For main facts about Schreier varieties of algebras see the articles [2], [11], [12], [28], [29], [36], [38], [39].

We start with inverse images of primitive elements of a (non-associative) algebra $F = F(X)$ which is free in one of the varieties described in the previous section.

Theorem 7 *Let u be an element of a free non-associative algebra $F = F(X)$, v a primitive element of F, and φ a monomorphism of F such that $\varphi(u) = v$. Then u itself is a primitive element of F.*

Now we consider a more general situation.

Theorem 8 *Let $u_1, \dots, u_k$ be elements of a free non-associative algebra $F = F(X)$ of rank n; $\{z_1, \dots, z_k\}$ a primitive system of F, $1 \leq k \leq n$, and let φ be a monomorphism of the algebra F such that $\varphi(u_i) = z_i$, $1 \leq i \leq k$. Then $\{u_1, \dots, u_k\}$ is a primitive system of the algebra F. Furthermore, if $k = n$, then φ is an automorphism of the algebra F.*

Now we consider inverse images of primitive systems under arbitrary endomorphisms.

Theorem 9 *Let $U = \{u_1, \dots, u_l\}$ be a subset of a (non-associative) algebra $F = F(X)$ which is free in one of the varieties described above; $\{z_1, \dots, z_l\}$ a primitive system of F; H the subalgebra of F generated by U, and φ an endomorphism of the algebra F such that $\varphi(u_i) = z_i$, $1 \leq i \leq l$. Then H is a retract of the algebra F.*

Finally, for endomorphisms of the free associative algebra of rank 2, we consider a related dynamical property and prove the following

Theorem 10 *Every endomorphism of the free associative algebra $A_2 = K\langle x, y\rangle$ that takes non-trivial linear combinations of x and y to primitive elements, is an automorphism.*

References

[1] Yu. A. Bahturin, A. A. Mikhalev, M. V. Zaicev, and V. M. Petrogradsky, *Infinite Dimensional Lie Superalgebras*. Walter de Gruyter Publ., Berlin–New York, 1992.

[2] P. M. Cohn, *Subalgebras of free associative algebras*. Proc. London Math. Soc. (3) **14** (1964), 618–632.

[3] P. M. Cohn, *Universal Algebra. 2nd Ed.* Reidel, Dordrecht, 1981.

[4] W. Dicks, *A commutator test for two elements to generate the free algebra of rank two*. Bull. London Math. Soc. **14** (1982), 48–51.

[5] W. Dicks, *Automorphisms of the polynomial ring in two variables*. Publ. Sec. Mat. Univ. Autonoma Barcelona **27** (1983), 155–162.

[6] V. Drensky and J.-T. Yu, *Test polynomials for automorphisms of polynomial and free associative algebras*. J. Algebra, in press.

[7] V. Drensky and J.-T. Yu, *Orbits in polynomial and relatively free algebras of rank two*. Commun. Algebra, in press.

[8] A. van den Essen and V. Shpilrain, *Some combinatorial questions about polynomial mappings*. J. Pure Appl. Algebra, **119** (1997), 47–52.

[9] S. V. Ivanov, *On endomorphisms of free groups that preserve primitivity*. Arch. Math., in press.

[10] Z. Jelonek, *A solution of the problem of van den Essen and Shpilrain*. Preprint IMUJ 1996/19, Krakow.

[11] A. G. Kurosh, *Nonassociative free algebras and free products of algebras*. Mat. Sb. **20** (1947), 239–262.

[12] J. Lewin, *On Schreier varieties of linear algebras*. Trans. Amer. Math. Soc. **132** (1968), 553–562.

[13] A. I. Malcev, *Algebraic Systems*. Nauka, Moscow, 1970.

[14] A. A. Mikhalev, *Subalgebras of free color Lie superalgebras*. Mat. Zametki **37** (1985), No. 5, 653–661. English translation: Math. Notes, **37** (1985), 356–360.

[15] A. A. Mikhalev, *Free color Lie superalgebras*. Dokl. Akad. Nauk SSSR **286** (1986), No. 3, 551–554. English translation: Soviet Math. Dokl. **33** (1986), 136–139.

[16] A. A. Mikhalev, *Subalgebras of free Lie p-superalgebras*. Mat. Zametki **43** (1988), No. 2, 178–191. English translation: Math. Notes **43** (1988), 99–106.

[17] A. A. Mikhalev, *Combinatorial aspects of the theory of Lie superalgebras*. First International Tainan–Moscow Algebra Workshop, Walter de Gruyter Publ., Berlin, 1996, 37–68.

[18] A. A. Mikhalev and J.-T. Yu, *Test elements and retracts of free Lie algebras*. Commun. Algebra **25** (1997), 3283–3289.

[19] A. A. Mikhalev and J.-T. Yu, *Automorphic orbits in free algebras of rank two*. In *Proc. 2nd Internat. Tainan-Moscow Algebra Workshop*, Springer-Verlag, Singapore, in press.

[20] A. A. Mikhalev, J.-T. Yu, and A. A. Zolotykh, *Images of coordinate polynomials*. Algebra Colloquium **4** (1997), no. 2, 159–162.

[21] A. A. Mikhalev and A. A. Zolotykh, *Endomorphisms of free Lie algebras preserving primitivity of elements are automorphisms*. Uspekhi Matem. Nauk **48** (1993), No. 6, 149–150. English translation: Russian Math. Surveys **48** (1993), No. 6, 189–190.

[22] A. A. Mikhalev and A. A. Zolotykh, *The rank of an element of the free color Lie (p-)superalgebra*. Doklady Akad. Nauk **334** (1994), 690–693. English translation: Russian Acad. Sci. Dokl. Math. **49** (1994), No. 1, 189–193.

[23] A. A. Mikhalev and A. A. Zolotykh, *Rank and primitivity of elements of free color Lie (p-)superalgebras*. Intern. J. Algebra and Computation **4** (1994), 617–656.

[24] A. A. Mikhalev and A. A. Zolotykh, *Automorphisms and primitive elements of free Lie superalgebras*. Commun. Algebra **22** (1994), 5889–5901.

[25] A. A. Mikhalev and A. A. Zolotykh, *Combinatorial Aspects of Lie Superalgebras*. CRC Press, Boca Raton, New York, 1995.

[26] A. A. Mikhalev and A. A. Zolotykh, *Test elements for monomorphisms of free Lie algebras and superalgebras*. Commun. Algebra **23** (1995), 4995–5001.

[27] J. Nielsen *Die Isomorphismen der allgemeinen, unendlichen Gruppe mit zwei Erzeugenden*. Math. Ann. **78** (1918), 385–397.

[28] A. I. Shirshov, *Subalgebras of free Lie algebras*. Mat. Sb. **33** (1953), 441–452.

[29] A. I. Shirshov, *Subalgebras of free commutative and free anti-commutative algebras*. Mat. Sb. **34** (1954), 81–88.

[30] V. Shpilrain, *Recognizing automorphisms of the free groups*. Arch. Math. **62** (1994), 385–392.

[31] V. Shpilrain, *Generalized primitive elements of a free group*. Arch. Math., in press.

[32] V. Shpilrain, *Combinatorial methods: from groups to polynomial algebras*, Preprint.

[33] A. S. Shtern, *Free Lie superalgebras*. Sibirsk. Mat. Zh. **27** (1986), 170–174. English translation: in Sib. Math. J.

[34] E. C. Turner, *Test words for automorphisms of free groups*. Bull. London Math. Soc. **28** (1996), 255–263.

[35] U. U. Umirbaev, *Primitive elements of free groups*. Uspekhi Mat. Nauk **49** (1994), no. 2, 175–176. English translation: in Russian Math. Surveys.

[36] U. U. Umirbaev, *On Schreier varieties of algebras*. Algebra i Logika **33** (1994), 317–340. English translation: in Algebra and Logic.

[37] U. U. Umirbaev, *On ranks of elements of free groups*. Fundamentalnaya i Prikladnaya Matematika **2** (1996), 313–315.

[38] U. U. Umirbaev, *Universal derivations and subalgebras of free algebras*. In *Proc. 3rd Int. Conf. in Algebra (Krasnoyarsk, 1993)*. Walter de Gruyter, 1996, 255–271.

[39] E. Witt, *Die Unterringe der freien Lieschen Ringe*. Math. Z. **64** (1956), 195–216.

Alexander A. Mikhalev: Department of Mechanics and Mathematics, Moscow State University, Moscow 119899, Russia

e-mail address: aamikh@cnit.math.msu.su

Vladimir Shpilrain and Jie-Tai Yu: Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong

e-mail addresses: shpil@hkusua.hku.hk yujt@hkusua.hku.hk