An Inequality Involving the Eigenvalues for Bipartite Distance-Regular Graphs

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Abstract

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, and intersection number $c_3 > 1$. Let θ, θ' denote eigenvalues of Γ other than $k, -k$, and let E, F denote the associated primitive idempotents. We obtain an inequality involving θ, θ' and the intersection numbers of Γ, and we show that equality holds exactly when the entry-wise product $E \circ F$ is a linear combination of at most two primitive idempotents of Γ. We then determine the pairs θ, θ' for which equality holds.

Résumé

Soit Γ un graphe bi-partie à distance-réguliere de diamètre $D \geq 4$, valence $k \geq 3$, et nombre d'intersection $c_3 > 1$. Soient θ, θ' des valeurs propres de Γ autres que $k, -k$, et soient E, F les idempotents primitifs associés. Nous obtenons une inégalité enveloppant θ, θ' et les nombres d'intersection de Γ, et nous montrons que nous avons l'égalité exactement dès lous que le produit Hadamard $E \circ F$ est une combinaison linéaire de deux idempotents primitives de Γ au plus. Nous déterminons les paires θ, θ' auxquelles il y a l'égalité.

1 Introduction

Let Γ denote a distance-regular graph with diameter $D \geq 3$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$ (formal definitions appear in Section 2). Let θ, θ' denote nontrivial eigenvalues of Γ. In [5], Jurisic, Koolen, and Terwilliger proved that the intersection numbers a_1, b_1 satisfy

$$\left(\theta + \frac{k}{a_1+1}\right)\left(\theta' + \frac{k}{a_1+1}\right) \geq \frac{-k a_1 b_1}{(a_1+1)^2}. \tag{1}$$

Note that if Γ is bipartite, then $a_1 = 0$, $\theta_D = -k$ and so (1) is trivially satisfied. In this paper we find the following analog of (1) for bipartite distance-regular graphs.

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Suppose $c_3 > 1$, and let θ, θ' denote nontrivial eigenvalues of Γ. We show the intersection numbers of Γ satisfy

$$\left(\theta^2 - \frac{b_2^2(c_3-1)}{\mu(\Delta + c_3 - \mu)}\right)\left(\theta'^2 - \frac{b_2^2(c_3-1)}{\mu(\Delta + c_3 - \mu)}\right) \geq \frac{-(k-1)\Delta b_2^2 b_3}{(\Delta + c_3 - \mu)^2}, \tag{2}$$

where $\Delta = (k-2)(c_3-1) - (\mu-1)p^2_{22}$.

Several authors have examined the case of equality in (1). Let Γ denote a distance-regular graph with diameter $D \geq 3$, and let θ, θ' denote nontrivial eigenvalues of Γ with associated primitive idempotents E, F, respectively. In [6], Pascasio showed that equality holds in (1) if and only if the entry-wise product $E \circ F$ is a scalar multiple of a primitive idempotent of Γ. Summarizing some results from [5] and [6], we have the following theorem.

Theorem 1.1. *[5],[6] Let Γ denote a distance-regular graph with diameter $D \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$.*

(i) Suppose Γ is bipartite. Then the set of nontrivial eigenvalues $\{\theta, \theta'\}$ for which equality holds in (1) is the empty set.

(ii) Suppose Γ is not bipartite. Then the set of nontrivial eigenvalues $\{\theta, \theta'\}$ for which equality holds in (1) is either the empty set or $\{\theta_1, \theta_D\}$.

In this paper we consider the case of equality in (2) and obtain a result analogous to Theorem 1.1. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, and intersection number $c_3 > 1$. Let θ, θ' denote nontrivial eigenvalues of Γ, and let E, F denote the associated primitive idempotents. We show equality holds in (2) if and only if $E \circ F$ is a linear combination of at most two primitive idempotents of Γ. We then obtain the following theorem.

Theorem 1.2. *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, intersection number $c_3 > 1$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$.*

(i) Suppose Γ is 2-homogeneous. Then equality holds in (2) for nontrivial eigenvalues θ, θ' if and only if at least one of θ, θ' is in $\{\theta_1, \theta_{D-1}\}$.

(ii) Suppose Γ is not 2-homogeneous, and D is odd. Then the set of nontrivial eigenvalues $\{\theta, \theta'\}$ for which equality holds in (2) is either the empty set or $\{\{\theta_1, \theta_d\}, \{\theta_1, \theta_{D-d}\}, \{\theta_{D-1}, \theta_d\}, \{\theta_{D-1}, \theta_{D-d}\}\}$, where $d = (D-1)/2$.

(iii) Suppose Γ is not 2-homogeneous, and D is even. Then the set of nontrivial eigenvalues $\{\theta, \theta'\}$ for which equality holds in (2) is either the empty set or $\{\{\theta_1, \theta_d\}, \{\theta_{D-1}, \theta_d\}\}$, where $d = D/2$.

2 Preliminaries

We now review some basic definitions and results concerning distance-regular graphs. For more information, we refer the reader to [1], [2], and [4]. Throughout, let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges, with vertex set X and edge set R. We define the *diameter* of Γ to be $D :=\max\{\partial(x, y) \mid x, y \in X\}$, where ∂ denotes the path-length distance function.

We define Γ to be *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$, and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}| \tag{3}$$

is independent of the choice of x and y. The numbers p^h_{ij} are called *intersection numbers* of Γ. It is conventional to abbreviate $c_i := p^i_{1i-1}$ $(1 \leq i \leq D)$, $a_i := p^i_{1i}$

$(0 \leq i \leq D)$, $b_i := p^i_{1i+1}$ $(0 \leq i \leq D-1)$, and define $c_0 := 0$, $b_D := 0$. We also write $k := b_0$ and $\mu := c_2$. We refer to k as the *valency* of Γ.

Let $\Gamma = (X, R)$ denote a distance-regular graph, and let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra of matrices over $\mathbb{R}$ with rows and columns indexed by X. For each integer i $(0 \leq i \leq D)$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in X). \tag{4}$$

We refer to A_i as the i^{th} *distance matrix* of Γ, and $A = A_1$ is called the *adjacency matrix* of Γ.

One can show that $\{A_0, A_1, \ldots, A_D\}$ is a basis for a commutative semisimple subalgebra M of $\mathrm{Mat}_X(\mathbb{R})$. M is known as the *Bose-Mesner algebra* for Γ. M has a second basis $\{E_0, E_1, \ldots, E_D\}$ such that

$$E_0 = |X|^{-1} J, \tag{5}$$

$$E_i E_j = \delta_{ij} E_i \qquad (0 \leq i, j \leq D), \tag{6}$$

$$E_0 + E_1 + \cdots + E_D = I. \tag{7}$$

We refer to E_i as the i^{th} *primitive idempotent for* Γ $(0 \leq i \leq D)$.

Let q^h_{rs} $(0 \leq h, r, s \leq D)$ denote real scalars such that

$$E_r \circ E_s = |X|^{-1} \sum_{h=0}^{D} q^h_{rs} E_h \qquad (0 \leq r, s \leq D), \tag{8}$$

where $\circ$ denotes the entry-wise matrix product. The q^h_{rs} are known as the *Krein parameters* of Γ. It is well-known that

$$q^h_{rs} \geq 0 \qquad (0 \leq h, r, s \leq D). \tag{9}$$

Let θ_i denote the eigenvalue of A associated with E_i $(0 \leq i \leq D)$, so that

$$A = \sum_{i=0}^{D} \theta_i E_i. \tag{10}$$

We refer to $\theta_0, \theta_1, \ldots, \theta_D$ as the *eigenvalues* of Γ. We assume the primitive idempotents are labelled so that $\theta_0 > \theta_1 > \ldots > \theta_D$. It is known that $\theta_0 = k$. An eigenvalue θ of Γ is said to be *nontrivial* whenever $\theta \neq k$, $\theta \neq -k$. We say a primitive idempotent E is nontrivial whenever its associated eigenvalue is nontrivial.

From now on, assume Γ is bipartite with diameter $D \geq 3$ and valency $k \geq 3$. Then

$$\theta_i = -\theta_{D-i} \quad (0 \leq i \leq D). \tag{11}$$

For convenience, we define

$$\Delta := (k-2)(c_3 - 1) - (\mu - 1) p^2_{22}. \tag{12}$$

By a result of Curtin [3], $\Delta \geq 0$.

Curtin showed in [3] that for any nontrivial eigenvalue θ of Γ,

$$(\mu - 1)\theta^2 \leq (k - \mu)(k - 2). \tag{13}$$

He then showed that the set of nontrivial eigenvalues θ of Γ for which equality holds in (13) is either (i) empty, or (ii) $\{\theta_1, \theta_{D-1}\}$. Curtin defined Γ to be *2-homogeneous* whenever case (ii) occurred. For more information on the 2-homogeneous distance-regular graphs, see [3].

3 An Inequality for Bipartite Graphs

Theorem 3.1. *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let θ, θ' denote nontrivial eigenvalues of Γ, and suppose $c_3 > 1$. Then*

$$\left(\theta^2 - \frac{b_2^2(c_3-1)}{\mu(\Delta + c_3 - \mu)}\right)\left(\theta'^2 - \frac{b_2^2(c_3-1)}{\mu(\Delta + c_3 - \mu)}\right) \geq \frac{-(k-1)\Delta b_2^2 b_3}{(\Delta + c_3 - \mu)^2}, \tag{14}$$

where Δ is from (12).

Sketch of proof. Let $\theta_0 > \theta_1 > \ldots > \theta_D$ denote the eigenvalues of Γ. We assume θ, θ' are nontrivial, so $\theta = \theta_r, \theta' = \theta_s$ for some integers r, s $(1 \leq r, s \leq D-1)$. Consider the matrix

$$G := \begin{pmatrix} \epsilon_0 & \epsilon_1 & \epsilon_2 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \epsilon_2 & \epsilon_3 & \epsilon_4 \end{pmatrix} \tag{15}$$

where

$$\epsilon_i := m_r^{-1} m_s^{-1} \sum_{h=0}^{D} q_{rs}^h m_h \theta_h^i \qquad (0 \leq i \leq 4), \tag{16}$$

and where $m_h = \text{rank}(E_h)$ $(0 \leq h \leq D)$. We show G is positive semi-definite. By (9), we find that for $i = 0, 1, 2$, there exists a vector v_i in $\mathbb{R}^{D+1}$ with h coordinate

$$\theta_h^i \sqrt{\frac{q_{rs}^h m_h}{m_r m_s}} \qquad (0 \leq h \leq D). \tag{17}$$

Observe the dot product $v_i \cdot v_j = \epsilon_{i+j}$ $(0 \leq i, j \leq 2)$, so G is the matrix of dot products for v_0, v_1, v_2. Now G is positive semi-definite, and it follows that $det(G) \geq 0$.

To compute $det(G)$, we use some standard results from graph theory to write $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ in terms of the eigenvalues and intersection numbers of Γ. We find $det(G)$ equals

$$\frac{(\theta_s - k)^2(\theta_s + k)^2(\theta_r - k)^2(\theta_r + k)^2 \mu(\Delta + c_3 - \mu)}{k^3(k-1)^3 b_2^2 b_3} \tag{18}$$

times

$$\left(\theta^2 - \frac{b_2^2(c_3-1)}{\mu(\Delta + c_3 - \mu)}\right)\left(\theta'^2 - \frac{b_2^2(c_3-1)}{\mu(\Delta + c_3 - \mu)}\right) + \frac{(k-1)\Delta b_2^2 b_3}{(\Delta + c_3 - \mu)^2}. \tag{19}$$

One can show the expression in (18) is positive, so (19) is nonnegative. □

Corollary 3.2. *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let θ, θ' denote nontrivial eigenvalues of Γ, and let E, F denote the corresponding primitive idempotents. Then the following are equivalent:*

(i) $E \circ F$ *is a linear combination of at most two primitive idempotents of* Γ.

(ii) $c_3 > 1$ *and equality holds in (14) for* θ, θ'.

4 Eigenvalues Satisfying Equality in the Bipartite Fundamental Bound

We now consider for which pairs of eigenvalues θ, θ' equality holds in (14). We treat separately the cases when $\Delta = 0$ and $\Delta \neq 0$. Using the form of (14) and the definition of 2-homogeneous, we obtain the following two results.

Lemma 4.1. *Let* Γ *denote a bipartite distance-regular graph with diameter* $D \geq 4$, *valency* $k \geq 3$, *and eigenvalues* $\theta_0 > \theta_1 > \ldots > \theta_D$. *For any nontrivial eigenvalues* θ, θ', *the following are equivalent.*

(i) Γ *is 2-homogeneous and at least one of* θ, θ' *is in* $\{\theta_1, \theta_{D-1}\}$.

(ii) $\Delta = 0$, $c_3 > 1$, *and equality holds in (14) for* θ, θ'.

We now turn our attention to the case when equality holds in (14) and $\Delta \neq 0$. To set the stage for the next result, suppose for the moment that D is odd. Then θ_d is the minimal positive eigenvalue of Γ, where $d = (D-1)/2$. If D is even, then $\theta_d = 0$, where $d = D/2$.

Lemma 4.2. *Let* Γ *denote a bipartite distance-regular graph with diameter* $D \geq 4$, *valency* $k \geq 3$, *and eigenvalues* $\theta_0 > \theta_1 > \ldots > \theta_D$. *Assume* $\Delta \neq 0$ *and* $c_3 > 1$, *and let* Φ *denote the set of pairs of nontrivial eigenvalues* $\{\theta, \theta'\}$ *for which equality holds in (14). Then exactly one of the following cases holds.*

(i) $\Phi = \emptyset$.

(ii) D *is odd, and* $\Phi = \{\{\theta_1, \theta_d\}, \{\theta_1, \theta_{D-d}\}, \{\theta_{D-1}, \theta_d\}, \{\theta_{D-1}, \theta_{D-d}\}\}$, *where* $d = (D-1)/2$.

(iii) D *is even, and* $\Phi = \{\{\theta_1, \theta_d\}, \{\theta_{D-1}, \theta_d\}\}$, *where* $d = D/2$.

We define Γ to be *taut* whenever $\Delta \neq 0$, $c_3 > 1$, and one of (ii),(iii) holds.

In view of Corollary 3.2, Lemma 4.1, and Lemma 4.2, we have the following result concerning the primitive idempotents of Γ.

Corollary 4.3. *Let* Γ *denote a bipartite distance-regular graph with diameter* $D \geq 4$, *valency* $k \geq 3$, *and eigenvalues* $\theta_0 > \theta_1 > \ldots > \theta_D$. *Let* E, F *denote nontrivial primitive idempotents of* Γ.

(i) *Suppose* Γ *is 2-homogeneous. Then* $E \circ F$ *is a linear combination of at most two primitive idempotents of* Γ *if and only if at least one of* E, F *is in* $\{E_1, E_{D-1}\}$.

(ii) *Suppose* Γ *is taut, and* D *is odd. Then* $E \circ F$ *is a linear combination of at most two primitive idempotents of* Γ *if and only if the set* $\{E, F\}$ *is one of* $\{E_1, E_d\}$, $\{E_1, E_{D-d}\}$, $\{E_{D-1}, E_d\}$, $\{E_{D-1}, E_{D-d}\}$, *where* $d = (D-1)/2$.

(iii) Suppose Γ is taut, and D is even. Then $E \circ F$ is a linear combination of at most two primitive idempotents of Γ if and only if the set $\{E, F\}$ is one of $\{E_1, E_d\}$, $\{E_{D-1}, E_d\}$ where $d = D/2$.

(iv) Suppose Γ is neither 2-homogeneous nor taut. Then $E \circ F$ is never a linear combination of at most two primitive idempotents of Γ.

References

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