Left-modular Elements

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Abstract

R. P. Stanley showed that if x is a modular element of a finite geometric lattice L then the characteristic polynomial of L, $\chi(L,t)$, is divisible by $\chi([\hat{0},x],t)$. We will generalize this result by replacing x with a left-modular element and using some other less strict conditions than a geometric lattice. The new generalization can be used to calculate the characteristic polynomials and the Möbius functions of non-crossing partition lattices and shuffle posets.

Résumé

R. P. Stanley a démontré le théorème suivant. Si x est un élément modular d'un treillis fini géométrique, L, donc le polynôme caractéristique de L, $\chi(L,t)$, est divisible par $\chi([\hat{0},x],t)$. Nous donnons une généralisation de ce résultat en remplaçant x par un élément modular-à-gauche et en utilisant quelques conditions moins strictes que celles pour un treillis géométrique. Cette généralisation permet la calculation des polynômes caractéristiques et fonctions de Möbius pour le treillis de partitions non-croisées et pour le poset de mélanges.

1 Left-modular elements

Throughout this paper L will denote a finite graded lattice and ρ will denote the rank function of L. Any relevant definitions not given can be found in Stanley's text [8]. We will use $\wedge$ for the meet (greatest lower bound) and $\vee$ for the join (least upper bound) in L. Since L is finite it has a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. We say y covers x, denoted by $x \prec y$, if $x < y$ and there is no element

$z \in L$ such that $x < z < y$. Denote by $A(L)$ the set of atoms of L. An element is call *atomic* if it is the join of some atoms. Let $J(L)$ be the set of atomic elements of L. The *characteristic polynomial* of L is defined by

$$\chi(L,t) = \sum_{x \in L} \mu(x) t^{n-\rho(x)}$$

where μ is the Möbius function of L. Since $\chi(L,t) = t^{\rho(L)-\rho(m)}\chi([\hat{0},m],t)$ where $m = \bigvee J(L)$, for convenience we will assume that $\hat{1}$ is atomic throughout this paper.

Two elements x, $y \in L$ form a *modular pair* (x,y) if

$$z \vee (x \wedge y) = (z \vee x) \wedge y$$

for any $z < y$. Note that (i) this relation is not symmetric; (ii) the modular inequality $z \vee (x \wedge y) \leq (z \vee x) \wedge y$ holds for any x and $z < y$; (iii) the equation holds whenever y or z is comparable to x. We call x a *left-modular element* if (x,y) is a modular pair for every $y \in L$, and we call x a *modular element* if both (x,y) and (y,x) are modular pairs for every $y \in L$. In a semimodular lattice, the pair (x,y) is modular if and only if $\rho(x \wedge y) + \rho(x \vee y) = \rho(x) + \rho(y)$ [1, p. 83]; this shows the relation of being a modular pair is symmetric in this case, i. e., left-modular elements are also modular.

We say that y is a *complement* of x if $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. A lattice is called *geometric* when it is atomic and semimodular. The following theorem is due to Stanley [6] and provides a characterization of modular elements. The corresponding properties of left-modular elements are shown next.

Theorem 1.1 *Let L be a geometric lattices. An element $x \in L$ is modular if and only if no two complements of x are comparable.* ■

Theorem 1.2 *Let x be an element of the lattice L. The following statements are equivalent.*

i. *The element x is left-modular.*

ii. *For any $z < y \in L$, we have $x \wedge z \neq x \wedge y$ or $x \vee z \neq x \vee y$.*

iii. *For any $z \prec y \in L$, we have $x \wedge z \neq x \wedge y$ or $x \vee z \neq x \vee y$.*

iv. *For any $z \prec y \in L$, we have either $x \wedge z = x \wedge y$ or $x \vee z = x \vee y$.*

v. *For every interval $[a,b]$ containing x, no two complements of x with respect to the sublattice $[a,b]$ are comparable.*

Proof. Here is the strategy: (v) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv). The proofs of (v) $\Leftrightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (iii) $\Leftarrow$ (iv) are trivial.

Suppose $z < y$. It is easy to check that $x \wedge y = x \wedge z$ if and only if $z \vee (x \wedge y) = z$, and $x \vee y = x \vee z$ if and only if $(z \vee x) \wedge y = y$. Now by the modular inequality we have

$$z \leq z \vee (x \wedge y) \leq (z \vee x) \wedge y \leq y.$$

Since $z \neq y$, at least one of $\leq$'s in the previous inequality should be "strictly less". Therefore (i) $\Rightarrow$ (ii).

Suppose $z \prec y$. Then exactly two of $\leq$'s should be "equal to" and the other one must be $\prec$. Thus (iii) $\Rightarrow$ (iv). To show (iii) $\Rightarrow$ (i) let us consider the contrapositive: if there are $u < v \in L$ such that $u \vee (x \wedge v) \neq (u \vee x) \wedge v$, then we will have $z \vee (x \wedge y) = z$ and $(z \vee x) \wedge y = y$ for any $z \prec y \in [u \vee (x \wedge v), (u \vee x) \wedge v]$. ■

Meet and join operations preserve the left-modularity in smaller sublattices. The proof follows directly from their definitions, so we leave it to the reader.

Proposition 1.3 *Let x be a left-modular element in the lattice L. Then for any $y \in L$*
1. the meet $x \wedge y$ is a left-modular element in $[\hat{0}, y]$, and
2. the join $x \vee y$ is a left-modular element in $[y, \hat{1}]$. ■

2 The characteristic polynomial

First let us say a few words about the motivation. There are two important factorization theorems of the characteristic polynomial, both of which were given by Stanley. These factorizations offer easier ways to calculate the characteristic polynomials of various lattices. The following theorem was given in [6] and shows that if L is a geometric lattice there is a factor of its characteristic polynomial arising from a modular element.

Theorem 2.1 (Stanley, 1971) *Let L be a geometric lattice of rank n. If x is a modular element of L, then*

$$\chi(L, t) = \chi([\hat{0}, x], t) \sum_{b : b \wedge x = \hat{0}} \mu(\hat{0}, b) t^{n - \rho(x) - \rho(b)}. \blacksquare$$

In another paper [7], Stanley defined a *supersolvable lattice* to be a pair (L, Δ) where L is a lattice, $\Delta : \hat{0} = x_0 \prec x_1 \prec \ldots \prec x_n = \hat{1}$ is a maximal chain of L, and Δ together with any other chain of L generates a distributive lattice. It was shown that supersolvable lattices are graded.

Theorem 2.2 (Stanley, 1972) *Let L be a supersolvable and semimodular lattice of rank n. Then*

$$\chi(L,t) = (t-a_1)(t-a_2)\cdots(t-a_n)$$

where a_i is the number of atoms of L that are below x_i but not below x_{i-1}. ■

Notice that, in both theorems, semimodularity is the only hypothesis in common. There was no other direct link between them, until the recent work [2] of A. Blass and B. Sagan generalizing the second factorization theorem. To replace supersolvability they defined a *left-modular lattice* L such that

L has a maximal chain Δ, all of whose elements are left-modular.

This hypothesis indicates that left-modularity is another connection. So we generalize the first factorization theorem by replacing x with a left-modular element. The new result also unifies all the previous ones. We will show the generalization in this section and the applications in the following two. To see the unification of two Stanley's theorems, please refer to [5].

Theorem 2.3 *Let L be a graded lattice and $\rho(L) = n$. If $x \in L$ is a left-modular element, then*

$$\chi(L,t) = \sum_{b\wedge x=\hat{0}} \left[\mu(\hat{0},b)t^{n-\rho(b\vee x)}\chi([b,b\vee x],t)\right]. \blacksquare \tag{1}$$

This theorem and its proof simply follow Stanley's [6, Theorem 2] with the help of Theorem 1.2 and Proposition 1.3 and the assumption that $\hat{1}$ is atomic. For the details, see [5].

In the sum (1), the term $\chi([b,b\vee x],t)$ depends on each b. To fulfill the factorization property, we need to drop the dependency by adding an extra hypothesis. We need some tools before we can accomplish that:

For any given a, b in a lattice, we define

$$\begin{aligned} \sigma_a &: [b, a\vee b] \to [a\wedge b, a] \text{ by } \sigma_a(u) = u\wedge a, \\ \tau_b &: [a\wedge b, a] \to [b, a\vee b] \text{ by } \tau_b(v) = v\vee b. \end{aligned}$$

Directly from definitions of $\wedge$ and $\vee$, we have $\tau_b\sigma_a(u) \le u$ and $\sigma_a\tau_b(v) \ge v$. Since τ_b is a natural map from $[\hat{0},x]$ to $[b,b\vee x]$, in order to show that $\chi([b,b\vee x],t) = \chi([\hat{0},x],t)$, at a minimum we need to require that τ_b is rank-preserving. The following lemma shows that this requirement is also a sufficient condition.

Lemma 2.4 *Let x, $b \in L$ be atomic elements with $b\wedge x = \hat{0}$. If x is left-modular and τ_b is rank-preserving on $J([\hat{0},x])$, then*

$$\chi([b,b\vee x],t) = \chi([\hat{0},x],t)$$

Proof. Let $\tau = \tau_b$, $\sigma = \sigma_x$, and define a map $\delta : L \to J(L)$ by

$$\delta(x) = \bigvee\{a \in A(L) \mid a \leq x\}.$$

The goal is to prove that $J([\hat{0}, x])$ and $J([b, b \vee x])$ are isomorphic. We will show that the restriction map $\tau|_{J([\hat{0},x])}$ is a bijection into $J([b, b \vee x])$ with its inverse $\delta\sigma$ also order-preserving. By isomorphism we have $\mu(\hat{0}, y) = \mu(b, b \vee y)$ for any $y \in J([\hat{0}, x])$. Also by the rank-preserving nature of τ we get

$$\begin{aligned}
\chi([\hat{0}, x], t) &= \sum_{y \in J([\hat{0},x])} \mu(\hat{0}, y) t^{\rho(x)-\rho(y)} \\
&= \sum_{y \in J([\hat{0},x])} \mu(b, b \vee y) t^{\rho(b,b\vee x)-\rho(b,b\vee y)} \\
&= \sum_{z \in J([b,b\vee x])} \mu(b, z) t^{\rho(b,b\vee x)-\rho(b,z)} \\
&= \chi([b, b \vee x], t).
\end{aligned}$$

The definitions show τ and σ are order-preserving with respect to $\leq$. Now $\tau\sigma(z) = b \vee (x \wedge z) = (b \vee x) \wedge z = z$ for any $z \in [b, b \vee x]$ by the left-modularity of x. This implies that τ is surjective; σ is injective and then order-preserving with respect to $<$.

Let $A_1 = A([\hat{0}, x])$ and $A_2 = A([b, b \vee x])$. Since the rank of $\tau(a)$ in $[b, b \vee x]$ is 1 for every $a \in A_1$, the restriction map $\tau : A_1 \to A_2$ is well-defined. The map is also surjective: For any $a' \in A_2$, we have $\tau^{-1}(a') \cap A_1 \neq \emptyset$ by the surjectivity and rank-preserving nature of τ.

Let $J_1 = J([\hat{0}, x])$ and $J_2 = J([b, b \vee x])$. For any atomic element $u = a_1 \vee \cdots \vee a_k \in J_1$ where $a_i \in A_1$, we find that $\tau(u) = (a_1 \vee b) \vee \cdots \vee (a_k \vee b) = \tau(a_1) \vee \cdots \vee \tau(a_k)$. By the previous result each $\tau(a_i)$ is an atom of $[b, b \vee x]$, so $\tau(u)$ is an atomic element of $[b, b \vee x]$. Therefore $\tau(J_1) \subseteq J_2$, i. e., the map $\tau : J_1 \to J_2$ is well defined.

Suppose $u, v \in J_1$ with $\tau(u) = \tau(v)$, so $\tau(u \vee v) = u \vee v \vee b = u \vee \tau(v) = u \vee \tau(u) = \tau(u)$. Since $u \vee v$ is atomic, we have $\rho(\hat{0}, u \vee v) = \rho(b, \tau(u \vee v)) = \rho(b, \tau(u)) = \rho(\hat{0}, u)$. It follows that $u = u \vee v$ or $u \geq v$. Similarly we get $v \geq u$. Thus the restriction map $\tau|_{J_1}$ is injective.

To prove the surjectivity of $\tau|_{J_1}$ onto J_2, it suffices to show that $\tau\delta\sigma|_{J_2} = id$. This result also implies that $\delta\sigma|_{J_2}$ is the inverse of $\tau|_{J_1}$. Let $A_{i,u} = \{a \in A_i \mid a \leq u\}$ for $u \in L$ and $i = 1, 2$. If $v \in J_2$, it is easy to check that $A_{1,\delta\sigma(v)} = A_{1,\sigma(v)} = A_{1,v}$. We also have $\tau(A_{1,v}) = A_{2,v}$ which follows from the surjectivity of $\tau|_{A_1}$ onto A_2. We conclude that

$$\tau\delta\sigma(v) \;=\; \tau(\bigvee A_{1,\delta\sigma(v)}) \;=\; \tau(\bigvee A_{1,v})$$

$$= \bigvee \tau(A_{1,v}) = \bigvee A_{2,v} = v.$$

It is trivial that $\delta\sigma|_{J_2}$ is order-preserving, and so the isomorphism follows. ∎

Using the previous lemma we achieve the main result.

Theorem 2.5 *Let L be a graded lattice with $\rho(\hat{1}) = n$, and $x \in L$ be an atomic element. If $x \in L$ is left-modular and the map τ_b is rank-preserving on $J[\hat{0}, x]$ for every atomic element b satisfying $b \wedge x = \hat{0}$, then*

$$\chi(L,t) = \chi([\hat{0},x],t) \sum_{\substack{b\wedge x=\hat{0} \\ b:\text{atomic}}} \mu(\hat{0},b)t^{n-\rho(x)-\rho(b)} \ ∎$$

Theorem 2.1 can be considered a corollary of this result:

Proof of Theorem 2.1 We need to show that if $b \wedge x = \hat{0}$ then $\rho(z \vee b) - \rho(b) = \rho(z) - \rho(\hat{0})$ for every $z \in [\hat{0}, x]$. Let $y = z \vee b$. Since (x, b) is a modular pair, we have $y \wedge x = z \vee (b \wedge x) = z$. By Proposition 1.3 $z = y \wedge x$ is a modular element in $[\hat{0}, y]$ and so (z, b) is a modular pair in L. Recall that, in a semimodular lattice, the pair (z, b) is modular if and only if $\rho(z \wedge b) + \rho(z \vee b) = \rho(z) + \rho(b)$. Noticing that $z \wedge b = \hat{0}$, the proof follows. ∎

3 Non-crossing Partition Lattices

The non-crossing partition lattice was first studied by Kreweras [4] who showed its Möbius function to be a Catalan number. By using NBB sets Blass and Sagan [2] combinatorially explained this fact. In this section we will calculate the characteristic polynomial of the non-crossing partition lattice and then offer another explanation for the value of its Möbius function.

We say a partition $\pi \vdash [n]$ is *non-crossing* if there do not exist $i, k \in B$ and $j, l \in C$ for two distinct blocks B, C of π with $i < j < k < l$. Otherwise π is crossing. The set of non-crossing partitions of $[n]$ forms a meet-sublattice NC_n of Π_n with the same rank function. However unlike Π_n, the lattice NC_n is not semimodular in general. For example, let $\pi = 13$ and $\sigma = 24$ in NC_n then $\pi \wedge \sigma = \hat{0}$ and $\pi \vee \sigma = 1234$, so $\rho(\pi) + \rho(\sigma) = 2 < 3 = \rho(\pi \wedge \sigma) + \rho(\pi \vee \sigma)$.

Another way to view non-crossing partitions will be useful. Let $G = (V, E)$ be a graph with vertex set $V = [n]$. Then G is *non-crossing* if, when the vertices are arranged in their natural order clockwise around a circle and the edges are drawn as straight line segments, no two edges of G cross geometrically. Given a partition π we can form a graph G_π by representing each block $B = \{i_1 < i_2 < \ldots < i_l\}$ as a

cycle with edges $i_1i_2, i_2i_3, \ldots, i_li_1$. Then it is easy to see that π is non-crossing as a partition if and only if G_π is non-crossing as a graph.

Let $\pi = 12\ldots(n-1)$. The pair (Π_n, Δ) forms a supersolvable lattice with the maximal chain $\Delta : \hat{0} < 12 < 123 < \ldots < 12\ldots(n-1) < \hat{1}$. By Stanley [7, Proposition 2.2] Δ is a left-modular chain, and so π is left-modular in Π_n. Let $a < b \in NC_n$ with both incomparable to π. It is clear that $a \vee \pi = b \vee \pi = \hat{1}$ in Π_n as well as NC_n. (Note that the join operations of Π_n and NC_n are not the same in general.) We now refer to Theorem 1.2 and get $a \wedge \pi < b \wedge \pi$ in Π_n. Since NC_n is a meet-sublattice of Π_n, the above inequality for the two meets still holds in NC_n. By Theorem 1.2 again π is left-modular in NC_n.

Proposition 3.1 *The characteristic polynomial of the non-crossing partition lattice is*

$$\chi(NC_n, t) = t\chi(NC_{n-1}, t) - \sum_{i=1}^{n-1} \chi(NC_i, t)\chi(NC_{n-i}, t).$$

Proof. Let $\pi = 12\ldots(n-1)$. By Theorem 2.3 we have

$$\chi(NC_n, t) = \sum_{b \wedge \pi = \hat{0}} \left[\mu(b) t^{(n-1)-\rho(b\vee\pi)} \chi([b, b\vee\pi], t)\right].$$

Note that $b \wedge \pi = \hat{0}$ if and only if any two numbers of $[n-1]$ are in different blocks of b, so $b = \hat{0}$ or $b = mn$ with $1 \le m \le n-1$.

If $b = \hat{0}$, then $\chi([b, b\vee\pi], t) = \chi([\hat{0}, \pi], t) = \chi(NC_{n-1}, t)$. Thus we get the first term of the formula. Now let $b = mn$. It is clear that $b \vee \pi = \hat{1}$, so we need to consider the sublattice $[b, \hat{1}]$. Given any $\omega \in [b, \hat{1}]$, the edge mn in G_ω geometrically separates the graph G_ω into two parts, $G_{\omega,1}$ and $G_{\omega,2}$, which are induced by vertex sets $\{1, 2, \ldots, m, n\}$ and $\{m, m+1, \ldots, n-1, n\}$ respectively. By contracting the edge mn in both subgraphs, we get two non-crossing graphs $\bar{G}_{\omega,1}$ and $\bar{G}_{\omega,2}$. It is easy to check that the map $f : [b, \hat{1}] \to NC_m \times NC_{n-m}$ defined by $f(G_\omega) = (\bar{G}_{\omega,1}, \bar{G}_{\omega,2})$ is bijective. Therefore

$$\chi([b, b\vee\pi], t) = \chi(NC_m, t)\chi(NC_{n-m}, t),$$

and the proof is complete. ∎

For any $\omega = B_1/B_2/\ldots/B_k \in NC_n$, the interval $[\hat{0}, \omega] \cong \prod_i NC_{|B_i|}$. Hence to compute the Möbius function of NC_n, it suffices to do this only for $\hat{1}$. By Proposition 3.1 we have the recurrence relation

$$\mu(NC_n) \;=\; \chi(NC_n, 0) \;=\; -\sum_{i=1}^{n-1} \chi(NC_i, 0)\chi(NC_{n-i}, 0)$$

$$= \quad -\sum_{i=1}^{n-1} \mu(NC_i)\mu(NC_{n-i}),$$

with the initial condition $\mu(NC_1) = 1$. This recurrence is similar to that for Catalan number, so we get a explicit formula

$$\mu(NC_n) = (-1)^{n-1}\frac{1}{n}\binom{2n-1}{n-1}.$$

4 Shuffle Posets

The poset of shuffles was introduced by Greene [3], and a formula for its characteristic polynomial

$$\chi(\mathcal{W}_{m,n}, t) = (t-1)^{m+n}\sum_{i\geq 0}\binom{m}{i}\binom{n}{i}\left(\frac{1}{1-t}\right)^i$$

was given in the same paper. In this section we will construct a new formula using Theorem 2.5. Before doing that we need to recall some definitions and results which were given by Greene. Let $\mathcal{A}$ be a set, called the *alphabet of letters*. A *word* over $\mathcal{A}$ is a sequence $\mathbf{u} = u_1u_2\ldots u_n$ of elements of $\mathcal{A}$. All of our words will consist of distinct letters and we will sometimes also use $\mathbf{u}$ to stand for the set of letters in the word, depending upon the context. A *subword* of $\mathbf{u}$ is $\mathbf{w} = u_{i_1}\ldots u_{i_l}$ where $i_1 < \ldots < i_l$. If $\mathbf{u}$, $\mathbf{v}$ are any two words then the *restriction* of $\mathbf{u}$ to $\mathbf{v}$ is the subword $\mathbf{u_v}$ of $\mathbf{u}$ whose letters are exactly those of $\mathbf{u} \cap \mathbf{v}$. A *shuffle* of $\mathbf{u}$ and $\mathbf{v}$ is any word $\mathbf{s}$ such that $\mathbf{s} = \mathbf{u} \uplus \mathbf{v}$ as sets (disjoint union) and $\mathbf{s_u} = \mathbf{u}$, $\mathbf{s_v} = \mathbf{v}$ as words.

Given nonnegative integers m, n Greene defined the *poset of shuffles* $\mathcal{W}_{m,n}$ as follows. Fix disjoint words $\mathbf{x} = x_1\ldots x_m$ and $\mathbf{y} = y_1\ldots y_n$. The elements of $\mathcal{W}_{m,n}$ are all shuffles $\mathbf{w}$ of a subword of $\mathbf{x}$ with a subword of $\mathbf{y}$. The partial order is $\mathbf{v} \leq \mathbf{w}$ if $\mathbf{v_x} \supseteq \mathbf{w_x}$, $\mathbf{v_y} \subseteq \mathbf{w_y}$ as sets and $\mathbf{v_w} = \mathbf{w_v}$ as words. The covering relation $\mathbf{v} \prec \mathbf{w}$ is more intuitive: the two words $\mathbf{v}$ and $\mathbf{w}$ are the same except for one extra letter which is either an x_i in $\mathbf{v}$ or a y_j in $\mathbf{w}$. It is easy to see that $\mathcal{W}_{m,n}$ has minimal element $\hat{0} = \mathbf{x}$, maximal element $\hat{1} = \mathbf{y}$ and is graded with rank function

$$\rho(\mathbf{w}) = (m - |\mathbf{w_x}|) + |\mathbf{w_y}|.$$

It was shown by Greene that every shuffle poset is actually a lattice. To describe the join operation in $\mathcal{W}_{m,n}$, Greene defined the crossed letters as follows. Given $\mathbf{u}, \mathbf{v} \in \mathcal{W}_{m,n}$ then $x \in \mathbf{u} \cap \mathbf{v} \cap \mathbf{x}$ is *crossed* in $\mathbf{u}$ and $\mathbf{v}$ if there exist letters y_i, $y_j \in \mathbf{y}$ such that $i \leq j$ and x appears before y_i in one of the two words but after y_j in the other. For example, if $\mathbf{x} = def$ and $\mathbf{y} = DEF$ then for the pair $\mathbf{u} = dDEe$, $\mathbf{v} = Fdef$

the only crossed letter is d. The join of $\mathbf{u}$, $\mathbf{v}$ will be the unique word $\mathbf{w}$ greater than both $\mathbf{u}$, $\mathbf{v}$ such that

$$\begin{aligned} \mathbf{w_x} &= \{x \in \mathbf{u_x} \cap \mathbf{v_x} \mid x \text{ is not crossed}\} \\ \mathbf{w_y} &= \mathbf{u_y} \cup \mathbf{v_y}. \end{aligned}$$

In the previous example, $\mathbf{u} \vee \mathbf{v} = DEFe$. This example also shows that $\mathcal{W}_{m,n}$ is not semimodular in general because $\rho(\mathbf{u})+\rho(\mathbf{v}) = 3+1 < 5 = \rho(\mathbf{u}\vee\mathbf{v}) \leq \rho(\mathbf{u}\vee\mathbf{v})+\rho(\mathbf{u}\wedge\mathbf{v})$. Since $(\mathcal{W}_{n,m})^* = \mathcal{W}_{m,n}$ the meet operation in $\mathcal{W}_{m,n}$ is as same as the join operation in $\mathcal{W}_{n,m}$. So to find the meet in the analog way we need to consider those crossed letters $y \in \mathbf{u} \cap \mathbf{v} \cap \mathbf{y}$ in $\mathbf{u}$ and $\mathbf{v}$.

Greene also showed that subwords of $\mathbf{x}$ and subwords of $\mathbf{y}$ are modular elements of $\mathcal{W}_{m,n}$. In particular, the empty set $\emptyset$ is modular. It is also atomic since $[\hat{0}, \emptyset] \cong B_m$. Now we claim a new formula for the characteristic polynomial of $\mathcal{W}_{m,n}$.

Proposition 4.1 *The characteristic polynomial of the shuffle poset is*

$$\chi(\mathcal{W}_{m,n}, t) = (t-1)^m \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{m+i}{i} t^{n-i}. \tag{2}$$

Proof. Let $\mathbf{u} \wedge \emptyset = \mathbf{w}$ in $\mathcal{W}_{m,n}$ then $\mathbf{w_x} = \mathbf{u_x} \cup \emptyset_{\mathbf{x}} = \mathbf{u_x}$. So the meet $\mathbf{u} \wedge \emptyset = \hat{0}$ if and only if $\mathbf{x}$ is a subword of $\mathbf{u}$, i. e., the element $\mathbf{u}$ is a shuffle of $\mathbf{x}$ with a subword of $\mathbf{y}$. For such $\mathbf{u}$ and any given $\mathbf{v} \in [\hat{0}, \emptyset]$, there is no crossed letter x in them, and then $(\mathbf{u} \vee \mathbf{v})_{\mathbf{x}} = \mathbf{u_x} \cap \mathbf{v_x} = \mathbf{v}$ and $(\mathbf{u} \vee \mathbf{v})_{\mathbf{y}} = \mathbf{u_y} \cup \mathbf{v_y} = \mathbf{u_y}$ as sets. So we have

$$\begin{aligned} \rho(\mathbf{u} \vee \mathbf{v}) - \rho(\mathbf{u}) &= [(m - |\mathbf{v}|) - |\mathbf{u_y}|] - [(m-m) - |\mathbf{u_y}|] \\ &= m - |\mathbf{v}| \;=\; \rho(\mathbf{v}) - \rho(\hat{0}). \end{aligned}$$

Thus the map $\tau_{\mathbf{u}} : [\hat{0}, \emptyset] \to [\mathbf{u}, \emptyset \vee \mathbf{u}]$ is rank-preserving.

Since $[\hat{0}, \emptyset] \cong B_m$, by Theorem 2.5 we get

$$\chi(\mathcal{W}_{m,n}, t) = (t-1)^m \sum_{\mathbf{u} \wedge \emptyset = \hat{0}} \mu(\mathbf{u}) t^{(m+n)-m-\rho(\mathbf{u})}.$$

It is easy to see that the interval $[\hat{0}, \mathbf{u}]$ is isomorphic to B_i where $i = |\mathbf{u_y}|$. So $\mu(\mathbf{u}) = (-1)^{|\mathbf{u_y}|} = (-1)^{\rho(\mathbf{u})}$. Now we conclude that

$$\begin{aligned} \chi(\mathcal{W}_{m,n}, t) &= (t-1)^m \sum_{i=0}^{n} \begin{bmatrix} \text{the number of ways to} \\ \text{shuffle } \mathbf{x} \text{ with } i \text{ letters of } \mathbf{y} \end{bmatrix} (-1)^i t^{n-i} \\ &= (t-1)^m \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{m+i}{i} t^{n-i}. \quad \blacksquare \end{aligned}$$

To determine the Möbius function of $\mathcal{W}_{m,n}$ it suffices to compute $\mu(\hat{1})$ since for any $\mathbf{w} \in \mathcal{W}_{m,n}$ the interval $[\hat{0}, \mathbf{w}]$ is isomorphic to a product of $\mathcal{W}_{p,q}$'s for certain $p \leq m$ and $q \leq n$. Simply plugging $t = 0$ into formula (2) gives us the Möbius function $\mu(\mathcal{W}_{m,n})$.

Corollary 4.2 (Greene) *We have*

$$\mu(\mathcal{W}_{m,n}) = (-1)^{(m+n)} \binom{m+n}{n}. \blacksquare$$

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