A New Inequality for Bipartite Distance-Regular Graphs

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Abstract. *Our main result is the following: Let* Γ *denote a bipartite distance-regular graph with diameter* $D \geq 4$ *and valency* $k \geq 3$. *Let* $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_D$ *denote a nontrivial cosine sequence of* Γ. *Then*

$$(\sigma_1 - \sigma_{i-1})(\sigma_1 - \sigma_{i+1}) \geq (1 - \sigma_i)(\sigma_2 - \sigma_i) \qquad (1 \leq i \leq D-1),$$

and equality holds for $i = 3$ *if and only if equality holds for all* i. *Moreover, these equalities hold with* $\sigma_D \neq 1$ *if and only if* Γ *is* Q*-polynomial with respect to* $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_D$.

Résumé. *Soit* Γ *un graphe bi-partie à distance-réguliere de diamètre* $D \geq 4$ *et valence* $k \geq 3$. *Soit* $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_D$ *une séquence cosinus nontrivial de* Γ. *Nous montrons que*

$$(\sigma_1 - \sigma_{i-1})(\sigma_1 - \sigma_{i+1}) \geq (1 - \sigma_i)(\sigma_2 - \sigma_i) \qquad (1 \leq i \leq D-1).$$

Nous avons egalité avec $i = 3$ *si et seulement si nous avons egalité pour* $1 \leq i \leq D-1$. *Nous avons egalité avec* $i = 3$ *et* $\sigma_D \neq 1$ *si et seulement si* Γ *a la propriété de polynòme* Q *pour* $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_D$.

Preliminaries

Let $\Gamma = (X, R)$ denote a finite connected undirected graph with no loops or multiple edges. X denotes the vertex set and R denotes the adjacency relation. Let ∂ denote the usual path-length distance function for Γ and let D denote the diameter of the graph.

We say Γ is *distance-regular* if, for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all vertices x, y with $\partial(x, y) = h$, the scalar

$$p^h_{ij} = |\{z \in X | \partial(x, z) = i, \partial(y, z) = j\}|$$

is independent of the choice of x and y. The p^h_{ij} are called the *intersection numbers* of Γ. For notational convenience we set $b_i := p^i_{1\,i+1}$ and $c_i := p^i_{1\,i-1}$.

Assume Γ is distance-regular. For each integer i $(0 \leq i \leq D)$, let A_i denote the matrix in $Mat_X(\mathbb{R})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in X).$$

We call this the ith *distance matrix*. It is known that $A_0, A_1, A_2, ..., A_D$ form a basis for a subalgebra M of $Mat_X(\mathbb{R})$, called the *Bose-Mesner algebra*.

M has a second basis $E_0, E_1, E_2, ..., E_D$ consisting of its primitive idempotents. For a given idempotent E, let $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_D$ denote scalars such that

$$E = m|X|^{-1} \sum_{i=0}^{D} \sigma_i A_i,$$

where $m = \text{rank } E$. We call $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_D$ the *cosine sequence* for E. It turns out that $\sigma_1 = \sigma_0$ implies $\sigma_i = 1$ $(0 \leq i \leq D)$ and $\sigma_1 = -\sigma_0$ implies $\sigma_i = (-1)^i$ $(0 \leq i \leq D)$. In either case, the sequence $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_D$ (and E) is said to be *trivial*.

Main Results

Theorem 1. *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$. Let $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_D$ denote a cosine sequence of Γ. Then*

$$(\sigma_1 - \sigma_{i-1})(\sigma_1 - \sigma_{i+1}) \geq (1 - \sigma_i)(\sigma_2 - \sigma_i), \tag{1}$$

for $1 \leq i \leq D - 1$.

In the next theorem we examine what happens when equality is attained in (1) for a particular value of i.

For a vertex $x \in X$, let $\hat{x}$ denote the column vector in $\mathbb{R}^X$ with a 1 in coordinate x and 0 in all other coordinates.

Theorem 2. *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Let E denote any nontrivial primitive idempotent of Γ, and let $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_D$ denote the corresponding cosine sequence. Then for all integers i $(1 \leq i \leq D - 1)$, the following are equivalent.*

(i) Equality holds in (1).

(ii) *For all $x, y \in X$ such that $\partial(x,y)=i$,*

$$\sum_{\substack{z \in X \\ \partial(x,z)=1 \\ \partial(y,z)=i-1}} E\hat{z} \quad - \sum_{\substack{z' \in X \\ \partial(x,z')=i-1 \\ \partial(y,z')=1}} E\hat{z}' \;=\; c_i \frac{\sigma_1 - \sigma_{i-1}}{1-\sigma_i}(E\hat{x} - E\hat{y}).$$

(iii) *There exists $x, y \in X$ such that $\partial(x,y)=i$, and such that*

$$E\hat{x} - E\hat{y}, \qquad \sum_{\substack{z \in X \\ \partial(x,z)=1 \\ \partial(y,z)=i-1}} E\hat{z} \quad - \sum_{\substack{z' \in X \\ \partial(x,z')=i-1 \\ \partial(y,z')=1}} E\hat{z}'$$

are linearly dependent.

(iv) *For all $x, y \in X$ such that $\partial(x,y)=i$,*

$$\sum_{\substack{z \in X \\ \partial(x,z)=1 \\ \partial(y,z)=i+1}} E\hat{z} \quad - \sum_{\substack{z' \in X \\ \partial(x,z')=i+1 \\ \partial(y,z')=1}} E\hat{z}' \;=\; b_i \frac{\sigma_1 - \sigma_{i+1}}{1-\sigma_i}(E\hat{x} - E\hat{y}).$$

(v) *There exists $x, y \in X$ such that $\partial(x,y)=i$, and such that*

$$E\hat{x} - E\hat{y}, \qquad \sum_{\substack{z \in X \\ \partial(x,z)=1 \\ \partial(y,z)=i+1}} E\hat{z} \quad - \sum_{\substack{z' \in X \\ \partial(x,z')=i+1 \\ \partial(y,z')=1}} E\hat{z}'$$

are linearly dependent.

It is clear that equality holds in (1) for $i = 1, 2$. We now consider what happens when equality holds in (1) at all i $(1 \leq i \leq D-1)$.

Theorem 3. *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_D$ denote a nontrivial cosine sequence of Γ. The following are equivalent.*

(i) *The equivalent statements (i)-(v) of Theorem 2 hold for all i $(1 \leq i \leq D-1)$.*

(ii) *The equivalent statements (i)-(v) of Theorem 2 hold for $i = 3$.*

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. Let E denote a primitive idempotent of Γ, with cosine sequence $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_D$. Γ is said to be *Q-polynomial* with respect to E whenever there exist polynomials $q_0, q_1, q_2, ..., q_D \in \mathbb{R}[\lambda]$ with deg $q_i = i$ $(0 \leq i \leq D)$, and an ordering $E_0, E = E_1, E_2, ..., E_D$ of the primitive idempotents of Γ, such that

$$E_i = \sum_{j=0}^{D} q_i(\sigma_j) A_j \quad (0 \leq i \leq D).$$

Theorem 4. *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let E denote a nontrivial primitive idempotent, and let $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_D$ denote the corresponding cosine sequence. The following are equivalent.*

(i) The equivalent conditions of Theorem 3 hold with $\sigma_D \neq 1$.

(ii) Γ is Q-polynomial with respect to E.

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