DOMINO TABLEAUX, SCHÜTZENBERGER INVOLUTION, AND THE ACTION OF THE SYMMETRIC GROUP

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ABSTRACT

We construct a bijection between the set of domino tableaux and the set of self-evacuating Young tableaux using the realization of both sets as fixed points of involutions $\mathbf{D}$ and $\mathbf{S}$ respectively. Namely, we discover that these involutions are naturally conjugated, so their fixed point sets are in a natural one-to-one correspondence. Our bijection gives another proof of the well-known result by J. Stembridge ([12]) which states that the number of self–evacuating tableaux of given shape and weight, is equal to the number of domino tableaux of the same shape and the corresponding weight. Using this bijection we define an action of the symmetric group $S_{[\frac{n}{2}]}$ on the sets of self-evacuating and domino tableaux. This action gives another proof of the well-known fact that the number $K^{(2)}_{\lambda,\beta'}$ of domino tableaux of shape λ and weight β', does not depend on permutations of the β'.

RÉSUMÉ

Nous définissons une action du groupe symétrique $S_{[n/2]}$ sur l'ensemble des tableaux domino ('domino tableaux') et prouvons que le nombre de tableaux domino de poids β' ne dépend pas de la permutation du poids β'. Une preuve bijective du résultat bien connu de J. Stembridge, voulant que le nombre de "self–evacuating tableaux" d'une forme donnée et de poids $\beta = \left(\beta_1, \ldots, \beta_{[(n+1)/2]}, \beta_{[(n+1)/2]}\right)$, est donnée.

0. Introduction

Domino tableaux play a significant role in the representation theory of the symmetric group S_n, and the theory of symmetric polynomials. The shortest definition of *semi-standard domino tableaux* is given in [10], p. 139. In particular, the generating function of the number of domino tableaux of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$, and weight $\beta' = (\beta_1, \ldots, \beta_{[\frac{n+1}{2}]})$ is given by specializing the Schur polynomial $s_\lambda(x)$ at $x \mapsto y$ where $y_n = -y_{n-1}$, $y_{n-2} = -y_{n-3}, \ldots$:

$$\pm s_\lambda(y) = \sum_{\beta'} K^{(2)}_{\lambda,\beta'}\ y_n^{2\beta_1} y_{n-2}^{2\beta_2} \cdots \quad , \tag{0.1}$$

where the sign $\pm$ depends only on λ.

Recently the domino tableaux were studied in connection with the representation theory of GL_n. In particular, in [12] John Stembridge proved the following conjecture by Richard Stanley (who stated it for $\beta_1 = \beta_2 = \cdots = 1$ that is, for *standard* tableaux).

Theorem 0.1. *([12], Theorem 3.1.). The number of self-evacuating tableaux of shape λ and weight $\beta = (\beta_1, \ldots, \beta_{[\frac{n+1}{2}]}, \beta_{[\frac{n}{2}]}, \ldots, \beta_1)$, equals the number of domino tableaux $K^{(2)}_{\lambda,\beta'}$ of shape λ and weight β'.*

The *self-evacuating tableaux* are defined as fixed points of the *Schützenberger evacuation involution* $\mathbf{S}$ on the *semi-standard tableaux* (see [11], [12]). The proof of Theorem 0.1 in [12] uses properties of the canonical basis for the GL_n-module V_λ. A key ingredient of the proof is the result of [3], Theorem 8.2 which states that $\mathbf{S}$ is naturally identified with the longest element of the symmetric group S_n acting on the canonical basis of V_λ.

There are combinatorial algorithms (see e.g., [5] or [9]) which lead to a bijective proof of Theorem 0.1 for the *standard* tableaux. For semi-standard tableaux and even n a similar algorithm appeared in [7], remark on the page 399.

The first result of the present paper is an *algebraic* bijective proof of Theorem 0.1 for any n (Theorem 1.2). The main idea of the bijection is that (semi-standard) domino tableaux are naturally identified with those ordinary (semi-standard) tableaux which are fixed under a certain involution $\mathbf{D}$ (see the Appendix).

We discover that the involutions $\mathbf{D}$ and $\mathbf{S}$ are conjugate as automorphisms of all tableaux, say $\mathbf{S} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ (see Lemma 1.3). Thus the conjugating automorphism $\mathbf{P}$ bijects the domino tableaux onto the self-evacuating tableaux.

It follows from (0.1) that the number $K^{(2)}_{\lambda,\beta'}$ does not depend on the permutation of the weight β'. This is analogous to the property of the ordinary Kostka numbers $K_{\lambda,\beta}$. Recently, Carre and Leclerc constructed in [4] an appropriate action of the symmetric group $S_{\frac{n}{2}}$ (for even n) which realizes this property. Their construction used a generalized RSK-type algorithm ([4], Algorithm 7.1, and Theorem 7.8).

The second result of the present paper is *algebraic* construction of an $S_{[\frac{n}{2}]}$-action on $Tab_n^{\mathbf{D}}$. This action is parallel to the action of S_n on the ordinary tableaux studied in [2] and has a natural interpretation in terms of self-evacuating tableaux (Theorem 1.8). One can conjecture that for even n our action coincides with the Carre-Leclerc's action.

The material of the paper is arranged as follows. In Section 1 we list main results and construct the bijection.

Section 2 contains the proofs of Theorems 1.6 and 1.8 on the action of the symmetric group $S_{[\frac{n}{2}]}$.

Appendix contains some definitions and basic properties of domino tableaux.

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1. Main results

Following [10], denote by Tab_n the set of all (semi-standard Young) tableaux with entries not exceeding n. For a partition λ with at most n rows, and for an integer vector $\beta = (\beta_1, \ldots, \beta_n)$ denote by $Tab_\lambda(\beta)$ the set of $T \in Tab_n$ of shape λ and weight β (for the reader's convenience we collect all necessary definitions in the Appendix).

Let $t_i : Tab_n \to Tab_n$, $(i = 1, \ldots, n-1)$ be the *Bender-Knuth* automorphisms which we define in the Appendix (see also [1],[2] and [6]).

Define the automorphism $\mathbf{D} : Tab_n \to Tab_n$ by the formula:

$$\mathbf{D} = \mathbf{D}_n := t_{n-1} t_{n-3} \cdots , \tag{1.1}$$

i.e., $\mathbf{D} = t_{n-1} t_{n-3} \cdots t_2$ for odd n, and $\mathbf{D} = t_{n-1} t_{n-3} \cdots t_1$ for even n.

Denote by of $Tab_n^{\mathbf{D}}$ the set of tableaux $T \in Tab_n$ such that $\mathbf{D}(T) = T$. For every shape λ and weight $\beta' = (\beta_1, \ldots, \beta_{[\frac{n+1}{2}]})$ denote $Tab_\lambda^{\mathbf{D}}(\beta') = Tab_\lambda(\beta) \cap Tab_n^{\mathbf{D}}$, where $\beta = (\beta_{[\frac{n+1}{2}]}, \cdots, \beta_k, \beta_k, \cdots, \beta_2, \beta_2, \beta_1, \beta_1)$.

The *semi-standard domino tableaux* of shape λ and weight β' were studied by several autors (see, e.g., [4], [7], [9], [10]). We also define them in the Appendix. Our first "result" is another definition of domino tableaux.

Proposition-Definition 1.1. *For every λ and β' as above the domino tableaux of shape λ and weight β' are in a natural one-to-one correspondence with the set $Tab_\lambda^{\mathbf{D}}(\beta')$.*

We prove this proposition in the Appendix. From now on we identify the domino tableaux with the elements of $Tab_n^{\mathbf{D}}$.

Let us introduce the last bit of notation prior to the next result.

Let $p_i := t_1 t_2 \cdots t_i$ for $i = 1, \ldots, n-1$ (this p_i is the inverse of the i-th *promotion operator* defined in [11]). Define the automorphism $\mathbf{P} : Tab_n \to Tab_n$ by the formula:

$$\mathbf{P} = \mathbf{P}_n := p_{n-1} p_{n-3} \cdots , \tag{1.2}$$

so $\mathbf{P} = p_{n-1} p_{n-3} \cdots p_2$ for odd n, and $\mathbf{P} = p_{n-1} p_{n-3} \cdots p_1$ for even n.

It is well-known (see e.g., [2], [6]) that the Schützenberger evacuation involution factorizes as follows.

$$\mathbf{S} = \mathbf{S}_n = p_{n-1} p_{n-2} \cdots p_1 \tag{1.3}$$

We regard (1.3) as a definition of $\mathbf{S}$.

Denote by $Tab_n^{\mathbf{S}} \subset Tab_n$ the set of the *self-evacuating* tableaux, that is all the tableaux $T \in Tab_n$ fixed under $\mathbf{S}$. Similarly, denote $Tab_\lambda^{\mathbf{S}}(\beta) = Tab_\lambda(\beta) \cap Tab_n^{\mathbf{S}}$. It is well-known that this set is empty unless $\beta_i = \beta_{n+1-i}$ for $i = 1, 2, \ldots, [\frac{n+1}{2}]$.

Theorem 1.2. *The above automorphism* $\mathbf{P}$ *of* $Tab_n^{\mathbf{D}}$ *induces the bijection*

$$Tab_n^{\mathbf{D}} \cong Tab_n^{\mathbf{S}} \ . \tag{1.4}$$

More precisely, $\mathbf{P}$ *induces the bijection* $Tab_\lambda^{\mathbf{D}}(\beta') \cong Tab_n^{\mathbf{S}}(\beta)$ *for any shape* λ *and the weight* $\beta' = (\beta_1, \ldots, \beta_{[\frac{n+1}{2}]})$*, where* $\beta = (\beta_1, \ldots, \beta_{[\frac{n+1}{2}]}, \beta_{[\frac{n}{2}]}, \ldots, \beta_1)$.

Remark. Theorem 1.2 is, the "realization" of Theorem 0.1.

The proof of this theorem is so elementary that we present it here.

Proof of Theorem 1.2. The statement (1.4) immediately follows from the surprisingly elementary lemma below.

Lemma 1.3. $\mathbf{S} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.

Indeed, if $T \in Tab_n^{\mathbf{D}}$ then $T = \mathbf{D}(T) = \mathbf{P}^{-1}\mathbf{S}\mathbf{P}(T)$ by Lemma 1.3, so $\mathbf{S}(\mathbf{P}(T)) = \mathbf{P}(T)$, that is, $\mathbf{P}(T) \in Tab_n^{\mathbf{S}}$. This proves the inclusion $\mathbf{P}(Tab_n^{\mathbf{D}}) \subset Tab_n^{\mathbf{S}}$. The opposite inclusion also follows.

Remark. By definition (see the Appendix), the involutions $t_1, \ldots, t_{n-1}$ involved in Lemma 1.3, satisfy the following obvious relations (see e.g. [2]):

$$t_i^2 = \mathrm{id}, \quad t_i t_j = t_j t_i \ , \qquad 1 \le i, j \le n-1, \quad |i-j| > 1 \ . \tag{1.5}$$

In fact, the factorization in Lemma 1.3 is based only on this relations (as we can see from the proof below), and, hence, makes sense in any group with the relations (1.5).

It also follows from the definition of t_i that

$$t_i : Tab_\lambda(\beta) \cong Tab_\lambda((i,i+1)(\beta)) \ .$$

This proves the second assertion of Theorem 1.2 provided that Lemma 1.3 is proved.

Proof of Lemma 1.3. We will proceed by induction on n. If $n = 2$ then $\mathbf{S} = t_1$ while $\mathbf{P} = t_1$, and $\mathbf{D} = t_1$. If $n = 3$ then $\mathbf{S}_3 = t_1 t_2 t_1$ while $\mathbf{D} = t_2$, and $\mathbf{P} = t_1$. Then, for any $n > 3$ the formula (1.3) can be rewritten as $\mathbf{S}_n = p_{n-1}\mathbf{S}_{n-1}$. It is easy to derive from (1.5) that $\mathbf{S}_n = \mathbf{S}_{n-1}p_{n-1}^{-1}$. Applying both of the above relations, we obtain

$$\mathbf{S}_n = p_{n-2}\mathbf{S}_{n-2}p_{n-1}^{-1} \ . \tag{1.6}$$

Finally, by (1.5) and the inductive assumption for $n-2$,

$$\mathbf{S}_n = p_{n-2}\mathbf{S}_{n-2}p_{n-1}^{-1} = p_{n-2}\mathbf{P}_{n-2}\mathbf{D}_{n-2}\mathbf{P}_{n-2}^{-1}p_{n-1}^{-1} = p_{n-2}\mathbf{P}_{n-2}t_{n-1}\mathbf{D}_n\mathbf{P}_{n-2}^{-1}p_{n-1}^{-1}$$

since $\mathbf{D}_{n-2} = t_{n-1}\mathbf{D}_n$. Finally, $\mathbf{P}_{n-2}^{-1}p_{n-1}^{-1} = \mathbf{P}_n^{-1}$, and $p_{n-2}\mathbf{P}_{n-2}t_{n-1} = p_{n-2}t_{n-1}\mathbf{P}_{n-2} = p_{n-1}\mathbf{P}_{n-2} = \mathbf{P}_n$.

Lemma 1.3 is proved.

Theorem 1.2 is proved.

Recall (see the Appendix) that the weight of a domino tableau $T \in Tab_n^{\mathbf{D}}$ is the vector $\beta' = (\beta_1, \ldots, \beta_{[\frac{n+1}{2}]})$, where β_1 is the number of occurences of n, β_2 is that of $n-2$, and so on. Denote by $Tab_n^{\mathbf{D}}(\beta')$ the set of all $T \in Tab_n^{\mathbf{D}}$ of weight β'. Our next main result is the following

Theorem 1.4. *There exists a natural faithful action of the symmetric group $S_{[\frac{n}{2}]}$ on $Tab_n^{\mathbf{D}}$ preserving the shape and acting on weight by permutation.*

In order to define the action precisely, let us recall one of the results of [2].

Theorem 1.5 ([2]). *There is an action of S_n on Tab_n given by $(i,i+1) \mapsto s_i$, where*

$$s_i = \mathbf{S}_{i+1}t_1\mathbf{S}_{i+1} \ , \qquad i = 1, \ldots, n-1 \ , \tag{1.7}$$

that is, the s_i are involutions, and $(s_j s_{j+1})^3 = \mathrm{id}$, $s_i s_j = s_j s_i$ for all i, j such that $|j-i| > 1$. (Here $\mathbf{S}_{i+1}$ given by (1.3) acts on Tab_n).

Definition. For $n \geq 4$ define the automorphisms $\sigma_i : Tab_n \to Tab_n$, $i = 2, \ldots, n-2$ by the formula

$$\sigma_i := t_i s_{i-1} s_{i+1} t_i \ . \tag{1.8}$$

Remark. The automorphisms s_i were first defined by Lascoux and Schützenberger in [8] in the context of plactique monoid theory. The definition (1.7) of s_i first appeared in [2] in a more general situation. As a matter of fact, for the restriction of s_i to the set of tableaux, (1.7) is implied in [8].

Theorem 1.4 follows from a much stronger result, namely the following theorem:

Theorem 1.6.
(a) $\mathbf{D}\sigma_i = \sigma_i \mathbf{D}$ *if and only if* $i \equiv n \pmod 2$.
(b) The automorphisms $\sigma_2, \sigma_3, \cdots \sigma_{n-2}$ *are involutions satisfying the Coxeter relations* $(\sigma_i\sigma_j)^{n_{ij}} = \mathrm{id}$ *for* $i, j = 2, \ldots, n-2$, *where*

$$n_{ij} = \begin{cases} 3 & \text{if } |j-i| = 1 \text{ or } 2 \\ 6 & \text{if } |j-i| = 3 \\ 2 & \text{if } |j-i| > 3 \end{cases} . \tag{1.9}$$

We will prove Theorem 1.6 in Section 2.

Remark. One can conjecture that the relations (1.8) (together with $\sigma_i^2 = \mathrm{id}$) give a presentation of the group Σ_n generated by the σ_i.

Indeed, the action of $S_{[\frac{n}{2}]}$ on Tab_n from Theorem 1.4 can be defined by

$$(i, i+1) \mapsto \sigma_{n-2i} , \qquad i = 2, \ldots, n-2 .$$

According to Theorem 1.6(a), this action preserves $Tab_n^{\mathbf{D}}$. By definition (1.7), this action preserves shape of tableaux and permutes the weights as follows:

$$\sigma_{n-2i} : Tab_\lambda^{\mathbf{D}}(\beta') \cong Tab_\lambda^{\mathbf{D}}((i, i+1)(\beta')) . \tag{1.10}$$

Let us try to get the action of $S_{[\frac{n}{2}]}$ on $Tab_n^{\mathbf{S}}$ using the following nice property of the involutions $s_1, \ldots, s_{n-1}$ (which can be found, e.g., in [2], Proposition 1.4):

$$\mathbf{S}s_i\mathbf{S} = s_{n-i} , \qquad i = 1, \ldots, n-1 . \tag{1.11}$$

Let $\tau_1, \ldots, \tau_{[\frac{n}{2}]-1}$ be defined by $\tau_k := s_k s_{n-k}$. The following proposition is obvious.

Proposition 1.7.
(a) $\mathbf{S}\tau_k = \tau_k\mathbf{S}$ *for all* k.
(b) The group generated by the τ_k *is isomorphic to* $S_{[\frac{n}{2}]}$ *via*

$$(k, k+1) \mapsto \tau_k , \qquad k = 1, \ldots, \left[\frac{n}{2}\right] - 1 . \tag{1.12}$$

Restricting the action (1.12) to $Tab_n^{\mathbf{S}}$ we obtain the desirable action of $S_{[\frac{n}{2}]}$ on $Tab_n^{\mathbf{S}}$.

Our last main result compares the latter action of $S_{[\frac{n}{2}]}$ with that on $Tab_n^{\mathbf{D}}$. The following theorem confirms naturality of our choice of σ_j in (1.8).

Theorem 1.8. *For* $k = 1, \ldots, [\frac{n}{2}] - 1$, *we have* $\mathbf{P}^{-1}\tau_k\mathbf{P} = \sigma_{n-2k}$.

2. Proof of Theorems 1.6, 1.8 and related results

We start with a collection of formulas relating s_i and t_j. First of all,

$$s_i t_j = t_j s_i \tag{2.1}$$

for all i, j with $|j-i| > 1$ (This fact follows from [8], and is implied in [2], Theorem 1.1). Then, by definition (1.8) of σ_i,

$$\sigma_i t_j = t_j \sigma_i \tag{2.2}$$

whenever $|j-i| > 2$. Furthermore, (1.3) and (1.7) imply that

$$s_j = p_j s_{j-1} p_j^{-1}\ , \qquad j = 1, \ldots, n-1\ . \tag{2.3}$$

More generally,

$$s_i = p_j s_{i-1} p_j^{-1}\ , \qquad 1 \le i \le j < n\ . \tag{2.4}$$

Finally, (2.3) and (2.1) imply that

$$t_{j-1} s_j t_{j-1} = t_j s_{j-1} t_j\ . \tag{2.5}$$

Proof of Theorem 1.6(a). Let i be congruent n modulo 2. Then by (2.2) and the definition (1.1) of $\mathbf{D}$,

$$\mathbf{D}\sigma_i\mathbf{D} = t_{i+1} t_{i-1} \sigma_i t_{i+1} t_{i-1}\ . \tag{2.6}$$

We now prove that (2.6) is equal to σ_i. Indeed,

$$\mathbf{D}\sigma_i\mathbf{D} = t_{i+1} t_{i-1} \sigma_i t_{i+1} t_{i-1} = t_{i+1} t_{i-1} t_i s_{i-1} s_{i+1} t_i t_{i+1} t_{i-1}\ .$$

Now we will apply the relations (2.5) in the form $t_{i-1} t_i s_{i-1} = s_i t_{i-1} t_i$, and $s_{i+1} t_i t_{i+1} = t_i t_{i+1} s_i$. We obtain

$$\mathbf{D}\sigma_i\mathbf{D} = t_{i+1}(s_i t_{i-1} t_i)(t_i t_{i+1} s_i) t_{i-1} = t_{i+1} s_i t_{i-1} t_{i+1} s_i t_{i-1} = t_{i+1} s_i t_{i+1} t_{i-1} s_i t_{i-1}\ .$$

Applying (2.5) with $j = i-1, i$ once again, we finally obtain

$$\mathbf{D}\sigma_i\mathbf{D} = (t_i s_{i+1} t_i)(t_i s_{i-1} t_i) = t_i s_{i+1} s_{i-1} t_i = t_i s_{i-1} s_{i+1} t_i = \sigma_i$$

by Theorem 1.5, which proves that $\mathbf{D}\sigma_i = \sigma_i\mathbf{D}$. The implication "if" of **Theorem 1.6(a)** is proved. Prove the "only if" implication. Let ρ be the canonical homomorphism from the group generated by $t_1, \ldots, t_{n-1}$ to S_n defined by $t_j \mapsto (j, j+1)$. Then $\rho(\sigma_i) = (i, i+1)(i-1, i)(i+1, i+2)(i, i+1)$ for $i = 2, \ldots, n-2$, and $\rho(\mathbf{D}) = (n-1, n)(n-3, n-2)\cdots$.

If $i \not\equiv n$ $(mod\ 2)$ then $\rho(\sigma_i)$ and $\rho(\mathbf{D})$ do not commute. Hence $\sigma_i \mathbf{D} \neq \mathbf{D}\sigma_i$ for such i. Thus **Theorem 1.6(a)** is proved.

Proof of Theorem 1.6(b). We are going to consider several cases when $j-i=1,2,3$ and >3. In each case we compute the product $\sigma_i\sigma_j$ up to conjugation, and express it as a product of several s_k. These products are always of finite order, because, according to Theorem 1.5, these s_k's generate the group isomorphic to S_n.

The easiest case is $j-i>3$. Then all the ingredients of σ_j in (1.7) (namely t_j, s_{j-1}, s_{j+1}) commute with those of σ_i (namely t_i, s_{i-1}, s_{i+1}). Thus σ_j commutes with σ_i, that is, $(\sigma_i\sigma_j)^2=\mathrm{id}$, and we are done in this case.

In what follows we will write $a\equiv b$ if a is conjugate to b.

Case $j-i=3$, i.e., $j=i+3$. We have

$$\sigma_i\sigma_{i+3} = t_i s_{i-1} s_{i+1} t_i t_{i+3} s_{i+2} s_{i+4} t_{i+3} \equiv s_{i-1} s_{i+1} s_{i+2} s_{i+4}$$

by the above commutation between remote s_k and t_l.

By Theorem 1.5, $(\sigma_i\sigma_{i+3})^6 \equiv (s_{i-1})^6 (s_{i+1}s_{i+2})^6 (s_{i+4})^6 = \mathrm{id}$. So $(\sigma_i\sigma_{i+3})^6=\mathrm{id}$ and we are done with this case.

Case $j-i=2$, i.e., $j=i+2$. We have

$$\sigma_i\sigma_{i+2} = t_i s_{i-1} s_{i+1} t_i t_{i+2} s_{i+1} s_{i+3} t_{i+2} \equiv t_{i+2} t_i s_{i-1} s_{i+1} t_i t_{i+2} s_{i+1} s_{i+3}$$

$$= t_i s_{i-1} t_{i+2} s_{i+1} t_{i+2} t_i s_{i+1} s_{i+3} = t_i s_{i-1} t_{i+1} s_{i+2} t_{i+1} t_i s_{i+1} s_{i+3}$$

by (2.5) with $j=i+2$. Conjugating the latter expression with t_i, we obtain,

$$\sigma_i\sigma_{i+2} \equiv s_{i-1} t_{i+1} s_{i+2} t_{i+1} t_i s_{i+1} s_{i+3} t_i = s_{i-1} t_{i+1} s_{i+2} t_{i+1} t_i s_{i+1} t_i s_{i+3}$$

$$= s_{i-1} t_{i+1} s_{i+2} t_{i+1} t_{i+1} s_i t_{i+1} s_{i+3} = s_{i-1} t_{i+1} s_{i+2} s_i t_{i+1} s_{i+3}$$

$$\equiv t_{i+1} s_{i-1} t_{i+1} s_{i+2} s_i t_{i+1} s_{i+3} t_{i+1} = s_{i-1} s_{i+2} s_i s_{i+3} = s_{i-1} s_i s_{i+2} s_{i+3}\ .$$

Thus, $(\sigma_i\sigma_{i+2})^3 \equiv (s_{i-1}s_i s_{i+2} s_{i+3})^3 = (s_{i-1}s_i)^3 (s_{i+2}s_{i+3})^3 = \mathrm{id}$, and we are done with this case too.

Case $j-i=1$, i.e., $j=i+1$. We have

$$\sigma_i\sigma_{i+1} = t_i s_{i-1} s_{i+1} t_i t_{i+1} s_i s_{i+2} t_{i+1} = t_i s_{i-1} s_{i+1} t_i t_{i+1} s_i t_{i+1} t_{i+1} s_{i+2} t_{i+1}$$

$$= t_i s_{i-1} s_{i+1} t_i t_i s_{i+1} t_i t_{i+2} s_{i+1} t_{i+2} = t_i s_{i-1} t_i t_{i+2} s_{i+1} t_{i+2}$$

$$= t_{i-1} s_i t_{i-1} t_{i+2} s_{i+1} t_{i+2} \equiv s_i t_{i+2} s_{i+1} t_{i+2} \equiv s_i s_{i+1}\ .$$

Thus, $(\sigma_i\sigma_{i+1})^3 \equiv (s_i s_{i+1})^3 = \mathrm{id}$, and we are done with this final case.

This finishes the proof of **Theorem 1.6**.

Proof of Theorem 1.8. First of all we prove Theorem 1.8 with $k=1$, that is,

$$\mathbf{P}_n^{-1} s_1 s_{n-1} \mathbf{P}_n = \sigma_{n-2} \ . \tag{2.7}$$

Since $\mathbf{P}_n = p_n \mathbf{P}_{n-2}$, the left hand side of (2.7) equals

$$\mathbf{P}_{n-2}^{-1} p_{n-1}^{-1} s_1 p_{n-1} p_{n-1}^{-1} s_{n-1} p_{n-1} \mathbf{P}_{n-2} = \mathbf{P}_{n-2}^{-1} (p_{n-1}^{-1} s_1 p_{n-1}) s_{n-2} \mathbf{P}_{n-2} \ . \tag{2.8}$$

Now we prove by induction on n that

$$p_{n-1}^{-1} s_1 p_{n-1} = p_{n-2} s_{n-1} p_{n-2}^{-1} \tag{2.9}$$

Indeed, if $n=2$, (2.9) is obvious (we agree that $p_0 = t_0 = \mathrm{id}$). Then using (2.9) as inductive assumption, we obtain

$$p_n^{-1} s_1 p_n = t_n (p_{n-1}^{-1} s_1 p_{n-1}) t_n = t_n (p_{n-2} s_{n-1} p_{n-2}^{-1}) t_n$$

$$= p_{n-2} t_n s_{n-1} t_n p_{n-2}^{-1} = p_{n-2} t_{n-1} s_n t_{n-1} p_{n-2}^{-1} = p_{n-1} s_n p_{n-1}^{-1}$$

by (2.5), and we are done with (2.9).

Finally, in order to finish the proof of (2.7), let us substitute (2.9) into (2.8). Thus, taking into account that $\mathbf{P}_{n-2} = p_{n-3}\mathbf{P}_{n-4} = p_{n-2} t_{n-2} \mathbf{P}_{n-2}$, (2.8) is equal to

$$\mathbf{P}_{n-2}^{-1} (p_{n-2} s_{n-1} p_{n-2}^{-1}) s_{n-2} \mathbf{P}_{n-2} = \mathbf{P}_{n-4}^{-1} t_{n-2} p_{n-2}^{-1} (p_{n-2} s_{n-1} p_{n-2}^{-1}) s_{n-2} p_{n-2} t_{n-2} \mathbf{P}_{n-4}$$

$$= \mathbf{P}_{n-4}^{-1} t_{n-2} s_{n-1} (p_{n-2}^{-1} s_{n-2} p_{n-2}) t_{n-2} \mathbf{P}_{n-4} = \mathbf{P}_{n-4}^{-1} t_{n-2} s_{n-1} s_{n-3} t_{n-2} \mathbf{P}_{n-4}$$

$$= t_{n-2} s_{n-1} s_{n-3} t_{n-2} = \sigma_{n-2} \ ,$$

which proves **Theorem 1.8** for $k=1$.

Now using induction on n we will prove the general case, namely, the formula

$$\mathbf{P}_n^{-1} s_k s_{n-k} \mathbf{P}_n = \sigma_{n-2k} \ , 1 \le k \le \left[\frac{n}{2}\right] - 1. \tag{2.10}$$

Indeed,

$$\mathbf{P}_n^{-1} s_k s_{n-k} \mathbf{P}_n = \mathbf{P}_{n-2}^{-1} (p_{n-1}^{-1} s_k s_{n-k} p_{n-1}) \mathbf{P}_{n-2}. \tag{2.11}$$

But $p_{n-1}^{-1} s_k s_{n-k} p_{n-1} = (p_{n-1}^{-1} s_k p_{n-1})(p_{n-1}^{-1} s_{n-k} p_{n-1}) = s_{k-1} s_{n-k-1}$ by (2.4) applied with $j := n-1, i = k$, and $i = n-k$. Thus (2.11) is equal to

$$\mathbf{P}_{n-2}^{-1} s_{k-1} s_{n-k-1} \mathbf{P}_{n-2} = \sigma_{n-2-2(k-1)} = \sigma_{n-2k}$$

by the inductive assumption (2.10) with $n-2$. **Theorem 1.8** is proved.

3. Appendix. Definitions of domino tableaux

By definition from [10], a *(semi-standard Young) tableau T with entries not exceeding n* is an ascending sequence of Young diagrams $T = (\{\emptyset\} = \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_n)$ such that $\Delta_i = \Lambda_i \setminus \Lambda_{i-1}$ is a *horizontal strip* for $i = 1, \ldots, n$ (we allow $\Lambda_{i+1} = \Lambda_i$). Denote by Tab_n the set of all (semi-standard Young) tableaux with entries not exceeding n. Define the *shape* of T by $\lambda = \Lambda_n$, and the *weight* $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ of T by $\beta_i = |\Delta_i|$ for $i = 1, \ldots, n$. Denote by $Tab_n(\beta)$ the set of all $T \in Tab_n$ having weight β.

One visualizes the tableau T as filling of each box of Δ_i with the number i for $i = 1, \ldots, n$.

Let $T = (\Lambda_0 \subset \Lambda_1 \subset \Lambda_2)$ be a *skew* 2*-tableau*, that is, $\Lambda_1 \setminus \Lambda_0$ and $\Lambda_2 \setminus \Lambda_1$ both are horizontal strips. For such a skew 2-tableau T define the skew 2-tableau $t(T) = T = (\Lambda_0 \subset \Lambda_1' \subset \Lambda_2)$ as follows. First of all we visualize T as the shape $D = \Lambda_2 \setminus \Lambda_0$ filled with the letters **a** and **b** where the letters **a** fill the horizontal strip $\Lambda_1 \setminus \Lambda_0$, and the letters **b** fill the remaining horizontal strip $\Lambda_2 \setminus \Lambda_1$. Then $t(T)$ will be a new filling of D with the same letters **a** and **b** which we construct below.

Let D_k be the longest connected sub-row of the k-th row of D, whose boxes have no horizontal edges in common with other boxes of D. D_k contains say, l boxes on the left filled with **a**, and r boxes (on the right) filled with **b**. We will replace this filling of D_k with the r letters **a** on the left, and l letters **b** on the right. We do this procedure for every k, and leave unchanged the filling of the complement of the union of all the D_k in D. It is easy to see that the new filling of D is a skew 2-tableau of the form $t(T) = T = (\Lambda_0 \subset \Lambda_1' \subset \Lambda_2)$. Clearly the correspondence $T \mapsto t(T)$ is an involutive automorphism of the set of skew 2-tableaux.

The following lemma is obvious.

Lemma A1. *Let D be a skew Young diagram. Then the following are equivalent:*

(i) There is a (unique) skew 2-tableau $T = (\Lambda_0 \subset \Lambda_1 \subset \Lambda_2)$ with $D = \Lambda_2 \setminus \Lambda_0$ such that $t(T) = T$.

(ii) There is a (unique) covering of D by (non-overlapping) dominoes such that every domino has no common horizontal edges with other boxes of D.

Define the Bender-Knuth involution ([1]) $t_i : Tab_n \to Tab_n$ as follows. $T = (\cdots \subset \Lambda_{i-1} \subset \Lambda_i \subset \Lambda_{i+1} \subset \cdots) \in Tab_n$. Define $T_i(T) \in Tab_n$ by

$$t_i(T) = (\cdots \subset \Lambda_{i-1} \subset \Lambda_i' \subset \Lambda_{i+1} \subset \cdots)$$

where Λ_i' is defined by $(\Lambda_{i-1} \subset \Lambda_i' \subset \Lambda_{i+1}) = t(\Lambda_{i-1} \subset \Lambda_i \subset \Lambda_{i+1})$, and other Λ_j remain unchanged.

Denote the resulting tableau by $t_i(T)$. It is easy to see that the correspondence $T \mapsto t_i(T)$ is a well-defined involutive automorphism $t_i : Tab_n \to Tab_n$, and $t_i t_j = t_j t_i$ whenever $|j-i| > 1$.

A *domino tableau* $\mathbf{T}$ with entries not exceeding $\frac{n+1}{2}$ is an ascending sequence of the Young diagrams $\mathbf{T} = (\Lambda_\varepsilon \subset \Lambda_{\varepsilon+2} \subset \cdots \subset \Lambda_{n-2} \subset \Lambda_n)$ with $\varepsilon = n - 2[\frac{n}{2}]$, such that

(i) $\Lambda_\varepsilon = \emptyset$ if n is even, (i.e., $\varepsilon = 0$), and Λ_ε is a connected horizontal strip if n is odd, (i.e., $\varepsilon = 1$);

(ii) Each difference $D_k = \Lambda_{n-2k} \setminus \Lambda_{n-2k+2}$ is subject to condition (ii) of Lemma A1.

The *shape* of $\mathbf{T}$ is defined to be Λ_n. We also define the *weight* $\beta' = (\beta_1, \ldots, \beta_{[\frac{n+1}{2}]})$ of $\mathbf{T}$ by $\beta_k = \frac{|D_k|}{2}$ for $k = 1, \ldots, [\frac{n+1}{2}]$.

Thus the above definitions together with Lemma A1 prove Proposition-Definition 1.1.

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