A RECURSION FOR CHARGE ON WORDS

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This paper studies a statistic on words called charge that was discovered by Lascoux and Schützenberger in connection with the Kostka-Foulkes polynomials $K_{\lambda\mu}(q)$. We are interested in properties of charge when applied to words of a given content. We give a combinatorial proof of a recursion for charge that arises from a recursion on the Hilbert series of the graded S_n-modules. Our proof also gives an extension of the properties of cyclage to the cycling of 1's.

Cet article étudie une statistique de mots appelée "charge" découverte par Lascoux et Schützenberger en relation avec les polynômes $K_{\lambda\mu}(q)$ de Kostka-Foulkes. Nous sommes intéressés aux proprietés de la charge lorsqu'elle est appliquée à des mots au contenu donné. Nous donnons une démonstration combinatoire d'une récursion pour la charge provenant d'une récursion sur la série d'Hilbert des S_n-modules gradués. Notre démonstration donne aussi une extension des propriétés de cyclage aux cyclage de uns.

1. Introduction

The Kostka-Foulkes polynomials are the connection coefficients between the Schur symmetric functions and the classical Hall-Littlewood polynomials. That is, if $s_\lambda(\mathbf{x})$ is the Schur function on the variables $\mathbf{x} = \{x_1, x_2, \dots\}$ and $P_\mu(\mathbf{x}; q)$ are the Hall-Littlewood polynomials (see [6] for definitions), then the Kostka-Foulkes polynomials $K_{\lambda\mu}(q)$ are defined by

$$s_\lambda(\mathbf{x}) = \sum_\mu K_{\lambda\mu}(q) P_\mu(\mathbf{x}; q). \tag{1}$$

When $q \to 1$, Equation (1) becomes

$$s_\lambda(\mathbf{x}) = \sum_\mu K_{\lambda\mu} m_\mu(\mathbf{x}),$$

the standard description of the Schur function in terms of the monomial symmetric functions.

Thus, Foulkes [2] conjectured that the $K_{\lambda\mu}(q)$ were polynomials in q with non-negative coefficients, and, in fact, could be realized by q-counting a statistic, ch, on column-strict tableaux. That is,

$$K_{\lambda\mu}(q) = \sum_{T \in \mathcal{T}_{\lambda\mu}} q^{ch(T)}. \tag{2}$$

where $\mathcal{T}_{\lambda\mu}$ is the set of column-strict tableaux of shape λ and content μ. Note that when q is set to 1 in Equation (2), we obtain $K_{\lambda\mu}$, the classical Kostka number, which counts column-strict tableaux of shape λ and content μ.

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In their seminal work [4], Lascoux and Schützenberger found the statistic ch, called *charge*. This work led directly to a body of literature investigating the algebraic and combinatorial consequences and extensions of their results. Of special note in this literature is the work of Butler [1], which went far to elucidate the work of Lascoux and Schützenberger. Another important contribution to this literature, and the work central to the results of this abstract, is the paper of Garsia and Procesi [3]. These authors derive properties of the Kostka-Foulkes polynomials by studying certain graded S_n-modules. These modules have a recursive structure which leads directly to recursions for the Kostka-Foulkes polynomials.

Our interest is in properties of the charge statistic as it is applied to words of a given content, and is motivated by the recursion on the Hilbert series of the graded S_n-modules in [3]. In this abstract, we will sketch a combinatorial proof of this recursion. Our proof depends on the extension of properties of cyclage (defined by Lascoux and Schützenberger in [5]) to the cycling of 1's.

2. Partitions, Words and the Plactic Action

In this section we set down some of the basic definitions for partitions and words. We also define the plactic action of Lascoux and Sch"utzenberger.

If $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 0$, we say μ is a *partition*. We will require several special definitions involving partitions. For a partition μ, define $\mu^{(i)} = (\mu_1, \ldots, \mu_{i-1}, \mu_i - 1, \mu_{i+1}, \ldots, \mu_n)$. For example, if $\mu = (3, 2, 2, 1)$, then $\mu^{(2)} = (3, 1, 2, 1)$.

Also, for any partition μ, define $r_{i,\mu} = \#\{j > i : \mu_j = \mu_i\}$. For example, if $\mu = (6, 5, 5, 4, 3, 3, 3, 3, 1, 1)$, then $r_{1,\mu} = 0$, $r_{2,\mu} = 1$, $r_{5,\mu} = 3$ and $r_{6,\mu} = 2$.

If μ is a partition and each part of μ is a multiple of d, then define $\mu/d = (\mu_1/d, \mu_2/d, \ldots, \mu_k/d)$. For example, if $\mu = (18, 12, 12, 6)$, then $\mu/3 = (6, 4, 4, 2)$ and $\mu/6 = (3, 2, 2, 1)$.

Define a *j-protopartition* to be $(a, \nu_1, \nu_2, \ldots, \nu_k)$, where $(\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{j-1} \geq a > \nu_j \geq \cdots \geq \nu_k)$ is a partition. Thus, for example, $(2, 4, 3, 2, 2, 1)$ is a 5-protopartition and $(4, 4, 4, 2)$ is a 3-protopartition.

Now define a *word* $\mathbf{w} = w_n \cdots w_2 w_1$ to be any arrangement of a multiset of numbers and define the *content* $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of a word $\mathbf{w}$ so that μ_i is the number of i's in the word $\mathbf{w}$. We will often refer to these numbers as *letters*. We say that a word $\mathbf{w}$ has *partition content* if μ is a partition. The *length* of a word $\mathbf{w}$ with content μ is $n = \mu_1 + \mu_2 + \cdots + \mu_k$, i.e., the number of letters in $\mathbf{w}$.

Let W_μ be the set of all words of content μ and $W_\mu^{(1)}$ be the set of all words of content μ with a 1 in the leftmost position, i.e., $w_n = 1$.

Given any word $\mathbf{w}$ of content μ we can select any 1, any 2, any 3, $\ldots$, and any k, and write these letters down in the order in which they appear in $\mathbf{w}$. We will call this new word a *standard subword*.

For example, let

$$\mathbf{w} = 1365374212.$$

Then $w_9 w_8 w_7 w_5 w_4 w_2 w_1 = 3657412$ would be a standard subword of length 7 and $w_9 w_4 w_3 w_2 = 3421$ would be a standard subword of length 4.

Given any word $\mathbf{w}$ of length n, define $ORB(\mathbf{w})$ to be the orbit of $\mathbf{w}$ under the action of the cyclic group $\mathbb{Z}_n$. For $\mathbf{w} \in W_\mu$, define

$$ORB_1^+(\mathbf{w}) = ORB(\mathbf{w}) \cap W_\mu^{(1)}.$$

That is, $ORB_1^+(\mathbf{w})$ is the set of μ_1 elements of $ORB(\mathbf{w})$ which have a one in the leftmost position. Similarly, define $ORB_1^-(\mathbf{w})$ to be the set of μ_1 elements of $ORB(\mathbf{w})$ which have a one in the rightmost position.

For example, suppose

$$\mathbf{w} = 215321431.$$

Then

$$\begin{aligned} ORB(\mathbf{w}) = \{&121532143, 312153214, 431215321,\\ &143121532, 214312153, 321431215,\\ &532143121, 153214312, 215321431\}. \end{aligned}$$

Also,

$$ORB_1^+(\mathbf{w}) = \{121532143, 143121532, 153214312\}$$

and

$$ORB_1^-(\mathbf{w}) = \{215321431, 431215321, 532143121\}.$$

Define the *crank* of $\mathbf{w}$, $cr(\mathbf{w})$, to be the operation of cycling the leftmost letter of $\mathbf{w}$ to the rightmost position. Thus crank restricts to a bijection from $ORB_1^+(\mathbf{w})$ to $ORB_1^-(\mathbf{w})$. For example, suppose

$$\mathbf{w} = 121532143.$$

Then

$$cr(\mathbf{w}) = 215321431.$$

Next we define an action of S_k on words using the letters $1, 2, \ldots, k$ called the plactic action. We first define the action of an adjacent transposition σ_i, which will swap the multiplicities of the i's and the $i+1$'s in the following way: look at only the subword of i's and $i+1$'s. Then below each i write a right parenthesis ")" and below each $i+1$ write a left parenthesis "(". Next pair the parentheses by matching left parentheses with right parentheses until we are left with a string of unmatched left parentheses to the right of a string of unmatched right parentheses. Suppose there are l unmatched right parentheses and m unmatched left parentheses. If $l > m$ switch the rightmost $l - m$ unmatched right parentheses to left parentheses. If $m > l$ switch the leftmost $m - l$ unmatched left parentheses to right parentheses. Leave all of the matched parentheses unaltered. Now write down an i for every right parenthesis and an $i+1$ for every left parenthesis. This gives us a word of with the multiplicities of the i's and $i+1$'s swapped.

For example, let

$$\mathbf{w} = 2131122123131222112331$$

be a word of content $\mu = (9, 8, 5)$. To obtain $\sigma_2\mathbf{w}$ of content $(9, 5, 8)$, first write down the subword of 2's and 3's, $\mathbf{w}_{23}$, and below it write a right parenthesis under every 2 and a left parenthesis under every 3. Below the parentheses we will label with the same number the left and right parentheses which pair:

$$\begin{array}{rccccccccccccc} \mathbf{w}_{23} = & 2 & 3 & 2 & 2 & 2 & 3 & 3 & 2 & 2 & 2 & 2 & 3 & 3 \\ &) & (&) &) &) & (& (&) &) &) &) & (& (\\ & & {\scriptstyle 1} & {\scriptstyle 1} & & & {\scriptstyle 3} & {\scriptstyle 2} & {\scriptstyle 2} & {\scriptstyle 3} & & & & \end{array}$$

The unpaired parentheses form the following sequence:

$$) \quad) \quad) \quad) \quad) \quad (\quad ($$

When we switch the multiplicities we have the sequence:

$$) \quad) \quad (\quad (\quad (\quad (\quad ($$

Thus our new word has the following parentheses and corresponding 23 subword.

$$\begin{array}{rccccccccccccc} &) & (&) &) & (& (& (&) &) & (& (& (& (\\ \sigma_2 \mathbf{w}_{23} = & 2 & 3 & 2 & 2 & 3 & 3 & 3 & 2 & 2 & 3 & 3 & 3 & 3 \end{array}$$

This gives us the word

$$\sigma_2 \mathbf{w} = 2131122133131223113331$$

with content $\sigma_2 \mu = (9, 5, 8)$.

The following proposition is due to Lascoux and Schützenberger [5].

Proposition 1. *The action of the transpositions σ_i on words using letters $1, 2, \ldots, k$ is an action of S_k. That is,*

$$\sigma_i^2 \mathbf{w} = \mathbf{w},$$

$$\sigma_i \sigma_j \mathbf{w} = \sigma_j \sigma_i \mathbf{w} \qquad \textit{for } |i - j| \geq 2,$$

and

$$\sigma_i \sigma_{i+1} \sigma_i \mathbf{w} = \sigma_{i+1} \sigma_i \sigma_{i+1} \mathbf{w}.$$

We will call this action of S_k the *plactic action*.

Proposition 2. *The plactic action commutes with crank, i.e., $cr(\sigma \mathbf{w}) = \sigma cr(\mathbf{w})$.*

Of special interest to us will be the j-cycle $\rho_j = (123 \ldots j) = \sigma_1 \sigma_2 \ldots \sigma_{j-1} \in S_k$.

Words with cyclic symmetry cause certain technical difficulties in the statements of some of the results of this abstract. Therefore, we will say a word $\mathbf{w}$ is d-cyclic if d is the smallest positive integer such that $cr^d(\mathbf{w}) = \mathbf{w}$. Thus, $d = \#ORB(\mathbf{w})$. Acyclic words of length n are thus n-cyclic.

3. Charge

In this section we define the statistic discovered by Lascoux and Sch"utzenberger, charge. We follow Macdonald [6] and define charge on standard words, and then extend to words of partition type. Finally, we define charge on words of non-partition type by extending through the plactic action.

Let $\pi = \pi_n \pi_{n-1} \cdots \pi_1$ be a permutation of $[n]$ (i.e., the integers 1 to n each appear exactly once in π). For each i define the *charge value* of i to be $c_i(\pi) = \#\{j : j \leq i$ and j is to the right of $j - 1\}$. Then we define the *charge* of π as $ch(\pi) = \sum_i c_i(\pi)$.

For example, if

$$\pi = \begin{array}{cccccc} 5 & 6 & 2 & 4 & 1 & 3 \\ {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} \end{array},$$

then

$$ch(\pi) = 1 + 2 + 1 + 1 = 5,$$

where $c_1 = 1$, $c_2 = 0$, $c_3 = 1$, $c_4 = 0$, $c_5 = 2$ and $c_6 = 1$.

We now extend the definition of charge to words with content λ where λ is any partition. If $\mathbf{w}$ is a word of partition content $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, then select standard subwords, $\mathbf{w}^1$, $\mathbf{w}^2$, ..., $\mathbf{w}^{\lambda_1}$, as described below. Then define the *charge* of $\mathbf{w}$ by

$$ch(\mathbf{w}) = ch(\mathbf{w}^1) + ch(\mathbf{w}^2) + \cdots + ch(\mathbf{w}^{\lambda_1}).$$

To select the subwords, read $\mathbf{w} = w_n w_{n-1} \cdots w_2 w_1$ from right to left, i.e. starting with w_1. Mark with a superscript $_1$ the first 1 which occurs, then mark with a superscript $_1$ the first 2 which occurs to the left of 1^1, then mark with a superscript $_1$ the first 3 which occurs to the left of 2^1, and so on. If at any stage there is no $i+1$ to the left of i^1, then search for the first $i+1$ beginning at w_1, reading from right to left. Continue until the letter λ_1' is marked with a superscript $_1$. The letters in $\mathbf{w}$ marked with a superscript $_1$ form the standard subword $\mathbf{w}^1$.

For example,

$$\mathbf{w} = 215^14^1253^112^147^16^131^1,$$

so

$$\mathbf{w}^1 = 5432761.$$

Now select the subword $\mathbf{w}^2$ in the same manner, only mark with a superscript $_2$ and ignore the letters already marked with a superscript $_1$.

As in the example before,

$$\mathbf{w} = 215^14^12^25^23^11^22^14^27^16^13^21^1,$$

so

$$\mathbf{w}^2 = 25143.$$

Repeat for $\mathbf{w}^3, \ldots, \mathbf{w}^{\lambda_1}$. Then

$$ch(21542531247631) = ch(5432761) + ch(25143) + ch(21) = 2 + 3 + 0 = 5.$$

For $\mathbf{w}$ of partition content, we define $c_i(\mathbf{w})$ to be the charge value of the letter in position i of the word $\mathbf{w}$, i.e., $c_i(\mathbf{w})$ is the charge value on w_i in its standard subword. For example, let $\mathbf{w} = 32514132$. Then $c_1 = c_2 = c_4 = c_6 = 1$ and $c_i = 0$ for all other i.

Now we extend the definition of charge to words of non-partition type by choosing $\sigma \in S_k$ so that $\sigma\mathbf{w}$ has partition content, and then defining $ch(\mathbf{w}) = ch(\sigma\mathbf{w})$. For example, let

$$\mathbf{w} = 24132352431232$$

be a word of content $\mu = (2, 5, 4, 2, 1)$. In order to calculate the charge of $\mathbf{w}$, note that $\sigma\mathbf{w}$ will have content $(5, 4, 2, 2, 1)$ if $\sigma = \sigma_2\sigma_1$. Then

$$\sigma\mathbf{w} = 24121352421131$$

and

$$\begin{aligned} ch(\sigma\mathbf{w}) &= ch(43521) + ch(2413) + ch(21) + ch(21) + ch(1) \\ &= 1 + 2 + 0 + 0 + 0 = 3 = ch(\mathbf{w}). \end{aligned}$$

It is easy to see that when $x \neq 1$ is cycled from the rightmost position in a word of partition content to the leftmost position in the word the charge value of x in the word decreases by one.

Lemma 3 (Lascoux and Schützenberger). *Let* $\mathbf{w}$ *be a word of partition content with* $\mathbf{w} \notin W_\mu^1$. *Then*

$$ch(cr(\mathbf{w})) = ch(\mathbf{w}) + 1.$$

In fact,

$$c_i(cr(\mathbf{w})) = c_{i-1}(\mathbf{w}) \qquad \text{for } 1 < i \leq n$$

and

$$c_1(cr(\mathbf{w})) = c_n(\mathbf{w}) + 1.$$

4. The Recursion

In this section we give the Garsia-Procesi recurrence of interest to us. We then outline a purely bijective proof through a series of lemmas and theorems.

In [3] Garsia and Procesi prove the following recurrence for the generating function for charge on words.

Theorem 4.

$$\sum_{\mathbf{w} \in W_\mu} q^{ch(\mathbf{w})} = \sum_i q^{r_{i,\mu}} \sum_{\mathbf{w} \in W_{\mu^{(i)}}} q^{ch(\mathbf{w})}$$

In fact, Garsia and Procesi prove the cocharge version of this recurrence, which arises as a recurrence on the Hilbert series for certain graded S_n-modules. Their proof follows from a more general recurrence on the modules themselves.

We will outline a direct combinatorial proof. To begin, the following two theorems describe what happens to charge on a word when a 1 is cranked.

Theorem 5. *Let* $\mathbf{w} \in W_\mu^{(1)}$ *be a word of partition content. Then there is a* j *and a standard subword* $\mathbf{v}$ *of* $\mathbf{w}$ *of length* μ'_j *such that*

i. *The* 1 *in the leftmost position in* $\mathbf{w}$ *is in* $\mathbf{v}$.
ii. *If* w_i *is a letter in* $\mathbf{v}$ *not equal to* 1, *then* $c_i(cr(\mathbf{w})) = c_{i-1}(\mathbf{w}) - 1$.
iii. *If* w_i *is any other letter in* $\mathbf{w}$, *then* $c_i(cr(\mathbf{w})) = c_{i-1}(\mathbf{w})$.
iv. $ch(cr(\mathbf{w})) = ch(\mathbf{w}) - \mu'_j + 1$.

Theorem 6. *Let* μ *be a partition of* n *and let* $\mathbf{w} \in W_\mu$ *be an* n/d*-cyclic word. There is a one-to-one correspondence between* $ORB_1^+(\mathbf{w})$ *and the* μ_1/d *columns of the Ferrers diagram of* μ/d *such that when* j *corresponds to* $\mathbf{z_j} \in ORB_1^+(\mathbf{w})$, *then*

$$ch(\mathbf{z_j}) = ch(cr(\mathbf{z_j}))) + \mu'_j - 1.$$

Proof. Let $\mathbf{w} = w_1 w_2 \ldots w_n$. Let k be the largest letter in $\mathbf{w}$ and choose an i so that $w_i = k$. Apply cr iteratively to $\mathbf{w}$. According to Lemma 3 and Theorem 5, Parts ii and iii, the only such iteration which results in an increase in the c-value of that k is when that k cycles from the leftmost position to the rightmost position. Therefore, by Parts ii and iii of Theorem 5, there must be exactly one word $\mathbf{z_1} \in ORB_1(\mathbf{w})$ such that $cr(\mathbf{z_1})$ decreases the c-value of that k. The result now follows from Theorem 5, Part iv, and induction. ☐

Theorem 7. *Let* μ *be a partition of* n *and let* $\mathbf{w} \in W_\mu$ *be an* n/d*-cyclic word. There is a bijection* ϕ *between the elements of* $ORB(\mathbf{w})$ *and the cells of the Ferrers diagram of* μ/d *satisfying*

i. *If* $\mathbf{z_j} \in ORB_1^+(\mathbf{w})$ *corresponds to column* j *according to Theorem 6 then* $\phi(\mathbf{z_j}) = (1, j)$.

ii. *If* $\mathbf{u}, \mathbf{v} \in ORB(\mathbf{w})$ *with* $\phi(\mathbf{u}) = (i-1, j)$ *and* $\phi(\mathbf{v}) = (i, j)$, $i \neq 1$, *then* $ch(\mathbf{v}) = ch(\mathbf{u}) - 1$.

Thus Part i says that elements of $ORB_1^+(\mathbf{w})$ appear at the top of the columns of μ/d, while Part ii says that charge values decrease by one down columns.

For example, if $\mathbf{w} = 215321431 \in W_{(32211)}$, then ϕ maps $ORB(\mathbf{w})$ to the cells of the Ferrers diagram of shape (32211) as follows (charge on each word is in parentheses):

1 5 3 2 1 4 3 1 2 (4)	1 4 3 1 2 1 5 3 2 (3)	1 2 1 5 3 2 1 4 3 (3)
2 1 5 3 2 1 4 3 1 (3)	2 1 4 3 1 2 1 5 3 (2)	
3 1 2 1 5 3 2 1 4 (2)	3 2 1 4 3 1 2 1 5 (1)	
4 3 1 2 1 5 3 2 1 (1)		
5 3 2 1 4 3 1 2 1 (0)		

Let $RW_{i,\mu}$ denote all the words in W_μ which map to row i under ϕ.

Choose any word $\mathbf{v}$ in $ORB(\mathbf{w})$ and suppose $\phi(\mathbf{v}) = (j, l)$. Define $\alpha : ORB(\mathbf{w}) \to ORB_1^+(\mathbf{w})$ by $\alpha(\mathbf{v}) = \mathbf{z}$ where $\phi(\mathbf{z}) = (1, l)$. That is, α "projects" $ORB(\mathbf{w})$ onto $ORB_1^+(\mathbf{w})$ up the column of the Ferrers diagram.

Now define β from words $1\mathbf{u}$ of content (μ_j, λ) where $\mu_j \neq 0$ and λ is a partition by $\beta(1\mathbf{u}) = \mathbf{u}$.

Theorem 8. *There is a bijection* $\psi = \rho_j^{-1} \circ \beta \circ \rho_j \circ \alpha$ *from* $RW_{i,\mu}$ *to* $W_{\mu^{(i)}}$ *such that if* $\mathbf{w} \in RW_{i,\mu}$ *and* $\tilde{\mathbf{w}} = \psi(\mathbf{w}) \in W_{\mu^{(i)}}$, *then*

$$ch(\mathbf{w}) = ch(\tilde{\mathbf{w}}) + r_{i,\mu}.$$

Note that Theorem 4 is a direct consequence of Theorem 8.

That ψ is a well-defined bijection follows from the following lemma.

Lemma 9. *Let* $\mathbf{v} \in ORB(\mathbf{w})$ *and suppose* $\phi(\mathbf{v}) = (j, l)$. *Then* $\rho_j(\alpha(\mathbf{v}))$ *has a* 1 *in the leftmost position.*

For example, let $\mathbf{v} = 431215321$ so that $\phi(\mathbf{v}) = (4, 1)$ as given above. Then $\alpha(\mathbf{v}) = 153214312$. The permutation $\rho_3(153214312)$ swaps the multiplicities in 153214312 from content $(3, 2, 2, 1, 1)$ to content $(1, 3, 2, 2, 1)$. We obtain the word $\mathbf{s} = 154324322$ which has a 1 in the leftmost position.

That ψ changes charge by the correct amount follows from the next lemma.

Lemma 10. *If* ν *is a* j*-protopartition of* μ *and* $1\mathbf{z} \in W_\nu$, *then* $ch(1\mathbf{z}) = ch(\mathbf{z}) + j - 1$.

To prove this lemma we will define a new statistic on words of protopartition type, as in the lemma. Define $\widetilde{ch}$ on such a word by finding the smallest letter reading right to left, assigning it a charge of zero, and then building a subword from it as we do in the definition of charge. We then remove this subword from the word and repeat the process. We calculate charge on the subwords in the usual way, starting with the smallest letter (not necessarily a one in this description). The statistic $\widetilde{ch}$ will be the sum of the charge calculations on the subwords.

To continue with our example, let $\mathbf{s} = 154324322$. The first subword that we select and then remove is $\mathbf{w}^1 = 15432$ leaving 4322. The second subword is $\mathbf{w}^2 = 432$ and the third subword is $\mathbf{w}^3 = 2$. When we calculate charge on the subwords (starting with the smallest letter having a charge of zero) we find that $\mathbf{w}^1$ has a charge of 4, $\mathbf{w}^2$ has a charge of 0 and $\mathbf{w}^3$ has a charge of 0. Thus $\widetilde{ch}(\mathbf{s}) = 4$.

Theorem 11. *If* $\mathbf{w}$ *is of protopartition type,* $\widetilde{ch}(\mathbf{w}) = ch(\mathbf{w})$.

Notice that if we swap multiplicities to partition content in the word $\mathbf{s}$ above, we obtain $\rho_3^{-1}\mathbf{s} = 153214312$, which has charge equal to 4.

Also note that we can define $\widetilde{ch}$ on j-protopartitions and then extend to all types through the plactic action, as we did with charge.

The proof of Lemma 10 now follows immediately. Once we have Lemma 9 and Lemma 10, the proof of Theorem 8 is clear.

5. A Simple Description

In this section we describe a simple description of the bijection given in Theorem 8. This description bypasses the plactic action required by ρ_j and ρ_j^{-1} and uses instead a charge-like calculation.

For partition μ, let $C(\mu) = \{i : \mu'_i > \mu'_{i+1}\}$. Thus, $C(\mu)$ is the set of column indices of the outer corners of the Ferrers diagram of μ. For example, if $\mu = (4, 4, 3, 3, 3, 1, 1)$, then $C(\mu) = \{1, 3, 4\}$.

Suppose $\mathbf{w} \in W^1_\mu$. For each $i \in C(\mu)$, we now describe a special standard subword of $\mathbf{w}$, which we call $\Delta_i(\mathbf{w})$.

We begin by finding the standard subwords $\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^{i-1}$ as in the charge calculation in Section 3. We call this set of standard subwords the i-th *charge set*. Now underline the unique μ'_i in $\mathbf{w}$ which is not in the i-th charge set. Next, underline the first $\mu'_i - 1$ which occurs to the right of the underlined μ'_i and is not in the i-th charge set. Continue by underlining the first $\mu'_i - 2$ which occurs to the right of the underlined $\mu'_i - 1$ and is not in the i-th charge set, and so on. If at any stage there is no $l - 1$ to the right of the underlined l, then search for the first $l - 1$ beginning at w_n, reading left to right. Continue until a 1 is underlined. The standard subword of underlined letters is $\Delta_i(\mathbf{w})$.

Note that the construction of $\Delta_i(\mathbf{w})$ is just the opposite of the construction of the charge words: the scanning takes place from left to right, and the numbers scanned for are chosen in descending order.

As an example, let

$$\mathbf{w} = 1\,2\,1\,4\,3\,5\,2\,1\,3\,1\,7\,5\,2\,6\,5\,2\,4\,4\,3.$$

Note that $\mathbf{w}$ has type $(4, 4, 3, 3, 3, 1, 1)$.

Below, we underline the three possible $\Delta_i(\mathbf{w})$, and note with superscripts the i-th charge set:

$$\Delta_1(\mathbf{w}) : \quad 1\,\underline{2}\,\underline{1}\,4\,3\,5\,2\,1\,3\,1\,\underline{7}\,5\,2\,\underline{6}\,\underline{5}\,2\,\underline{4}\,4\ ,\underline{3},$$

$$\Delta_3(\mathbf{w}) : \quad \underline{1}\,2^2\,1\,4^1\,3^1\,\underline{5}\,2^1\,1^2\,\underline{3}\,1^1\,7^1\,5^2\,\underline{2}\,6^1\,5^1\,2\,\underline{4}\,4^2\,3^2,$$

and

$$\Delta_4(\mathbf{w}) : \quad \underline{1}\,2^2\,1^3\,4^1\,3^1\,5^3\,2^1\,1^2\,3^3\,1^1\,7^1\,5^2\,\underline{2}\,6^1\,5^1\,2^3\,4^3\,4^2\,3^2.$$

The location of the 1 in $\Delta_i(\mathbf{w})$ determines the location of $\mathbf{w}$ in $\phi(\mathbf{w})$ described in Theorem 7. We will say a standard subword of $\mathbf{w} \in W^1_\mu$ is *pinned* if the 1 of the standard subword is w_n.

Theorem 12. *Let μ be a partition of n and let $\mathbf{w} \in W^1_\mu$ be an n/d-cyclic word. Suppose $\phi(\mathbf{w}) = (1, j)$. Then $\Delta_i(\mathbf{w})$ is pinned if and only if $dj \leq i$.*

¿From the example, Theorem 12 means that $\phi(\mathbf{w}) = (1, 2)$ or $(1, 3)$.

Corollary 13. *If* $\mathbf{w} \in W_\mu^1$ *has partition type and* $\Delta_i(\mathbf{w})$ *is pinned, then* $\Delta_l(\mathbf{w})$ *is pinned for each* $l \in C(\mu)$ *with* $l \geq i$.

Furthermore, we can describe a portion of the bijection in Theorem 8. If $\Delta_i(\mathbf{w})$ is pinned, then define $\tau_i(\mathbf{w})$ by removing w_n from $\mathbf{w}$ and subtracting 1 from the other letters in $\Delta_i(\mathbf{w})$.

Theorem 14. *If* $\Delta_i(\mathbf{w})$ *is pinned, then*

$$\rho_{\mu_i'}^{-1} \circ \beta \circ \rho_{\mu_i'}(\mathbf{w}) = \tau_i(\mathbf{w})$$

In the example,

$$\tau_3(\mathbf{w}) = 2\,1\,4\,3\,4\,2\,1\,2\,1\,7\,5\,1\,6\,5\,2\,3\,4\,3$$

and

$$\tau_4(\mathbf{w}) = 2\,1\,4\,3\,5\,2\,1\,3\,1\,7\,5\,1\,6\,5\,2\,4\,4\,3.$$

The reader may verify that these two words are the results of applying $\rho_5^{-1} \circ \beta \circ \rho_5$ and $\rho_2^{-1} \circ \beta \circ \rho_2$, respectively, to $\mathbf{w}$. Furthermore, the reader may wish to check that $\rho_7(\mathbf{w})$ does not have a 1 in the leftmost position, as required by Theorem 12 and the fact that $\Delta_1(\mathbf{w})$ is not pinned.

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