The class algebra of the symmetric group

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Abstract

An algorithm for the evaluation of products of arbitrary conjugacy class-sums in the symmetric group is conjectured. Let the support-size of a conjugacy class be the number of indices that are not fixed points. The algorithm proposed implies that the coefficient of the class-sum C in the product of the class-sums A and B is given in terms of a well-defined enumeration problem within the symmetric group S_p, where p is the smallest of the support-sizes of A, B and C.

Reśumé

Un algorithme est conjecturé pour l'évaluation de produits de sommes-des-classes-de-conjugaison arbitraires dans le groupe symétrique. Le nombre des indices qui ne sont pas des points fixés dans une classe-de-conjugaison donnéra la mesure de support de cette classe. L'algorithme proposé implique que le coefficient de la somme-de-la-classe C dans le produit des sommes-des-classes A and B est donnée par un problème d'énumération défini dans le groupe symétrique S_p, ou p est la plus petite des mesures du support des classes A, B et C.

1 Introduction

The conjugacy class-sums of a finite group span the center of the corresponding group algebra, and consequently play a major role in its representation theory. Many of the known results concerning the structure of the class-algebra of the symmetric group (e.g., [1] and references therein) are based on the application of the representation theoretic expression for the coefficient of the class-sum C in the product of the class-sums A and B,

$$(A \cdot B)_C = \frac{|A||B|}{n!} \sum_{\Gamma} \frac{\chi_A^\Gamma \chi_B^\Gamma \chi_C^\Gamma}{|\Gamma|} \quad . \tag{1}$$

Here, χ_A^Γ is the (ordinary) character of the class A in the irreducible representation (irrep) Γ, $|\Gamma|$ is the dimension of the irrep, and $|A|$ is the number of elements in the class. Several attempts to derive specific coefficients in the class-algebra of the symmetric group using their combinatorial significance were reported (e.g., [2] and references therein). A comprehensive review was compiled sometime ago [3].

This article reports a (conjectural) generalization of a combinatorial algorithm recently developed for the evaluation of the products of class-sums in the algebra of the symmetric group. The original algorithm deals with products of two class-sums, at least one of which consists of a single cycle (and an appropriate number of fixed points). The general features of that algorithm are reviewed in ref. [4]. It presents the product of a class-sum of the form $[(p)(1)^{n-p}]_n$, that is, of the formal sum of the permutations that consist of a single cycle of length p, with an arbitrary class-sum in S_n, in terms of the action of a set of "reduced class-sums" on the arbitrary class-sum of interest. The reduced class-sums are labelled by pairs of partitions of p and by some connectivity data. Each of these reduced class-sums is multiplied by a constant (*i.e.,* n independent) *reduced class-coefficient* (RCC). A conjecture was proposed in ref. [5], that allows the evaluation of all the p-RCCs as an enumeration problem within S_p. This procedure is generalized in the present article, to allow the evaluation of arbitrary class-sum products within the symmetric group algebra. The product of the class-sum C', that consists of k cycles of lengths $p_1,\ p_2,\ \cdots,\ p_k$, ($p_i > 1;\ i = 1,\ 2,\ \cdots,\ k$) and of $n - \sum_{i=1}^{k} p_i$ fixed points, with an arbitrary class-sum, is given in terms of an expression of the form reviewed above. The main distinction has to do with a modification of the connectivity conditions that the non-vanishing RCCs ought to satisfy.

The algorithm is stated in terms of a "global conjecture" that specifies the overall form of the expression for the product $[C']_n \cdot [*]_n$, where $*$ stands for an arbitrary conjugacy class. This expression consists of a linear combination of operators that depend only on C, the part of C' that is obtained by discarding the fixed points. The coefficients of the various terms in this linear combination are given by means of the "central conjecture". The "global conjecture" as formulated in Section 2 is a generalization of the "global conjecture" presented in ref. [4], that applies to the special case in which C is a single-cycle conjugacy class. Similarly, the "central conjecture" presented in Section 3 generalizes the single-cycle "central conjecture", that was presented in ref. [5]. Sections 4-7 serve to illustrate several facets of the formalism.

2 Class-sum products: The "global conjecture"

Let C' be a conjugacy class in S_n, $C' \equiv (1)^{m_1}(2)^{m_2}\cdots(n)^{m_n}$, $\sum_{i=1}^n im_i = n$. The support-size of C' is the number of indices in the permutations belonging to the conjugacy class C' that are not fixed points, i.e., $p = \sum_{i=2}^n im_i$. $C \equiv (2)^{m_2}\cdots(n)^{m_n}$ is the conjugacy class in S_p that is obtained from C' by discarding the fixed points. The parity of the members of the conjugacy class C (and C') is given by $p_C = \sum_{i=2}^n (i-1)\ell_i = p - \sum_{i=2}^n \ell_i$. A partition of p into k non-vanishing parts will be denoted by $(p_1, p_2, \cdots, p_k) \vdash p$.

The symbol

$$< (\overbrace{r_1, r_2, \cdots, r_{p_1}}^{p_1})(\overbrace{r_{p_1+1}, \cdots}^{p_2})\cdots(\overbrace{\cdots, r_p}^{p_k});(\overbrace{\cdots}^{p'_1})(\overbrace{\cdots}^{p'_2})\cdots(\overbrace{\cdots}^{p'_\ell})* > , \tag{2}$$

where $* \equiv (1)^{\ell_1}(2)^{\ell_2}\cdots(n)^{\ell_n}$ stands for an arbitrary conjugacy class, will be referred to as the **reduced class-sum**. It is specified by two set-partitions of the indices $r_1, r_2, \cdots, r_p$. Each part of each of the set-partitions specifies a cycle whose length is equal to the sum of its indices. The number of indices r_i in a given cycle will be referred to as this cycle's **index-length**. For each set of values of the indices $r_1, r_2, \cdots, r_p$, the reduced class-sum, eq. (2), stands for a linear combination of two class-sums: In one of them, a set of cycles of the lengths specified by one of the two sets of index-sums is eliminated from the conjugacy class $*$ and a set specified by the second is inserted. The other class-sum is generated by interchanging the roles of the two sets of cycles. If either set contains a cycle of a length that does not appear in the original class $*$, the term involving the elimination of that set vanishes. When the two sets of indexed cycles are related to one another by means of a permutation of indices that occupy equivalent positions, the term is referred to as self-associated, and is defined to operate only once on the original class $*$. Self-associated reduced class-sums are written with double angular brackets. The indices r_i are often designated by i.

The definition of the reduced class-sum is taken to imply that each class generated in the procedure described above is multiplied by the factor

$$\prod_{i=1}^{n}\left[i^{\lambda_i}\cdot\lambda_i!\binom{\ell_i+\lambda_i-\mu_i}{\lambda_i}\right]$$

where λ_i is the number of cycles of actual length i inserted, μ_i the number of cycles of that length eliminated, and ℓ_i the number of cycles of the same length that were present in the original class $*$.

Each reduced class sum is associated with a **reduced class coefficient** (RCC) that depends on the distribution of the two sets of p indices. It is denoted by

$$[(\overbrace{1,2,\cdots,p_1})(\overbrace{(p_1+1),\cdots}^{p_2})\cdots(\overbrace{\cdots,p}^{p_k});(\overbrace{\cdots}^{p'_1})(\overbrace{\cdots}^{p'_2})\cdots(\overbrace{\cdots}^{p'_\ell})]$$

Self-associated RCCs are written with double square brackets. The commas are sometimes omitted.

Using the definitions made above we formulate

THE GLOBAL CONJECTURE: *The product $[C']_n\cdot[*]_n$ is given by an expression of the form:*

$$[C']_n\cdot[*]_n=\sum_{\substack{k,\ell\\(k+\ell+p_C=\text{even})}}\quad\sum_{(p_1,p_2,\cdots,p_k)\vdash p}\quad\sum_{(p'_1,p'_2,\cdots,p'_\ell)\vdash p}\quad\sum_{\substack{\text{distinct}\\\text{distributions}}}$$

$$[(\overbrace{1,2,\cdots,p_1}^{p_1})(\overbrace{(p_1+1),\cdots}^{p_2})\cdots(\overbrace{\cdots,p}^{p_k});(\overbrace{\cdots}^{p'_1})(\overbrace{\cdots}^{p'_2})\cdots(\overbrace{\cdots}^{p'_\ell})]$$

$$\cdot \sum_{\substack{r_1,r_2,\cdots,r_p \\ (r_1+r_2+\cdots+r_p\leq n)}} < (\overbrace{r_1,\, r_2,\, \cdots,\, r_{p_1}}^{p_1})(\overbrace{r_{p_1+1},\, \cdots}^{p_2})\cdots(\overbrace{\cdots,\, r_p}^{p_k});(\overbrace{\cdots}^{p'_1})(\overbrace{\cdots}^{p'_2})\cdots(\overbrace{\cdots}^{p'_\ell})* >$$

The condition $k+\ell+p_C =$ even is a consequence of parity considerations.

3 The RCCs: The "central conjecture"

When C is a single-cycle conjugacy class of the form $(p)(1)^{n-p}$ several types of procedures are available for the evaluation of specific families of RCCs, *cf.* ref. [4]. A generally applicable procedure for the evaluation of the p-RCCs in terms of an enumeration problem within S_p was presented in ref. [5]. We now present the generalization of that procedure to the case of the C-RCCs, where C is a multi-cycle conjugacy class.

Let $a_{i,j}$ denote a subset of i indices out of the set $\{1,\, 2,\, \cdots,\, p\}$, i.e., $a_{i,j} \subset \{1,\, 2,\, \cdots,\, p\}$, $|a_{i,j}| = i$, such that $a_{i,j} \cap a_{i',j'} = \phi$ for $(i,j) \neq (i',j')$. We assume that $A = \{a_{i,j} \mid i = 1,\, 2,\, \cdots,\, p\,;\, j = 1,\, 2,\, \cdots,\, \ell_i(A)\}$ forms a set-partition of $\{1,\, 2,\, \cdots,\, p\}$. $\ell_i(A)$ is the number of subsets of length i in A. Let $B = \{b_{i,j} \mid i = 1,\, 2,\, \cdots,\, p\,;\, j = 1,\, 2,\, \cdots,\, \ell_i(B)\}$ be similarly defined. Obviously, $\sum_{i=1}^{p} i\ell_i(A) = \sum_{i=1}^{p} i\ell_i(B) = p$. The pair of set-partitions A and B specifies a C-RCC.

Let $R_{I,J} = |a_{i,j} \cap b_{i',j'}|$, where $I = \sum_{k=1}^{i-1} \ell_k(A) + j$ and $J = \sum_{k=1}^{i'-1} \ell_k(B) + j'$, be the corresponding intersection array. Symbolically, $|A \cap B| = R$. Since $a_{i,j}$ is the j'th cycle of index-length i in A, arranging the indices in A in consecutive order the indices in $a_{i,j}$ are $\{t_{i,j}+1,\ t_{i,j}+2,\ \cdots,\ t_{i,j}+i\}$, where $t_{i,j} = \sum_{k=1}^{i-1} k \cdot \ell_k(A) + (j-1)\ell_i(A)$. With this (or any other well-defined) ordering A can be interpreted as an element of S_p, i.e., a permutation. We shall denote the conjugacy class specified by the set of cycles comprising the partition B by $\overline{B}$. The C-RCC specified by the two partitions A and B, that correspond to the intersection array R, will be denoted $[A;B]_C$.

The C-RCCs were observed to satisfy

THE CENTRAL CONJECTURE:

$$[A;B]_C = \frac{|A|}{n!}\left|\left\{X \in \overline{B} \;\middle|\; A\cdot X \in \overline{C}\;;\; |A\cap X| = R\right\}\right|$$

where

$$|A| = \frac{n!}{\prod_{i=1}^{p}\left(i^{\ell_i(A)}\ell_i(A)!\right)}$$

is the number of elements in the conjugacy class A. The intersection array R is defined up to transposition of (non-identical) columns that correspond to cycles of equal lengths in B.

Thus, the C-RCC $[A;B]_C$ can be evaluated by determining the number of distinct members within the conjugacy class $\overline{B}$ whose product with the permutation A belongs to the conjugacy class $\overline{C}$ and whose intersection array with A is either R or an array that differs from R by permutations of columns that have different individual entries but a common entry sum. It is sometimes convenient to rewrite the central conjecture in the form

$$[A;B]_C = \frac{1}{n!}\left|\left\{X \in \overline{A},\; Y \in \overline{B} \;\middle|\; X\cdot Y \in \overline{C}\;;\; |X\cap Y| = R\right\}\right|.$$

In terms of this expression the RCC $[A;B]_C$ is manifestly symmetric with respect to transposition of the set-partitions A and B.

4 Evaluation of C-RCCs: An illustration

In the present section we illustrate the application of the "central conjecture" to the evaluation of the RCCs that correspond to the simplest multi-cycle class-sum, $[(2)^2]_n$. For this case the specification of the intersection array turns out to be redundent so we suppress it. Denoting a cycle of length i by $((i))$ we obtain

$\left\{X \in \overline{((4))}\middle|(1234)\cdot X \in \overline{((2))^2}\right\} = \left\{(1234)\right\}$, i.e., $[(1234);(1234)]_{[(2)^2]} = \frac{1}{4}$.

$\left\{X \in \overline{((2))((1))^2}\middle|(1234)\cdot X \in \overline{((2))^2}\right\} = \left\{(13)(2)(4)\;;\;(24)(1)(3)\right\}$,

i.e., $[(1234);(12)(3)(4)]_{[(2)^2]} = \frac{2}{4} = \frac{1}{2}$.

$\left\{X \in \overline{((3))((1))}\middle|(123)(4)\cdot X \in \overline{((2))^2}\right\} = \left\{(124)(3)\;;\;(234)(1)\;;\;(314)(2)\right\}$,

i.e., $[(123)(4);(1)(234)]_{[(2)^2]} = \frac{3}{3} = 1$.

$\left\{X \in \overline{((1))^4}\middle|(12)(34)\cdot X \in \overline{((2))^2}\right\} = \left\{(12)(34)\right\}$, i.e., $[(12)(34);(1)(2)(3)(4)]_{[(2)^2]} = \frac{1}{2^2\cdot 2!} = \frac{1}{8}$.

$\left\{X \in \overline{((2))^2}\middle|(12)(34)\cdot X \in \overline{((2))^2}\right\} = \left\{(13)(24)\ ;\ (14)(23)\right\}$,

i.e., $[(12)(34);(13)(24)]_{[(2)^2]} = \frac{2}{2^2\cdot 2!} = \frac{1}{4}$.

$\left\{X \in \overline{((2))((1))^2}\middle|(12)(3)(4)\cdot X \in \overline{((2))^2}\right\} = \left\{(1)(2)(34)\right\}$,

i.e., $[(12)(3)(4);(1)(2)(34)]_{[(2)^2]} = \frac{1}{2\cdot 1^2\cdot 2!} = \frac{1}{4}$.

5 Evaluation of disconnected RCCs

Let C be a multi-cycle conjugacy class with support-size p. We say that a C-RCC is disconnected if the two set-partitions A and B that specify it can be written in the form $A = A_1 \cup A_2 \cup \cdots \cup A_\ell$ and $B = B_1 \cup B_2 \cup \cdots \cup B_\ell$, such that, for all $i = 1, 2, \cdots, \ell$, A_i and B_i are set-partitions of the same subset of $\{1, 2, \cdots, p\}$. Clearly, it follows from the "central conjecture" that this is only possible if $C = C_1 \cup C_2 \cup \cdots \cup C_\ell$ such that C_i is a set-partition of the same set of indices comprising A_i (and B_i). Moreover, denoting by $|C_i|$ the support-size of C_i it is obvious that $|C_1|, |C_2|, \cdots, |C_\ell|$ is a partition of p of which $p_1, p_2, \cdots, p_k$ is a refinement. It follows from the "central conjecture" that such a C-RCC can be evaluated in terms of the ℓ RCCs $[A_i; B_i]_{C_i}$. If these ℓ RCCs consist of m subsets of equivalent RCCs with ℓ_i RCCs in each, than

$$[A; B]_C = \frac{1}{\prod_{j=1}^{m} \ell_j!} \prod_{i=1}^{\ell} [A_i; B_i]_{C_i}\,. \qquad (3)$$

Such RCCs will sometimes be written in the form $[A_1, A_2, \cdots, A_\ell; B_1, B_2, \cdots, B_\ell]$ with commas separating the disconnected parts.

6 Some class-sum products

We now present the expressions for the products of some class-sums with low support-size, obtained by application of the conjectures stated above. The evaluation of the RCCs using the "central conjecture" is straightforward, if sometimes tedious. Terms with vanishing RCCs are omitted. Recall that within the symbols for the reduced class sums i stands for r_i.

$$[(2)]_n \cdot [*]_n = \sum_{r_1,r_2} \frac{1}{2} < (12); (1)(2)*>$$

$$[(3)]_n \cdot [*]_n = \sum_{r_1,r_2,r_3} \left\{ \frac{1}{3} \ll (123); (123)*\gg \right.$$
$$\left. + \frac{1}{3} < (123); (1)(2)(3)*> + 1 \ll (12)(3); (1)(23)*\gg \right\}$$

$$[(2)^2]_n \cdot [*]_n = \sum_{r_1,r_2,r_3,r_4} \left\{ \frac{1}{4} \ll (1234); (1234)*\gg \right.$$
$$+ \frac{1}{2} < (1234); (12)(3)(4)*> + 1 \ll (123)(4); (1)(234)*\gg$$
$$+ \frac{1}{8} < (12), (34); (1)(2), (3)(4)*> + \frac{1}{4} \ll (12)(34); (13)(24)*\gg$$
$$\left. + \frac{1}{4} \ll (12), (3)(4); (1)(2), (34)*\gg \right\}$$

$$[(2)(3)]_n \cdot [*]_n = \sum_{r_1,r_2,r_3,r_4,r_5} \left\{ 2 < (12345); (1234)(5)*> \right.$$
$$+1 < (12345); (12)(3)(4)(5)*> +1 < (12345); (123)(45)*>$$
$$+2 < (1234)(5); (1)(2)(345)*> +1 < (1234)(5); (1)(23)(45)*>$$
$$+ \frac{1}{6} < (123), (45); (123), (4)(5)*> +1 < (123)(45); (124)(3)(5)*>$$
$$+ \frac{1}{6} < (123), (45); (1)(2)(3), (4)(5)*> +1 < (123)(45); (14)(25)(3)*>$$
$$\left. + \frac{1}{6} < (123), (4)(5); (1)(2)(3), (45)*> + \frac{1}{2} < (12), (34)(5); (1)(2), (3)(45)*> \right\}$$

The disconnected RCCs in the expressions involving $[(2)^2]_n$ and $[(2)(3)]_n$ satisfy eq. 3.

7 Class-algebra structure constants: An illustration

We now illustrate the application of the expressions presented above to the evaluation of class-sum products.

The representation theoretic expression for the structure constant $(A \cdot B)\big|_C$, eq. 1, suggests that the renormalized structure constant $(A,B,C) \equiv |C|(A\cdot B)\big|_C$, which yields the number of S_n group elements of type C generated by multiplication of the conjugacy classes A and B, is symmetric with respect to the 3! permutations of the class-sums A, B and C.

Consider the three class-sums $A = [(2)^2]_n$, $B = [(2)(3)]_n$ and $C = [(3)(4)]_n$. To evaluate the structure constant $[(2)^2]_n \cdot [(2)(3)]_n\big|_{[(3)(4)]_n}$ we use the expression $[(2)^2]\cdot[*]_n$ and note that there are two contributions to the desired structure constant. One of them, $<(12)(34);(1)(2)(3)(4)*>$, involves the elimination of a two-cycle and a three-cycle along with two one-cycles, and the generation of a three-cycle and a four-cycle, by setting $r_1 = 3$, $r_3 = 2$ and $r_2 = r_4 = 1$ or any one of seven additional permutations of these indices, or otherwise eliminating the two-cycle along with two one-cycles via the reduced class-sum $<(1234);(12)(3)(4)*>$ in which we choose $r_1 = r_2 = r_3 = r_4 = 1$. The two contributions to the structure constant of interest are $\frac{1}{8}\cdot 8\cdot 3\cdot 4$ and $\frac{1}{2}\cdot 4$, respectively. Thus, $[(2)^2]_n \cdot [(2)(3)]_n\big|_{[(3)(4)]_n} = 14$ and

$$\big|[(3)(4)]_n\big| \cdot \Big([(2)^2]_n \cdot [(2)(3)]_n\Big)\Big|_{[(3)(4)]_n} = \frac{7}{6}\frac{n!}{(n-7)!}\,. \tag{4}$$

The same structure constant can be evaluated in the form $\Big([(2)(3)]_n \cdot [(2)^2]_n\Big)\Big|_{[(3)(4)]_n}$, using the expression $[(2)(3)]_n\cdot[*]_n$. Equivalent results can be obtained by evaluating any one of the structure constants $\Big([(2)^2]_n\cdot[(3)(4)]_n\Big)\Big|_{[(2)(3)]_n}$, $\Big([(2)(3)]_n\cdot[(2)^2]_n\Big)\Big|_{[(3)(4)]_n}$, or $\Big([(2)(3)]_n\cdot [(3)(4)]_n\Big)\Big|_{[(2)^2]_n}$.

8 Conclusions

A "global conjecture" that allows the presentation of the product of the class-sum $[C]_n$ with an arbitrary class-sum in S_n in terms of a linear combination of reduced class-sums, is formulated. A combinatorial interpretation of the coefficients in this linear combination, the C-RCCs, is provided by means of the "central conjecture". These two conjectures constitute a generalization of the treatment of products involving class-sums of the form $[C]_n = [(p)]_n$, *cf.* refs. [4, 5].

The observation made in Section 7, according to which the renormalized structure constant (A, B, C) is symmetric with respect to permutations of the classes A, B and C, implies that this structure constant can be evaluated by applying the conjectures presently discussed to any one of these three classes. In particular, one can choose to apply the conjectures to the class of smallest support-size. This choice would give rise to the smallest possible number of reduced class-sums; Furthermore, the required RCCs can be evaluated, using the central conjecture, within S_p, where p is the corresponding support size.

The actual evaluation of higher order RCCs using the presently proposed procedure still poses a rather severe enumeration problem. The elimination rules and related procedures developed in previous articles for the special case in which C is a single cycle class-sum can hopefully be extended to the general case presently discussed. Finally, we emphasize that the challenge of proving the two conjectures stated above is wide open.

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