# Shuffle posets of multisets

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**Abstract**

We study posets defined by Stanley as a multiset generalization of Greene's shuffle posets. We compute the flag $f$-vector encoded as a quasi-symmetric function $F_P$ which was defined by Ehrenborg in [2]. The result is a sum of complete symmetric functions with integer coefficients which is also expressible as a sum of Schur functions with nonnegative integer coefficients. We deduce structural properties of shuffle posets of multisets based on the computation of $F_P$, in addition to giving a symmetric group action on maximal chains with Frobenius characteristic related to $F_P$.

**Résumé**

Ce travail étudie les ensembles partiellement ordonnés, définis par Stanley comme étant une généralisation multi-ensembliste du mélange ("shuffle") d'ensembles partiellement ordonnés de Greene. On va donner une description explicite du $f$-vecteur drapeau encodé comme une fonction quasi-symétrique $F_P$, qui a été définie par Ehrenborg dans [2]. Le résultat est une somme de fonctions symétriques complètes avec coefficients entiers, que l'on peut aussi exprimer comme somme de fonctions de Schur avec coefficients entiers positifs. On en déduit des propriétés structurales des mélanges d'ensembles partiellement ordonnés de multi-ensembles, basées sur le calcul de $F_P$, en plus de donner une action du groupe symétrique sur les chaînes maximales dont la caractéristique de Frobenius est reliée à $F_P$.

## 1 Introduction

We introduce a class of finite, ranked posets called *shuffle posets of multisets*, suggested by Stanley, and we compute the flag $f$-vector and consequently determine the rank generating

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function, zeta polynomial and characteristic polynomial. We show that shuffle posets of multisets are supersolvable which implies they are EL-shellable and Cohen-Macaulay, and we determine homology of the order complex. First we review terminology and properties of quasi-symmetric functions and the flag $f$-vector.

The *flag $f$-vector* of a poset is the vector indexing chains through particular collections of ranks; for each collection $S = \{r_1, \ldots, r_k\}$, the corresponding coefficient of the flag $f$-vector, denoted $\alpha_P(S)$, counts chains passing through ranks $r_1, \ldots, r_k$. The *flag $h$-vector* is the vector related by the formula $\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T)$ where $\beta_P(S)$ is the coefficient indexed by $S$ in the flag $h$-vector. This is often a more convenient way of expressing the same data.

In [2], Ehrenborg introduced another encoding, denoted $F_P$, for the flag $f$-vector of a poset in terms of the Hopf algebra of quasi-symmetric functions which was defined by Gessel in [3]. A power series $q(x)$ is *quasi-symmetric* if the coefficient of $x_{a_1}^{k_1} \ldots x_{a_n}^{k_n}$ in $q(x)$ equals the coefficient of $x_{b_1}^{k_1} \ldots x_{b_n}^{k_n}$ for any $a_1 < \cdots < a_n$ and any $b_1 < \cdots < b_n$, together with any choice of $k_1, \ldots, k_n \in \mathbb{N}$. Summing over multichains, $F_P$ is defined for nontrivial, ranked posets as

$$F_P = \sum_{\hat{0} = t_0 \le t_1 \le \cdots \le t_{k-1} < t_k = \hat{1}} x_1^{\rho(t_0,t_1)} x_2^{\rho(t_1,t_2)} \ldots x_k^{\rho(t_{k-1},t_k)},$$

where $\rho(x, y)$ is the difference in the ranks of $x$ and $y$. Clearly, $F_P$ is quasi-symmetric. In [3], Gessel defined a quasi-symmetric analogue of monomial symmetric functions by

$$M_{(r_1, r_2 - r_1, \ldots, n - r_k)} = \sum_{i_1 < \cdots < i_{k+1}} x_{i_1}^{r_1} x_{i_2}^{r_2 - r_1} \ldots x_{i_{k+1}}^{n - r_k},$$

and these form a basis for the quasi-symmetric functions. $F_P$ encodes the flag $f$-vector in a natural way, namely, $F_P = \sum_{S \subseteq \{1, \ldots, n-1\}} \alpha_P(S) M_{(r_1, r_2 - r_1, \ldots, r_k - r_{k-1}, n - r_k)}$ for $S = \{r_1 < r_2 < \cdots < r_k\}$ and $P$ a finite poset of rank $n$.

When $F_P$ is a symmetric function, then the poset $P$ is called *flag-symmetric*. In this case, the coefficient of $M_{(s_1, \ldots, s_k)}$ in the expression for $F_P$ does not depend on the order of the elements $s_1, \ldots, s_k$ where $(s_1, \ldots, s_k)$ is a composition of $n$; monomial quasi-symmetric functions may be grouped into monomial symmetric functions, yielding $F_P = \sum_{\lambda \vdash n} \alpha_P(\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \cdots + \lambda_k) m_\lambda$ for a finite poset of rank $n$, summing over partitions rather than compositions of $n$. To compute $F_P$ for shuffle posets of multisets, we will rely on two nice properties of $F_P$, namely that $F_P = h_n$ for $P$ a chain of rank $n$ and the formula

$$F_{P \times Q} = F_P F_Q$$

observed in [11] for cartesian products of posets.

Gessel originally provided another basis for the quasi-symmetric functions, with elements which we denote by $L_{S,n}$, defined as

$$L_{S,n}(x) = \sum_{\substack{a_1 \le \cdots \le a_n \\ a_i < a_{i+1} \text{ if } i \in S}} x_{a_1} \ldots x_{a_n}.$$

This basis is related to the monomial basis of quasi-symmetric functions by $L_{S,n} = \sum_{T \subseteq S} M_T$, and it follows that $F_P = \sum \beta_P(S) L_{S,n}(x)$ for $P$ of rank $n$, as is shown in [11]. When a poset $P$ is Cohen-Macaulay, then the coefficients $\beta_P(S)$ are nonnegative, since $\beta_P(S)$ counts the dimension of the homology group of the order complex of $P$ restricted to ranks belonging to $S$, as discussed in [9]. It follows from results of Gessel in [3] that the coefficients $\beta_P(S)$ are also nonnegative whenever if $F_P$ is Schur-positive.

We show for shuffle posets of multisets that $F_P$ can be written as a sum of Schur functions with nonnegative integer coefficients. This raises the question of whether $F_P$ or $\omega F_P$ is the Frobenius characteristic of a symmetric group action on maximal chains, where $\omega$ is the involution swapping complete and elementary symmetric functions. Simion and Stanley recently computed $F_P$ for traditional shuffle posets in [7] by finding a particularly nice symmetric group action on maximal chains, computing its Frobenius characteristic $\phi_P$, and proving $F_P = \omega\phi_P$ by virtue of the symmetric group action and a chain labelling from which the action is induced. For all other shuffle posets of multisets, however, there is no symmetric group action on maximal chains with $F_P$ or $\omega F_P$ as Frobenius characteristic; $F_P$ is instead computed by finding a chain decomposition based on the topology of the order complex. This decomposition leads to a symmetric group action on maximal chains which specializes to the action of [7] and has Frobenius characteristic related to $F_P$ in general.

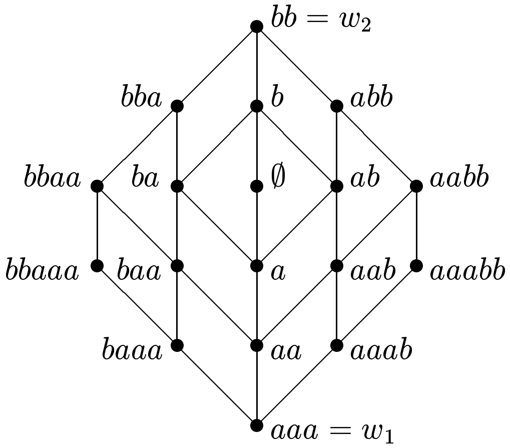


Figure 1: The shuffle poset of multisets $W_{3,2}$

For fixed compositions $\alpha$ and $\beta$, the shuffle poset of multisets, denoted by $W_{\alpha,\beta}$, is obtained by fixing a word $w_1$ of type $\alpha$ and a word $w_2$ of type $\beta$, chosen from disjoint alphabets, and putting a partial order on the ways of shuffling any subword of $w_1$ with any subword of $w_2$, subject to the constraint that letters of $w_1$ or $w_2$ occuring with multiplicity in a shuffled word must occur consecutively. For example, if $w_1 = aaab$ and $w_2 = c$ for $\alpha = (3,1)$ and $\beta = (1)$, then $aacb$ is a valid poset element while $acab, ba$ and $aaaa$ are not. The minimal element of $W_{\alpha,\beta}$ is $w_1$, the maximal element is $w_2$, and $W_{\alpha,\beta}$ has covering relation $u \leq v$ whenever $v$ is obtained from $u$ by either deleting a letter of $w_1$ or legally inserting a letter of $w_2$. Figure 1 illustrates the poset $W_{3,2}$ for $w_1 = aaa$ and $w_2 = bb$. The traditional shuffle posets are usually denoted $W_{m,n}$, but unfortunately they become $W_{1^m,1^n}$ in this notation which is convenient for shuffle posets of multisets.

We note that a different generalization of the shuffle posets based on a shuffling operation for lattices has been examined by Doran in [1].

## 2 Computing the flag-symmetric function $F_P$

We compute $F_P$ in Theorem 1, deriving an expression in terms of complete symmetric functions which will make it easy to show in Corollary 4 that $F_P$ is Schur-positive, although it is not in general $h$-positive or $e$-positive. We indicate throughout how this and other formulas specialize to results of [4] and [7] about traditional shuffle posets.

**Theorem 1.** *The poset $P = W_{\alpha,\beta}$ is flag-symmetric, satisfying*

$$F_P = \sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1\le a_1<\cdots<a_j\le l(\alpha)\\ 1\le b_1<\cdots<b_j\le l(\beta)}} \left(\prod_{i=1}^{j}(h_{\alpha_{a_i}}h_{\beta_{b_i}} - h_{\alpha_{a_i}+\beta_{b_i}})\right)\left(\prod_{\substack{i\notin\{a_1,\ldots,a_j\}\\ 1\le i\le l(\alpha)}} h_{\alpha_i}\right)\left(\prod_{\substack{i\notin\{b_1,\ldots,b_j\}\\ 1\le i\le l(\beta)}} h_{\beta_i}\right) \tag{1}$$

This yields

$$F_P = \sum_{j=0}^{\min(m,n)} \binom{m}{j}\binom{n}{j} e_2^j e_1^{n+m-2j}$$

for shuffle posets of the form $W_{1^m,1^n}$, i.e. traditional shuffle posets, since $h_1h_1 - h_2 = e_2$, $h_1 = e_1$, and $\alpha_i = \beta_i = 1$ for all $i$, so (1) becomes a sum of elementary symmetric functions, agreeing with a formula of [7].

Before proving Theorem 1, we give an example which suggests a recurrence; we verify this recurrence and then show how Theorem 1 follows from it.

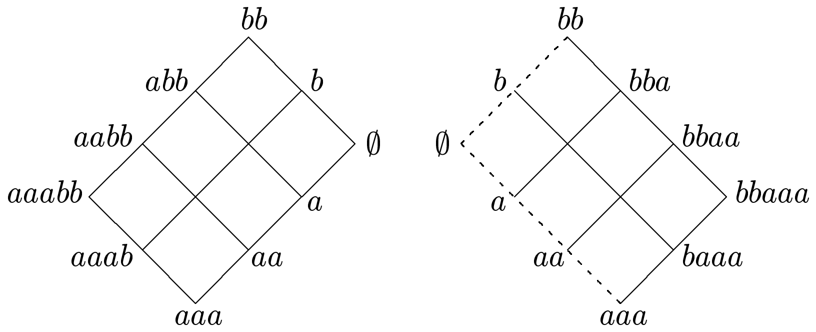


Figure 2: A chain decomposition for $W_{3,2}$

Let $w_1 = aaa$ and $w_2 = bb$ as in Figure 1. We decompose the space of maximal chains into two pieces and account for the contribution of each separately to $F_P$. Chains in which each element is a subword of $w_1 = aaa$ followed by a subword of $w_2 = bb$ comprise the product of a chain of 3 elements with a chain of 4 elements, shown on the left in Figure 2; this contributes $h_3h_2$ to $F_P$ because a chain $C_{r+1}$ of rank $r$ satisfies $F_P = h_r$, and applying $F_{P\times Q} = F_P F_Q$ yields $h_3h_2$ for $C_4 \times C_3$.

When we consider the remaining chains, namely those with the letter $b$ occuring immediately before the letter $a$ in some element of the chain, we obtain another copy of $h_3h_2$ for the product of chains from subwords of $bb$ followed by subwords of $aaa$, shown on the right in figure 2, but then we must subtract the contribution of chains in which $a$ and $b$ never appear together, as these were included twice, so we subtract $h_5$ to account for chains within the maximal chain $aaa < aa < a < \emptyset < b < bb$. Consequently, $F_P = h_3h_2 + (h_3h_2 - h_{3+2})$ for $P = W_{3,2}$.

Recall from [4] that a chain is $w_1$*-terminal* ($w_2$*-terminal*) if each chain element involving the last letter of $w_1$ ($w_2$) has this letter occuring last, i.e. after all letters of $w_2$ ($w_1$) appearing in the shuffled word.

**Lemma 2.** *The shuffle posets of multisets satisfy the recurrence*

$$F_{\alpha,\beta} = F_{\alpha-\alpha_k,\beta}h_{\alpha_k} + F_{\alpha,\beta-\beta_l}h_{\beta_l} - F_{\alpha-\alpha_k,\beta-\beta_l}h_{\alpha_k+\beta_l} \tag{2}$$

*for* $\alpha = (\alpha_1, \ldots, \alpha_k)$ *and* $\beta = (\beta_1, \ldots, \beta_l)$.

PROOF. Decompose the space of chains into those which are $w_1$-terminal and those which are $w_2$-terminal, subtracting for overlap from chains which never involve the last letters of $w_1$ and $w_2$ simultaneously and which are both $w_1$-terminal and $w_2$-terminal. Note that restricting to chains in which a letter always appears last yields the product of a chain of rank equalling the multiplicity with which that letter appears and a shuffle poset not involving that letter. We apply $F_{P\times Q} = F_P F_Q$ to obtain each of the three terms on the right side of (2), recalling that $F_P = h_r$ when $P$ is a chain of rank $r$. □

**Proof of theorem 1.** This follows from (2) using induction, by substituting (1) into the right side of (2). Note that $F_{\alpha-\alpha_k,\beta}h_{\alpha_k}$ yields the terms in (1) for $F_{\alpha,\beta}$ with $k \notin \{a_1, \ldots, a_j\}$, while $F_{\alpha,\beta-\beta_l}h_{\beta_l}$ accounts for those such that $l \notin \{b_1, \ldots, b_j\}$, so chains satisfying both conditions are included twice. Subtracting $F_{\alpha-\alpha_k,\beta-\beta_l}h_{\alpha_k+\beta_l}$ from the extra copy of $F_{\alpha-\alpha_k,\beta-\beta_l}h_{\alpha_k}h_{\beta_l}$ yields the terms of (1) satisfying $k \in \{a_1, \ldots, a_j\}$ and $l \in \{b_1, \ldots, b_j\}$. □

The recurrence given by (2) also yields the following expression.

**Theorem 3.** *Summing over compositions* $\alpha$ *and* $\beta$ *indexed by monomials in noncommutative variables* $u_i$ *and* $v_j$ *with relations* $u_iv_j = v_ju_i$ *for* $i, j > 0$ *yields*

$$\sum_{\alpha,\beta} F_{\alpha,\beta} u_{\alpha_1} \ldots u_{\alpha_k} v_{\beta_1} \ldots v_{\beta_l} = \frac{1}{1 - \sum_{i>0} u_i h_i - \sum_{j>0} v_j h_j + \sum_{i,j>0} u_i v_j h_{i+j}}.$$

Non-commutative variables are used because $F_P$ depends on the order of the parts of the compositions $\alpha$ and $\beta$. This agrees with a result of [7] when we let $u_1 = u$, $v_1 = v$ and set $u_i = v_i = 0$ for $i > 1$.

**Corollary 4.** *The flag-symmetric function for the shuffle poset of multisets is a Schur-positive symmetric function.*

PROOF. By applying the Jacobi-Trudi identity, we obtain

$$h_l h_k - h_{l+k} = s_{(l+k-1,k)/(k-1)} \tag{3}$$

for $l \geq k$. Alternatively, $s_{l+i,k-i} = h_{l+i}h_{k-i} - h_{l+i+1}h_{k-i-1}$ for each $0 \leq i < k$, implying

$$h_l h_k - h_{l+k} = s_{l,k} + s_{l+1,k-1} + \cdots + s_{l+k-1,1},$$

again for $l \geq k$. Substituting (3) and $h_n = s_n$ into (1) yields

$$\sum_{j=0}^{\min l(\alpha),l(\beta)} \sum_{\substack{1\leq a_1<\cdots<a_j\leq l(\alpha)\\ 1\leq b_1<\cdots<b_j\leq l(\beta)}} \left(\prod_{i=1}^{j} s_{(\alpha_{a_i}+\beta_{b_i}-1,\beta_{b_i})/(\beta_{b_i}-1)}\right)\left(\prod_{i\notin\{a_1,\ldots,a_j\}} s_{\alpha_i}\right)\left(\prod_{i\notin\{b_1,\ldots,b_j\}} s_{\beta_i}\right)$$

as an expression for $F_P$. This is Schur-positive because the Littlewood-Richardson coefficients are nonnegative. □

Formula (1) was determined by decomposing the space of chains in a way that will prove useful later. We associate to each chain a collection of ordered pairs of letters from the words $w_1$ and $w_2$, respectively, which are being shuffled, and we refer to the set of such pairs associated to a particular chain as the *interface* of the chain; this generalizes Greene's notion of the interface of a poset element in [4]. Partitioning chains according to their interface gives the chain decomposition we will use.

Recall that the interface of a traditional shuffle poset element, i.e. of a subword of $w_1$ shuffled with a subword of $w_2$, is the collection of pairs of letters $(u, v)$ such that $u$ belongs to $w_2$, $v$ belongs to $w_1$, and $u$ immediately precedes $v$ in the shuffled word, so the interface determines the degree to which the subwords are shuffled. Although letters may occur with multiplicity in shuffle posets of multisets, we refer to pairs of letters as belonging to the interface when we actually mean pairs of classes of identical letters, since identical letters always occur consecutively. Greene introduced the notion of interface to construct a symmetric chain decomposition for posets of shuffles in [4], and the same argument works for more general shuffle posets of multisets.

**Proposition 5.** *Shuffle posets of multisets have symmetric chain decompositions.*

PROOF. Partitioning poset elements according to interface gives a decomposition into symmetrically placed products of chains, each of which has a symmetric chain decomposition. In the case of traditional shuffle posets, these products of chains are boolean. □

We define the interface of a chain to also reflect the degree of shuffling which occurs in a chain. Each possible way of shuffling $w_1$ with $w_2$ yields a subposet consisting of all subwords of this shuffled word, and each such subposet is a product of chains. Every chain in $W_{\alpha,\beta}$ is contained in one or more of these products of chains. We partially order these products of chains by imposing a partial order on the shuffled words specifying them. We use the interval in the weak order from the permutation $n+1,\ldots,n+m,1,\ldots,n$ to the longest element, namely $n+m,\ldots,1$, or equivalently, the order $L(m+n,n)$. Partitioning chains according to interface amounts to assigning each chain to the product of chains containing it which comes earliest in this partial order, i.e. the one specified by the least shuffled word. The interface of a chain is obtained by taking the union of the interfaces of all chain elements, then eliminating those ordered pairs which are preempted by other "more shuffled" pairs arising elsewhere in the chain. One pair preempts another if it consists of the same letter or a later letter of $w_1$ and the same letter or an earlier letter of $w_2$. Following [4], letters not occuring in the interface of a chain comprise the *residue* of the chain.

Each summand in Formula 1 accounts for chains with a particular interface, specified by the sets $\{a_1,\ldots,a_j\}$ and $\{b_1,\ldots,b_j\}$ which index positions in $w_1$ and $w_2$, respectively. All chains with a particular interface are contained in the product of chains of subwords of the particular way of shuffling $w_1$ and $w_2$ with this interface, so we can restrict attention to chains in this product of chains. The order in which letters are inserted and deleted in travelling from $\hat{0}$ to $\hat{1}$ does not affect the interface of a chain except when two letters comprise an interface pair, so the contribution to $F_P$ by all chains with particular interface is a product over residue letters multiplied by a product over interface pairs where we repeatedly apply $F_{P\times Q}=F_PF_Q$. A residue letter with multiplicity $n$ contributes $h_n$ to this product while an interface pair with multiplicities $m$ and $n$ contributes $h_nh_m-h_{n+m}$ by the reasoning in the example given.

The assignment of interfaces to chains also generalizes the chain labelling for shuffle posets in [7] which leads to a symmetric group action on maximal chains. The decomposition according to interface will likewise lead to a symmetric group action for shuffle posets of multisets.

# 3 A symmetric group action on maximal chains

One might hope to find a symmetric group action on maximal chains with Frobenius characteristic $F_P$, since $F_P$ is a sum of Schur functions with nonnegative integer coefficients, and because of the following observation of Stanley in [11]. The coefficient of $m_{1^n}$ in the monomial basis expansion of a symmetric function is the dimension of the representation with this symmetric function as Frobenius characteristic, while the coefficient of $m_{1^n}$ in $F_P$ for $P$ flag-symmetric of rank $n$ is the number of maximal chains in $P$, so $F_P$ is the Frobenius image of a representation of the proper dimension to come from an action on maximal chains.

However, each complete symmetric function is the Frobenius characteristic of a transitive action, thereby containing exactly one copy of the trivial representation, so by taking differences of complete symmetric functions in (1), we eliminate all but one copy of the Schur function which gives rise to the trivial representation from $F_P$, so $F_P$ cannot be the Frobenius characteristic of a symmetric group action unless it has only one orbit. Since the number of maximal chains does not in general divide the order of the symmetric group, a transitive action on maximal chains is impossible. When $\alpha=1^m$ and $\beta=1^n$ for $m,n\in\mathbb{N}$, there is a symmetric group action on maximal chains given in [7] with Frobenius characteristic $\omega F_P$, but the multiplicity of the trivial representation in $\omega F_P$ is 0 for all other choices of compositions $\alpha$ and $\beta$, so there cannot in general be a symmetric group action with Frobenius characteristic $\omega F_P$.

There is a symmetric group action on maximal chains defined in terms of a labelling of chains (which is not a chain-labelling in the sense of CL-shellability) based on the interface of a chain. This action has Frobenius characteristic

$$\phi_P = \sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1\le a_1<\cdots<a_j\le l(\alpha)\\ 1\le b_1<\cdots<b_j\le l(\beta)}} \left(\prod_{i=1}^{j}\left(\binom{\alpha_{a_i}+\beta_{b_i}}{\alpha_{a_i}}-1\right)h_{\alpha_{a_i}+\beta_{b_i}}\right)\left(\prod_{\substack{i\notin\{a_1,\ldots,a_j\}\\ 1\le i\le l(\alpha)}} h_{\alpha_i}\right)\left(\prod_{\substack{i\notin\{b_1,\ldots,b_j\}\\ 1\le i\le l(\beta)}} h_{\beta_i}\right)$$

which specializes to

$$\phi_P = \sum_{j=0}^{\min(m,n)} \binom{m}{j}\binom{n}{j} h_2^j h_1^{m+n-2j} = \omega F_P$$

when $\alpha = 1^m$ and $\beta = 1^n$ for $m, n \in \mathbb{N}$, and then the action is the one given in [7]. For any fixed chain, label each covering relation edge with the letter of $w_2$ being inserted or the letter of $w_1$ being deleted, except when the letter belongs to a pair in the interface of the chain, defined at the end of section 2; in this case, the edge is labelled with the letter in the interface pair which belongs to $w_2$. This gives the chain labelling of [7], but when multiplicity is introduced in the words being shuffled, then the interface of a chain may depend on the chain both above and below particular covering relations.

The action is induced from this labelling $\lambda$ by letting the adjacent transposition $s_i = (i, i+1)$ send the maximal chain $x$ to the unique maximal chain $s_i(x)$ which agrees with $x$ everywhere except possibly at rank $i$ and which satisfies $\lambda_{i+1}(s_i(x)) = \lambda_i(x)$ and $\lambda_i(s_i(x)) = \lambda_{i+1}(x)$. This action is what is defined in [11] to be a *local action* since adjacent transpositions only affect maximal chains locally. Each orbit consists of the maximal chains in a product of chains within the poset where the chains forming the product come from pairs of interface letters and from residue letters. The ranks of these chains are determined by the multiplicity with which the residue letters occur and by the sum of multiplicities for pairs of interface letters. Interface pairs give rise to single chains since the order in which the two types of letters are inserted and deleted (in travelling from $\hat{0}$ to $\hat{1}$ in the chain) is fixed for a particular orbit.

# 4 Structural properties of $W_{\alpha,\beta}$

In this section, we prove shellability, contractibility of the order complex when $(\alpha, \beta) \neq (1^m, 1^n)$, and we find combinatorial formulas describing shuffle posets of multisets. By convention, the *order complex* of $P$ is the simplicial complex of poset chains not involving $\hat{0}$ or $\hat{1}$.

**Proposition 5.** *$W_{\alpha,\beta}$ is a lattice.*

PROOF. Use Greene's argument to determine which letters occur in which order in $u \wedge v$ and $u \vee v$; the multiplicity with which letters occur if they occur at all is the minimum (resp. maximum) of their multiplicity in $u$ and in $v$. □

**Proposition 5.** *If $\alpha = 1^m$ and $\beta = 1^n$, then $\mu_{W_{\alpha,\beta}}(\hat{0},\hat{1}) = \sum_{j=0}^{\min(m,n)} (-1)^{m+n}\binom{m}{j}\binom{n}{j}$. Otherwise, $\mu_{W_{\alpha,\beta}}(\hat{0},\hat{1}) = 0$.*

PROOF. The former case is proven in [4] and [7]. Ehrenborg expressed $\mu_P(\hat{0},\hat{1})$ as a linear function of $F_P$ in [2]; let $g$ be this function. When $F_P$ is a symmetric function, $\mu_P(\hat{0},\hat{1})$ equals $(-1)^{m+n}$ multiplied by the coefficient of $h_{1^{n+m}}$ in the complete symmetric function expansion of $F_P$: note that $g((h_1)^{n+m}) = (-1)^{m+n}$ and $g(h_\alpha) = 0$ for all other $\alpha$ by evaluating $\mu_P(\hat{0},\hat{1})$ for $P$ a product of chains, and then express $F_P$ as a sum of complete symmetric functions and apply linearity of $g$. □

Furthermore, note that every interval is a product of shuffle posets of multisets, and the Möbius function is multiplicative, so $\mu(u,v)$ may be computed using Proposition 5 for arbitrary $u$ and $v$. A canonical way of associating products of shuffle posets to intervals is discussed in [7] and applies similarly to shuffle posets of multisets.

**Proposition 6.** *Shuffle posets of multisets are shellable.*

PROOF. Greene's EL-labelling for traditional shuffle posets given in [4] generalizes naturally. Label poset edges with the letter of $w_1$ being deleted or the letter of $w_2$ being inserted. Choose any total order for the alphabets of $w_1$ and $w_2$ such that all letters of $w_1$ are smaller than all letters of $w_2$. □

When $\mu(\hat{0},\hat{1}) = 0$, shellability implies the collapsibility of the order complex.

**Corollary 7.** *The order complex of $W_{\alpha,\beta}$ is contractible unless $\alpha = 1^m$ and $\beta = 1^n$ for $m,n \in \mathbb{N}$.*

Shuffle posets of multisets consist of overlapping products of chains, each given by all subwords of a particular way of shuffling $w_1$ and $w_2$. When $\alpha = 1^m$ and $\beta = 1^n$ for $m,n \in \mathbb{N}$ then these are boolean and give a homology basis for the order complex.

**Proposition 8.** *Shuffle posets of multisets are supersolvable.*

PROOF. We claim that any maximal chain containing the empty word is an $m$-chain. Each word $w$ that comes from shuffling $w_1$ with $w_2$ gives rise to a product of chains within $W_{\alpha,\beta}$ consisting of all subwords of $w$. Every maximal chain belongs to at least one such product of chains, and maximal chains including the empty word belong to all such products of chains. Hence, given any element of a chain involving the empty word and an arbitrary poset element, there is some product of chains which includes both elements; their meet and join in the entire poset are exactly their meet and join in this product of chains which is distributive, so the claim follows. $\square$

We conclude with a collection of formulas for shuffle posets of multisets which provide an analogue to Theorem 3.4 of Greene in [4]. We adhere to the notation used in [4], so $\Omega_{\alpha,\beta}$ is the number of poset elements, $\Omega_{\alpha,\beta}(q)$ is the rank generating function, $C_{\alpha,\beta}$ is the number of maximal chains, $Z_{\alpha,\beta}(s)$ is the zeta polynomial, counting multichains $\hat{0} \leq x_1 \leq \cdots \leq x_s = \hat{1}$ in $W_{\alpha,\beta}$ and $\chi_{\alpha,\beta}(t) = \sum_{u \in P} \mu(\hat{0}, u) t^{n-\rho(u)}$ for $P$ of rank $n$ and $\rho(u)$ equalling the rank of $u$. In the expression for $\Omega_{\alpha,\beta}(q)$, let $[n]_q = \frac{1-q^n}{1-q}$.

**Theorem 9.** *The following formulas hold for the shuffle poset of multisets $W_{\alpha,\beta}$ with $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_l)$, $m = \sum_{i=1}^k \alpha_i$ and $n = \sum_{i=1}^l \beta_i$.*

$$\begin{aligned}
\Omega_{\alpha,\beta} &= \sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1 \leq a_1 < \cdots < a_j \leq l(\alpha) \\ 1 \leq b_1 < \cdots < b_j \leq l(\beta)}} \left( \prod_{i=1}^{j} \alpha_{a_i} \beta_{b_i} \right) \left( \prod_{i \notin \{a_1,\ldots,a_j\}} (\alpha_i + 1) \right) \left( \prod_{i \notin \{b_1,\ldots,b_j\}} (\beta_i + 1) \right) \\
\Omega_{\alpha,\beta}(q) &= \sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1 \leq a_1 < \cdots < a_j \leq l(\alpha) \\ 1 \leq b_1 < \cdots < b_j \leq l(\beta)}} \left( \prod_{i=1}^{j} ([\alpha_{a_i} + 1]_q [\beta_{b_i} + 1]_q - [\alpha_{a_i} + \beta_{b_i} + 1]_q) \right) \\
&\qquad \left( \prod_{i \notin \{a_1,\ldots,a_j\}} [\alpha_i + 1]_q \right) \left( \prod_{i \notin \{b_1,\ldots,b_j\}} [\beta_i + 1]_q \right) \\
C_{\alpha,\beta} &= (m+n)! \sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1 \leq a_1 < \cdots < a_j \leq l(\alpha) \\ 1 \leq b_1 < \cdots < b_j \leq l(\beta)}} \left( \prod_{i=1}^{j} \left( \frac{1}{(\alpha_{a_i})!(\beta_{b_i})!} - \frac{1}{(\alpha_{a_i} + \beta_{b_i})!} \right) \right) \\
&\qquad \left( \prod_{i \notin \{a_1,\ldots,a_j\}} \left( \frac{1}{\alpha_i!} \right) \right) \left( \prod_{i \notin \{b_1,\ldots,b_j\}} \left( \frac{1}{\beta_i!} \right) \right) \\
Z_{\alpha,\beta}(s) &= \sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1 \leq a_1 < \cdots < a_j \leq l(\alpha) \\ 1 \leq b_1 < \cdots < b_j \leq l(\beta)}} \left( \prod_{i=1}^{j} \binom{\alpha_{a_i} + s - 1}{s-1} \binom{\beta_{b_i} + s - 1}{s-1} - \binom{\alpha_{a_i} + \beta_{b_i} + s - 1}{s-1} \right) \\
&\qquad \left( \prod_{i \notin \{a_1,\ldots,a_j\}} \binom{\alpha_i + s - 1}{s-1} \right) \left( \prod_{i \notin \{b_1,\ldots,b_j\}} \binom{\beta_i + s - 1}{s-1} \right) \\
\chi_{\alpha,\beta}(t) &= t^{m+n-l(\alpha)-l(\beta)} (t-1)^{l(\alpha)+l(\beta)} \sum_{j=0}^{\min(l(\alpha),l(\beta))} \binom{l(\alpha)}{j} \binom{l(\beta)}{j} \left( \frac{1}{1-t} \right)^j
\end{aligned}$$

PROOF. The formulas follow by substituting $d+1, [d+1]_q, 1/d!, \binom{d+s-1}{s-1}$ and $(1 - \frac{1}{t})$ for $h_d$ in formula (1), i.e. substituting the values of multiplicative functions on individual chains to obtain the values on products of chains. In the formula counting maximal chains, $C_{\alpha,\beta}$ is not multiplicative, so $C_{\alpha,\beta}/(m+n)!$ is used instead, introducing the factor of $(m+n)!$, recalling that $m+n$ is the rank of $W_{\alpha,\beta}$. Likewise, to obtain the characteristic polynomial, notice that $\sum_{u\in P}\mu(\hat{0},u)(1/t)^{\rho(u)}$ is a multiplicative function equalling $\chi_P(t)/t^{m+n}$, so in this case an extra factor of $t^{\rho(P)}$ is introduced. $\square$

**Theorem 10.** *Summing over compositions $\alpha$ and $\beta$ indexed by monomials consisting of noncommutative variables $u_i$ and $v_j$ satisfying relations $u_iv_j = v_ju_i$ for all $i,j>0$, the following identities hold.*

$$\begin{aligned}
\sum_{\alpha,\beta}\Omega_{\alpha,\beta}u_\alpha v_\beta &= \frac{1}{1-\sum_{k>0}(k+1)u_k-\sum_{l>0}(l+1)v_l+\sum_{k,l>0}(k+l-1)u_kv_l}\\
\sum_{\alpha,\beta}\Omega_{\alpha,\beta}(q)u_\alpha v_\beta &= \frac{1}{1-\sum_{k>0}[k+1]_qu_k-\sum_{l>0}[l+1]_qv_l+\sum_{k,l>0}[k+l-1]_qu_kv_l}\\
\sum_{\alpha,\beta}\left(\frac{C_{\alpha,\beta}}{(m+n)!}\right)u_\alpha v_\beta &= \frac{1}{1-\sum_{k>0}\left(\frac{1}{k!}\right)u_k-\sum_{l>0}\left(\frac{1}{l!}\right)v_l+\sum_{k,l>0}\left(\frac{1}{(k+l)!}\right)u_kv_l}\\
\sum_{\alpha,\beta}Z_{\alpha,\beta}(s)u_\alpha v_\beta &= \frac{1}{1-\sum_{k>0}\binom{s+k-1}{s-1}u_k-\sum_{l>0}\binom{s+l-1}{s-1}u_l+\sum_{k,l>0}\binom{s+k+l-1}{s-1}u_kv_l}\\
\sum_{\alpha,\beta}\chi_{\alpha,\beta}(t)u_\alpha v_\beta &= \frac{1}{1-\sum_{k>0}f(t,k)u_k-\sum_{l>0}f(t,l)v_l+\sum_{k,l>0}f(t,k+l)u_kv_l}
\end{aligned}$$

*for $f(t,j)=t^{j-1}(t-1)$.*

PROOF. To simplify notation, let $\alpha-\alpha_{a_i}=(\alpha_1,\ldots,\hat{\alpha}_{a_i},\ldots,\alpha_k)$ and $\beta-\beta_{b_i}=(\beta_1,\ldots,\hat{\beta}_{b_i},\ldots,\beta_l)$.

The identities follow from the following recurrences for $\alpha=(\alpha_1,\ldots,\alpha_k)$ and $\beta=(\beta_1,\ldots,\beta_l)$, proven similarly to lemma 2.

$$\begin{aligned}
\Omega_{\alpha,\beta} &= (\alpha_k+1)\Omega_{\alpha-\alpha_k,\beta}+(\beta_l+1)\Omega_{\alpha,\beta-\beta_l}-(\alpha_k+\beta_l+1)\Omega_{\alpha-\alpha_k,\beta-\beta_l}\\
\Omega_{\alpha,\beta}(q) &= [\alpha_k+1]_q\Omega_{\alpha-\alpha_k,\beta}(q)+[\beta_l+1]_q\Omega_{\alpha,\beta-\beta_l}(q)-[\alpha_k+\beta_l+1]_q\Omega_{\alpha-\alpha_k,\beta-\beta_l}(q)\\
C_{\alpha,\beta} &= \binom{m+n}{\alpha_k}C_{\alpha-\alpha_k,\beta}+\binom{m+n}{\beta_l}C_{\alpha,\beta-\beta_l}-\binom{m+n}{\alpha_k+\beta_l}C_{\alpha-\alpha_k,\beta-\beta_l}\\
Z_{\alpha,\beta}(s) &= \binom{s+\alpha_k-1}{s-1}Z_{\alpha-\alpha_k,\beta}(s)+\binom{s+\beta_l-1}{s-1}Z_{\alpha,\beta-\beta_l}(s)\\
&- \binom{s+\alpha_k+\beta_l-1}{s-1}Z_{\alpha-\alpha_k,\beta-\beta_l}(s)\\
\chi_{\alpha,\beta}(t) &= \left(t^{\alpha_k-1}(t-1)\right)\chi_{\alpha-\alpha_k,\beta}(t)+\left(t^{\beta_l-1}(t-1)\right)\chi_{\alpha,\beta-\beta_l}(t)\\
&- \left(t^{\alpha_k+\beta_l-1}(t-1)\right)\chi_{\alpha-\alpha_k,\beta-\beta_l}(t)
\end{aligned}$$

□

Finally, a related topic we have not considered is to generalize results of Simion and Stanley in [7] about the monoid of multiplicative functions to generating functions in several variables, e.g. the variables $u_i, v_j$ with relations $u_i v_j = v_j u_i$ for all positive integers $i$ and $j$.

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