BROWNIAN MOTION IN A WEYL CHAMBER, NON-COLLIDING PARTICLES, AND RANDOM MATRICES

MOUVEMENT BROWNIEN DANS UNE CHAMBRE DE WEYL, PARTICULES NON S'ENTRECHOQUANTES, ET MATRICES ALÉATOIRES

DAVID J. GRABINER

Abstract. Let n particles move in standard Brownian motion in one dimension, with the process terminating if two particles collide. This is a specific case of Brownian motion constrained to stay inside a Weyl chamber; the Weyl group for this chamber is A_{n-1}, the symmetric group. For any starting positions, we compute a determinant formula for the density function for the particles to be at specified positions at time t without having collided by time t. We show that the probability that there will be no collision up to time t is asymptotic to a constant multiple of $t^{-n(n-1)/4}$ as t goes to infinity, and compute the constant as a polynomial of the starting positions. We have analogous results for the other classical Weyl groups; for example, the hyperoctahedral group B_n gives a model of n independent particles with a wall at $x = 0$.

We can define Brownian motion on a Lie algebra, viewing it as a vector space; the eigenvalues of a point in the Lie algebra correspond to a point in the Weyl chamber, giving a Brownian motion conditioned never to exit the chamber, If there are m roots in n dimensions, this shows that the radial part of the conditioned process is the same as the $n + 2m$-dimensional Bessel process. The conditioned process also gives physical models, generalizing Dyson's model for A_{n-1} corresponding to $\mathfrak{su}_n$ of n particles moving in a diffusion with a repelling force between two particles proportional to the inverse of the distance between them.

Résumé. Soit n particules en mouvement Brownien ordinaire qui se termine quand deux particules s'entrechoquent. C'est un cas specifique de mouvement Brownien contraint à rester dans une chambre de Weyl; le groupe de Weyl pour cette chambre est A_{n-1}, le groupe symmetrique. Pour toutes positions initiales, nous calculons une formule determinantale qui donne la fonction de

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densité pour que les particules être aux positions specifiques au temps t sans s'être entrechoquées avant le temps t. Nous démontrons que la probabilité qu'il n'y ai pas de collision avant le temps t est asymptotique à un multiple constant de $t^{n(n-1)/4}$ quand $t \to \infty$, et nous calculons le constant comme fonction des positions initiales. Nous avons des resultats analogues pour les autres groupes de Weyl classiques; par exemple, le groupe B_n donne un modèle de n particules avec un mur à $x = 0$.

On peut definir un mouvement Brownien dans une algebre de Lie en considerant l'algebre comme un espace vectoriel. Les valeurs propres d'un point de l'algebre de Lie correspondent à un point dans la chambre de Weyl. Ceci donne un mouvement Brownien conditionné à ne jamais sortir de la chambre. S'il y a m racines en n dimensions, la partie radiale du processus conditionné est identique au processus de Bessel en $(n + 2m)$ dimensions. Le processus donne aussi des modèles physiques qui generalizent le modèle de Dyson pour A_{n-1} correspondant à $\mathfrak{su}_n$, avec n particules en diffusion et une force repulsive entre chaque pair de particules en proportion inverse à la distance entre eux.

Let n particles move in standard Brownian motion in one dimension, with the process terminating if two particles collide. Given the starting positions, we can use a reflection argument to calculate the density function for the particles to be at specified positions at time t without having collided by time t. Using this density function and the theory of Lie algebras, we can prove the following results; Theorems 3 and 4 were first proved by Dyson [3].

Theorem 1. *For any starting positions, the probability that there will be no collision up to time t is asymptotic to a constant multiple of $t^{-n(n-1)/4}$ as t goes to infinity; the constant is a known polynomial in the starting positions.*

Theorem 2. *Given that there is no collision up to time t, the distribution of the radius of the vector whose coordinates are the positions of the particles, divided by the square root of t, converges in measure to the distribution of the Bessel process with parameter $n(n+1)/2$ at time 1, which is the radial part of an $n(n+1)/2$-dimensional standard normal.*

Theorem 3. *We can construct an n-dimensional Brownian motion which is conditioned for no two particles ever to collide. If we take Brownian motion on the space of Hermitian matrices, the induced process on the eigenvalues is the same process. If the starting point is appropriately chosen at the starting radius, the process given by the radial part of the conditioned Brownian motion is the same process as the n^2-dimensional Bessel process.*

Theorem 4. *This conditioned process is identical to the process obtained by n particles moving in a one-dimensional diffusion with constant infinitesimal variance,* *with a repelling force between two particles proportional to the inverse of the distance between them; that is, its infinitesimal generator has* $\sigma_{ij}^2(\vec{x}, t) = \delta_{ij}$, $\mu_i(\vec{x}, t) = \sum_{i \neq j} 1/(x_i - x_j)$.

This is a specific case of Brownian motion in a Weyl chamber; the vector whose coordinates are the locations of the n particles is constrained to stay inside the chamber. The Weyl group is A_{n-1}, the symmetric group.

We have similar results for other Weyl groups. Theorems 1 and 2 have analogues for the classical Weyl groups, and Theorems 3 and 4 have analogues for all Weyl groups. We also have the following generalization of Theorem 4 for all finite Coxeter groups.

Theorem 5. *For any finite Coxeter group, Brownian motion conditioned to stay within a chamber is equivalent to the motion of a particle in a diffusion with constant infinitesimal velocity, and a repulsive force from every hyperplane of reflection (not merely the walls of the chamber) inversely proportional to the distance from that hyperplane.*

For example, the Weyl group B_n models n particles in independent Brownian motion with an absorbing wall at $x = 0$. In this case, the asymptotic probability is $ct^{-n^2/2}$ (and the constant is again known), with the radial part corresponding to the $n(n+1)$-dimensional Bessel process; the conditioned process has a radial part which is the $n(2n+1)$-dimensional Bessel process.

Our reflection argument is a generalization of the *reflection principle*, a standard argument in the analysis of both discrete random walks and Brownian motion. In the discrete case, it is used in the classical formula for the Catalan numbers, which enumerate the arrangements of n $+1$'s and n -1's so that none of the partial sums are negative. Similarly, it can be used to study Brownian motion in one dimension with an absorbing barrier at $x = 0$ and a known starting point.

Gessel and Zeilberger [4], and independently Biane [2], give a generalization to multiple reflections. Recall that a *Coxeter group* is generated by reflections in its root system, and that the hyperplanes of reflection divide space into *chambers.* If the group stabilizes a lattice, then it is a *Weyl group*; this is necessary in order for reflection of a random walk to be possible, but not necessary for Brownian motion.

A random walk-type is *reflectable* with respect to a particular Weyl group if the set of steps is symmetric under the Weyl group, and the starting point and chamber are such that the walk cannot go from the interior of the chamber to the exterior without stopping on a wall. For a reflectable walk, the reflection principle can be used to enumerate walks which stay within the Weyl chamber in the same way that it is used to compute the Catalan numbers.

The argument of Gessel and Zeilberger can be generalized to a diffusion in any Weyl chamber or chamber of a Coxeter group, with either absorbing or reflecting boundary conditions. The reflectability conditions for a diffusion are analogous to the conditions for a discrete walk. The diffusion must be symmetric under the Weyl group, which is true for Brownian motion, and must have continuous sample paths, which is part of the definition of a diffusion. The argument carries over naturally to the continuous case, giving the following result.

Theorem 6. *If $c_t(\mu)$ is the density function for a reflectable diffusion to go from the origin to μ at time t, and $b_t(\eta, \lambda)$ is the density function for the same diffusion, started at η, to be at λ at time t without leaving the chamber of W, then*

$$b_t(\eta, \lambda) = \sum_{w \in W} \operatorname{sgn}(w) c_t(w(\lambda) - \eta).$$

To prove this theorem, reflect the part of any path from η to any $w(\lambda)$ after it first touches a wall in that wall. These paths will cancel out in the sum, leaving only those paths which do not touch a wall.

We use this result to compute formulas for Brownian motion in the Weyl chambers of A_{n-1}, $B_n = C_n$, and D_n. For A_{n-1}, the chamber is $x_1 > x_2 > \cdots > x_n$, modeling n independent particles in one dimension which are not allowed to collide. For B_n, the chamber is $x_1 > x_2 > \cdots > x_n > 0$, modeling n independent particles with a wall at $x = 0$. For D_n, the chamber is $x_1 > x_2 > \ldots > x_n$, $x_{n-1} > -x_n$, which does not give a natural model for n particles in one dimension.

In each case, we get a determinant of normal distribution functions. The simplest formula is for A_{n-1}. The density function for Brownian motion starting at η to be at λ at time t without having left the chamber is

$$b_t(\eta, \lambda) = \det_{n \times n} \left| N_t(\lambda_i - \eta_j) \right|,$$

where N_t is the normal distribution with variance t. This formula was first developed by Karlin and MacGregor [5], using the model of n independent particles starting at $\eta_1, \ldots, \eta_n$ to be at $\lambda_1, \ldots, \lambda_n$ without having collided, rather than motion in a Weyl chamber.

The formula for B_n is

$$b_t(\eta, \lambda) = \det_{n \times n} \left| N_t(\lambda_i - \eta_j) - N_t(\lambda_i + \eta_j) \right|,$$

and for D_n,

$$\begin{aligned} b_t(\eta, \lambda)(x) &= \frac{1}{2}\Big[\det_{n \times n} \left| N_t(\lambda_i - \eta_j) - N_t(\lambda_i + \eta_j) \right| \\ &\quad + \det_{n \times n} \left| N_t(\lambda_i - \eta_j) + N_t(\lambda_i + \eta_j) \right|\Big]. \end{aligned}$$

These determinants factor into forms which can be easily analyzed; this allows us to find the asymptotic probability that there will be no collision up to time t for these three cases, for any starting point.

Writing the normal distribution functions explicitly as exponentials for A_n gives

$$b_t(\eta, \lambda) = \frac{1}{(2\pi t)^{n/2}} \det_{n\times n} \left| \exp\left(\frac{-\lambda_i^2 + 2\lambda_i\eta_j - \eta_j^2}{2t} \right) \right|,$$

which can be factored as

$$b_t(\eta, \lambda) = \frac{1}{(2\pi t)^{n/2}} \exp\left(\frac{-|\lambda|^2 - |\eta|^2}{2t} \right) \det_{n\times n} \left| [\exp(\lambda_i/t)]^{\eta_j} \right|.$$

We can approximate the η_j by rationals, and then re-scale by $x_i \to cx_i$, $t \to c^2 t$ to make them integers without affecting the probabilities. The determinant $\det_{n\times n} |[\exp(\lambda_i/t)]^{\eta_j}|$ is then a generalized Vandermonde determinant, which factors into the product of the ordinary Vandermonde determinant of the $\exp(\lambda_i/t)$ and a Schur function. Since the determinant is exponentially small unless λ_i is much less than t, we can approximate the Schur function by its value when all arguments are 1, which is

$$\prod_{i<j} (\eta_i - \eta_j) \Big/ \prod_{i<j} (j - i),$$

factor the Vandermonde determinant as $\prod_{i>j}(\lambda_i - \lambda_j)$, and integrate the product of this with $\exp(-|\lambda^2|/2t)$ over the Weyl chamber to get an asymptotic for the probability that there will be no collision up to time t. Since $\prod_{i>j}(\lambda_i - \lambda_j)$ is homogeneous of degree $n(n-1)/2$ and all other terms are spherically symmetric, we can factor the integral into spherical and radial terms, and integrate spherically first to get a constant multiple of the surface area of the sphere. This gives

$$\int_0^\infty C r^{n-1} t^{-n/2} \exp(-r^2/2t) (r/t)^{n(n-1)/2} dr,$$

and the factor of $(r/t)^{n(n-1)/2}$ causes this integral to be a constant multiple of $t^{-n(n-1)/4}$. This is the desired asymptotic.

The methods for B_n and D_n are similar. The density functions are only approximately equal to generalized Vandermonde determinants, but as with the Schur function approximation above, the error is very small if λ_i is much less than t, and the density function is exponentially small otherwise. The entries of the generalized Vandermonde determinants are slightly different polynomials, giving the product of the roots for B_n or D_n instead of the product $\prod(\lambda_i - \lambda_j)$ of the roots for A_{n-1}. The spherical and radial integration still works because the product of the roots is homogeneous. The Schur function argument does not work; instead, the time-reversibility of Brownian motion, which gives $b_t(\eta, \lambda) = b_t(\lambda, \eta)$, can be used to show the dependence of the constant factor in the asymptotic on the starting point.

In each case, the leading term is $ct^{-m/2}$, where m is the number of roots, and the coefficient c is equal to a constant multiple of the product of the roots, evaluated at the starting point η. The remaining constant can be computed from Selberg's integral [6, 7].

Instead of viewing our Weyl chamber as the chamber of a Weyl group, we can view it as the space of eigenvalues of the Lie algebra corresponding to that Weyl group, and then use the theory of Lie algebras to study it. This construction was first developed by Dyson [3] for $\mathfrak{su}_n$; he computed the properties of the Brownian motion from the specific data rather than using the general Lie theory.

Given a Lie group G and its Lie algebra $\mathfrak{g}$, we can define a normal distribution or Brownian motion on $\mathfrak{g}$ by viewing it as a vector space and taking an orthonormal basis with respect to a G-invariant inner product; for the classical Lie algebras, the norm is the Hilbert-Schmidt norm on the matrices. For example, for $\mathfrak{su}_n(\mathbb{C})$, the Lie algebra of skew-Hermitian matrices, we can take standard Brownian motion on the imaginary part of each diagonal entry M_{ii}, and $1/\sqrt{2}$ times standard Brownian motion on the real and imaginary parts of each matrix entry M_{ij} with $i < j$, with $M_{ji} = -\overline{M_{ij}}$.

Since the Brownian motion is invariant under conjugation by G, it induces a diffusion on the space of eigenvalues of the matrices. The Weyl chamber is in a correspondence with the space of eigenvalues, obtained by dividing the independent eigenvalues by i and arranging them in decreasing order.

To study this diffusion, we use a version of the Weyl Integration Formula for Lie algebras, which can be proved directly from the analogous formula for Lie groups. This gives a measure on the Weyl chamber which is induced by the standard measure on the Lie algebra. For δ the product of the roots, the measure at a point $(x_1, \ldots, x_n)$ in the Weyl chamber is proportional to $\delta\bar{\delta}$. In particular, we note that δ is positive real on the interior of the Weyl chamber, and zero on the walls; thus we have a diffusion which is conditioned never to leave the Weyl chamber.

This proves Theorem 3 and an analogous theorem for any Weyl group. If we view $\mathfrak{g}$ as a vector space, the Hilbert-Schmidt norm of a matrix is the radius of a vector. If there are m positive roots, then $\mathfrak{g}$ has dimension $n + 2m$. The Hilbert-Schmidt norm is thus given by the Bessel process in $n+2m$ dimensions. The Hilbert-Schmidt norm is unchanged under diagonalization, so the diffusion in the Weyl chamber has its radius given by the same process.

The simplest case of this theorem is for B_1, which is one-dimensional motion with an absorbing boundary at $x = 0$. In this case, the only root is x, so $m = 1$, and we have the result of Pitman [8] and Williams [9] that Brownian motion conditioned never to hit 0 is the same as the three-dimensional Bessel process. In our more general cases, the number of dimensions for the Bessel process is $n + 2m = n^2$ for A_{n-1}, $n(2n + 1)$ for B_n, and $n(2n - 1)$ for D_n.

Another way to construct the transformed Brownian motion is by h-transformation. This is a procedure which allows us to transform standard Brownian motion in any number of dimensions to get a conditioned diffusion satisfying certain conditions.

For any non-negative harmonic function h, the measure for the transformed Brownian motion to go to λ at time t after starting at η is $h(\lambda)$ times the measure for untransformed Brownian motion with the same starting point, renormalized by an appropriate constant so that the total measure on all paths is 1.

In particular, suppose that we have a connected region D and a harmonic function h which is zero on the boundary of the region and positive on the interior. Then the h-transformed Brownian motion will be conditioned to stay in the interior of the region.

The density function for this transformed motion will be h times the density function for the untransformed motion on the same region, normalized appropriately. In our case, we can take h to be δ itself. We can see δ is harmonic because δ and thus $\nabla^2\delta$ are skew-symmetric in the Weyl group and thus zero on every hyperplane of reflection, but $\nabla^2\delta$ is of degree $m-2$ and needs to be zero on all m hyperplanes [1].

Having constructed the conditioned process as a transformation of Brownian motion, we can construct it as a diffusion. In this form, both the multidimensional models and the models of n independent particles lead to the natural physical models of Theorems 4 and 5.

A diffusion is characterized by its *infinitesimal generator*, consisting of the drift vector $\mu(\vec{x})$ which describes the expected velocity of a particle at $\vec{x}$ and the infinitesimal variance matrix $\sigma(\vec{x})$ which describes the variances and covariances of the individual coordinates over small times when the particle is at $\vec{x}$. Standard Brownian motion has $\mu = 0$ and $\sigma = I_n$; each coordinate is given by an independent standard normal. h-transformed Brownian motion has $\mu = \nabla h/h$ and $\sigma = I_n$; it is like Brownian motion except that there is a force $\nabla h/h$, which we can also write as $\nabla(\log h)$.

Since our h is the product of the roots, $\log h$ is the sum of the logs of the roots. A root evaluated at $\vec{x}$ is a constant multiple of the distance of $\vec{x}$ from the wall; thus $\log h(\vec{x})$ is a constant plus the sum of the logarithms of the distances of $\vec{x}$ from all walls of all chambers (not merely the chamber containing $\vec{x}$). Its gradient is the sum of the reciprocals of the distances of $\vec{x}$ from these walls, directed away from the walls. For example, for A_n, we have $h(\vec{x}) = \prod_{i>j}(x_i - x_j)$, and thus the ith component of $\nabla(\log h)$ is $\sum_{i\neq j} 1/(x_i - x_j)$. This proves Theorems 4, and for other Coxeter groups, it proves 5.

The analogues of Theorem 4 for the other classical groups are more natural as models of eigenvalues than as models of repelling particles, because the eigenvalues of an element in the Lie algebras come as dependent pairs, and only the independent eigenvalues repel one another. For D_n, a matrix in the Lie algebra $\mathfrak{so}_{2n}$ has eigenvalues $\pm i\lambda_j$; the eigenvalues in different pairs $\pm i\lambda_j$ and $\pm i\lambda_k$ for $j \neq k$ repel one another, although the pair $\pm i\lambda_j$ do not repel each other. For B_n, a matrix in the Lie algebra $\mathfrak{so}_{2n+1}$ has eigenvalues $\pm i\lambda_j$ and 0; this model thus says that the unconnected eigenvalues $\pm i\lambda_j$ and $\pm i\lambda_k$ for $j \neq k$ repel each other, and each eigenvalue is also repelled by the fixed eigenvalue at 0.

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E-mail address: `grabiner@math.lsa.umich.edu`

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109