Construction of q-equations for convex polyominoes

Dominique Gouyou-Beauchamps
LRI, Bât. 490
Université de Paris Sud
91405 Orsay CEDEX, France

November 21, 1997

Abstract

For most of column-convex polyominoes we give a method that constructs algebraic functional q-equations verified by their generating functions. Our method is based on the Temperley methodology and on a result of Viennot and Roblet about multi-continued fractions.

Résumé

On propose une méthode générale pour construire des q-équations algébriques pour les fonctions génératrices de la plupart des familles de polyominos verticalement convexes. Cette méthode repose sur la méthodologie de Temperley et sur un résultat de Viennot et Roblet concernant les fractions multi-continuées.

1 Introduction

In last years there has been new interest in two-dimensional polyomino enumeration problems. These questions are not only interesting for combinatorialists, but are also of considerable importance in the study of lattice models in physics and chemistry. Usually, physicists consider equivalent objects which are named animals, self-avoiding polygons or polygons. These models give simplified descriptions of physical phenomena as equilibrium of polymers in dilute solution, cell-growth, or percolation. Physicists use different methods to show that these models have a phase transition, and locate the critical point.

We consider here only the square lattice and restricted classes of polyominoes. In the cartesian plane $\mathbb{N} \times \mathbb{N}$, a polyomino is a finite connected union of elementary cells (unit squares) without cut points and defined up to translation. A polyomino is said vertically convex (or column-convex) when its intersection with any vertical line is convex. We can define similarly a notion of horizontal convexity (or row-convexity). A polyomino is convex if it is both vertically and horizontally convex. A polyomino P is said to be directed when every cell of P can be reached from a distinguished cell, called the root, by a path that is contained in P and has only North and East steps. Thus we are interested only by 1-source directed polyominoes. Combining the two notions described above, we can define several families of polyominoes.

Several authors gave methods in order to obtain q-equations for convex polyominoes. We can cite Temperley [19], Klarner [12], Bhat, Bhan and Singh [1], Privman and Švrakić [17], Lin and Chang [13], Fédou [9], Feretić and Svrtan [11], Prellberg and Brak [16], Dutour [8].

In a recent paper, M. Bousquet-Mélou [2] presents a single method that gives, for any class of column-convex polyominoes, a functional equation that implicitly defines its generating function, and then solves it. With this method, she obtains linear q-differential equations expressing the generating function $X(x,s,q)$ in terms of $X(x,sq,q)$, $X(x,1,q)$ and $\partial X/\partial s(x,1,q)$, where x encodes the perimeter and s the left height and q the area. Roughly, she can conclude that the perimeter generating function $(X(x,s,1))$ for any usual family of convex or column-convex polyominoes is an algebraic series, whereas the area generating function involves q-series. The algebraic perimeter generating function can be derived from the linear q-equation by a method introduced by Feretić and Svrtan [11] and generalized by M. Bousquet-Melou.

The aim of this paper is to give another method that constructs a functional equation for the generating function $X(x,s,q)$. More precisely, we obtain an algebraic q-equation $P(X(x,s,q)) = 0$, where $P(t)$ is a polynomial with coefficient in $\mathbb{R}[[x,s,q]]$, the algebra of formal power series in the variables x,s,q, with real coefficients, and where the power $X^k(x,s,q)$, for all integer $k \geq 1$, is replaced by its q-analogue $X(x,s,q)X(x,sq,q)\cdots X(x,sq^{k-1},q)$. Thus, thanks to this method, it is trivial to show that the perimeter generating function $(X(x,s,1))$ is algebraic for any usual family of convex or column-convex polyominoes. Moreover, following T. Prellberg and R. Brak [16] and T. Prellberg [15], it seems that q-algebraic equations are more suitable for the study of the series $X(x,s,q)$ singularity structure than linear q-differential equations.

On one hand, like the Bousquet-Melou method, our method is based on the Temperley methodology [19]. On the other hand, it uses a result of Viennot [20] and Roblet [18] about continued fractions. This method gives two types of results: equations and enumeration formulas. Only equations are new. Most of the enumerative results are already known.

Our method is based the four following steps :

1. we establish an infinite system of linear equation using Temperley methodology,

2. we reduce the bandwith of the previous matrices in order obtain recurrences relations for the finite determinants,

3. using the result of Viennot and Roblet, we get the generating function as a multi-continued fraction (L-fraction),

4. from this L-fraction, we extract a q-equation system.

In the following section, we recall definitions and the main result that we need. Then, we present our method and we apply it to the parallelogram polyominoes in section 3. Section 4 is devoted respectively to diagonally convex directed polyominoes, directed column-convex polyominoes, column-convex polyominoes, stack polyominoes and directed convex polyominoes.

2 Definitions

Let P be a column-convex polyomino. The area of P, denoted by $a(P)$, is the number of elementary cells composing it. The width, denoted by $w(P)$, is the number of columns of P. The left height, denoted by $l(P)$, is the height of its leftmost column. We denote the number of vertical steps in its perimeter by $2v(P)$. Note that the perimeter of P is $2(v(P)+w(P))$. Let $\mathcal{P}$ be a set of column-convex polyominoes. We say that a line of equation $y+x=b$, for any integer b, is a diagonal of P if and only if it intersects the interior of at least one cell of P. We denote the number of diagonals of P by $d(P)$. The generating function of $\mathcal{P}$ is:

$$\sum_{P\in\mathcal{P}} s^{l(P)}u^{d(P)}x^{w(P)}y^{v(P)}q^{a(P)}.$$

Let K be a commutative field. The reciprocal polynomial of a polynomial $P(x)$ with coefficients in K is denoted by $P^*(x)$ and is equal to $x^{deg(P)}P(1/x)$ where $deg(P)$ denotes the degree of P.

Let $\lambda = (\lambda_{l,k})_{0\leq k\leq l}$ be a family of elements in K and let $(P_k(x))_{k\geq 0}$ be a sequence of unitary polynomials which satisfy the recurrence

$$P_{k+1}(x) = xP_k(x) - \sum_{i=0}^{k} \lambda_{k,k-i}P_{k-i}(x).$$

The family λ also defines the multi-continued fraction (L-fraction)

$$L(\lambda,t) = \cfrac{1}{1-\lambda_{0,0}t-\sum_{p=1}^{\infty}\lambda_{p,0}t^{p+1}\prod_{i=1}^{p}\cfrac{1}{1-\lambda_{i,i}t-\sum_{q=1}^{\infty}\lambda_{q+i,i}t^{q+1}\prod_{j=1}^{q}\cfrac{1}{1-\lambda_{i+j,i+j}t-\cdots}}}.$$

In $L(\lambda,t)$, if we substitute $\lambda_{i,j}$ with 0, for all $i\geq k+1$ and all $j\leq i$, we get a rational fraction $L_k(t)$, the so-called kth convergent of $L(\lambda,t)$.

Let δ be the correspondence from λ onto λ defined by $\delta\lambda_{k,l} = \lambda_{k+1,l+1}$.

Theorem 1 (Viennot [20] and Roblet [18]) *The kth convergent of $L(\lambda,t)$ is equal to*

$$L_k(t) = \frac{\delta P_k^*(t)}{P_{k+1}^*(t)},$$

and the following approximation is verified in $t=0$:

$$L(\lambda,t) = L_k(t) + o(t^{k+1}).$$

A convex polyomino is a parallelogram polyomino if it contains the south-west vertex and the north-east vertex of its minimal bounding rectangle.

We denote by $[n]_y$ the polynomial $1+y+y^2+\ldots+y^{n-1}$.

3 Our method

Now, we present the four steps of our method. Each step is applied to the set of parallelogram polyominoes (or staircase polyominoes).

3.1 Establishing equations

Our method is based on a very simple idea that is known as Temperley methodology [19] in statistical physics: a column-convex polyomino can be obtained by successively gluing columns, in a way that depends on the class of polyominoes one studies. This obvious remark leads to construct an infinite system of linear equations. In the staircase polyomino case, there are i ways of gluing a column of height k on the left of a parallelogram polyomino P such that $l(P) = i$ with $1 \leq i \leq k$. There are k ways of gluing a column of height k on the left of a parallelogram polyomino P such that $l(P) = i$ with $k < i$.

Let $p^{(k)}(x, y, q)$ be the generating function for the parallelogram polyominoes having a left height of k. We will often denote it, for short, by $p^{(k)}(x)$. This generating function verifies the following equation:

$$p^{(k)}(x) = xy^k q^k + \sum_{i=1}^{k} xy^{k-i}[i]_y q^k p^{(i)}(x) + \sum_{i>k} x[k]_y q^k p^{(i)}(x).$$

Therefore we obtain the infinite system:

$$\begin{pmatrix} qx-1 & qx & qx & qx & qx & \cdots \\ q^2xy & q^2x[2]_y-1 & q^2x[2]_y & q^2x[2]_y & q^2x[2]_y & \cdots \\ q^3xy^2 & q^3xy[2]_y & q^3x[3]_y-1 & q^3x[3]_y & q^3x[3]_y & \cdots \\ q^4xy^3 & q^4xy^2[2]_y & q^4xy[3]_y & q^4x[4]_y-1 & q^4x[4]_y & \cdots \\ q^5xy^4 & q^5xy^3[2]_y & q^5xy^2[3]_y & q^5xy[4]_y & q^5x[5]_y-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p^{(1)}(x) \\ p^{(2)}(x) \\ p^{(3)}(x) \\ p^{(4)}(x) \\ p^{(5)}(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} -qxy \\ -q^2xy^2 \\ -q^3xy^3 \\ -q^4xy^4 \\ -q^5xy^5 \\ \vdots \end{pmatrix}$$

Our goal is to get the generating function $p(x, y, q) = \sum_{k\geq 1} p^{(k)}(x, y, q)$ (denoted by p(x) for short). This function can be written as $n(x)/d(x)$, where $n(x)$ and $d(x)$ are determinants defined by:

$$d(x) = \begin{vmatrix} qx-1 & qx & qx & qx & qx & \cdots \\ q^2xy & q^2x[2]_y-1 & q^2x[2]_y & q^2x[2]_y & q^2x[2]_y & \cdots \\ q^3xy^2 & q^3xy[2]_y & q^3x[3]_y-1 & q^3x[3]_y & q^3x[3]_y & \cdots \\ q^4xy^3 & q^4xy^2[2]_y & q^4xy[3]_y & q^4x[4]_y-1 & q^4x[4]_y & \cdots \\ q^5xy^4 & q^5xy^3[2]_y & q^5xy^2[3]_y & q^5xy[4]_y & q^5x[5]_y-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

and

$$n(x) + d(x) = \begin{vmatrix} qx(1-y)-1 & qx(1-y) & qx(1-y) & qx(1-y) & \cdots \\ q^2xy(1-y) & q^2x([2]_y-y^2)-1 & q^2x([2]_y-y^2) & q^2x([2]_y-y^2) & \cdots \\ q^3xy^2(1-y) & q^3xy([2]_y-y^2) & q^3x([3]_y-y^3)-1 & q^3x([3]_y-y^3) & \cdots \\ q^4xy^3(1-y) & q^4xy^2([2]_y-y^2) & q^4xy([3]_y-y^3) & q^4x([4]_y-y^4)-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

3.2 Recurrence relations

This step is also part of Temperley methodology. Our purpose is to reduce the bandwidth of the previous matrices in order to simplify the determinant expansions. Here, we subtract the

ith column from the $(i+1)$th column for $i > 1$. Next, we multiply the ith row by yq and we subtract it from from the $(i+1)$th row for $i > 1$. Then, we get:

$$d(x) = \begin{vmatrix} qx-1 & 1 & 0 & 0 & 0 & \cdots \\ qy & q^2x-qy-1 & 1 & 0 & 0 & \cdots \\ 0 & qy & q^3x-qy-1 & 1 & 0 & \cdots \\ 0 & 0 & qy & q^4x-qy-1 & 1 & \cdots \\ 0 & 0 & 0 & qy & q^5x-qy-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

and

$$n(x)+d(x) = \begin{vmatrix} qx(1-y)-1 & 1 & 0 & 0 & 0 & \cdots \\ qy & q^2x-qy-1 & 1 & 0 & 0 & \cdots \\ 0 & qy & q^3x-qy-1 & 1 & 0 & \cdots \\ 0 & 0 & qy & q^4x-qy-1 & 1 & \cdots \\ 0 & 0 & 0 & qy & q^5x-qy-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

We denote the determinant formed by the first k rows and the first k columns of $d(x)$ (resp. $n(x)$) by $d_k(x)$ (resp. $n_k(x)$). Note that the series $p_k(x) = n_k(x)/d_k(x)$ is the generating function of parallelogram polyominoes with columns of height less than $k+1$. The polynomials $d_k(x)$ and $n_k(x)$ verify the following three term recurrences:

$$d_k(x) = (xq^k - yq - 1)d_{k-1}(x) - yqd_{k-2}(x)\ ,\ n_k(x) = (xq^k - yq - 1)n_{k-1}(x) - yqn_{k-2}(x)$$

with

$$\begin{array}{rcl} d_1(x) = qx - 1 & \text{and} & n_1(x) = -xyq \\ d_2(x) = x^2q^3 - xq^2(1+y) - xq + 1 & \text{and} & n_2(x) = -x^2yq^2 + xy^2q^2 + xyq. \end{array}$$

The higher degree monomial of $d_k^*(x)$ and $xn_k^*(x)/yq$ is $(-x)^k$. In order to get unitary polynomials, we multiply $d_k^*(x)$ and $n_k^*(x)q$ by $(-1)^k$ and divide $n_k^*(x)q$ by yq. Hence we obtain for $k > 2$:

$$\begin{aligned} d_k^*(x) &= xd_{k-1}^*(x) - (q^k - xyq)d_{k-1}^*(x) - x^2yqd_{k-2}^*(x) \\ n_k^*(x) &= xn_{k-1}^*(x) - (q^k - xyq)n_{k-1}^*(x) - x^2yqn_{k-2}^*(x) \end{aligned}$$

with

$$d_0^*(x) = 1 \quad , \quad n_0^*(x) = yq \quad , \quad d_1^*(x) = xd_0^* - qd_0^*(x) \quad , \quad n_1^*(x) = n_0^*(x) \quad ,$$
$$d_2^*(x) = xd_1^*(x) - (q^2 - xyq)d_1^*(x) - x^2yqd_0^*(x) \quad \text{and} \quad n_2^*(x) = xn_1^*(x) - (q^2 - xyq)n_1^*(x).$$

3.3 Continued fractions

The previous recurrences show that we are in the context of theorem 1, with $\lambda_{0,0} = q$, $\lambda_{i,i} = q^{i+1} - \frac{yq}{x}$, $\lambda_{i,i-1} = yq/x^2$ for $i > 0$ and $\lambda_{i,j} = 0$ everywhere else. Thus we can express $p(x) = n(x)/d(x)$ as a Jacobi continued fraction (J-Fraction):

$$p(x) = \cfrac{xyq}{1 - xq - \cfrac{yq}{1 + yq - xq^2 - \cfrac{yq}{1 + yq - xq^3 - \cfrac{yq}{1 + yq - xq^4 - \cfrac{yq}{\ddots}}}}}.$$

3.4 q-equations

From the previous continued fraction, we can extract the following q-equation system satisfied by $p(x)$:

$$\begin{cases} p(x) = \dfrac{xyq}{1 - xq - m(x)} \\ m(x) = \dfrac{yq}{1 + yq - xq^2 - m(xq)}. \end{cases}$$

A very easy calculation leads to the functional equation (see Bousquet-Mélou and Viennot [5], Bousquet-Mélou and Fédou [4] and T. Prellberg and R. Brak [15] for the solution)

$$p(x) = (y + p(x))\,(xq + p(xq))$$

We observe that this equation is not symmetric in x and y even though row and column play the same role. The symmetry becomes apparent if we use diagonals instead of columns.

Let i be the height of the first diagonal of a parallelogram polyomino P. If $i = k-1$ or $k+1$ (resp. $i = k$) there is one way (resp. two ways) of gluing a diagonal on the left of P. Thus the generating functions $p^{(k)}(u, x, y, q)$ ($p^{(k)}(u)$ for short) for parallelogram polyominoes whose first diagonal is of height k verify the following equation:

$$p^{(k)}(u) \quad = \quad uq^k\left(xyp^{(k-1)}(u) + (x+y)p^{(k)}(u) + p^{(k+1)}(u)\right)$$

Then we obtain the infinite system (where the polynomial $u(x+y)$ is denoted by f):

$$\begin{pmatrix} qf-1 & uq & 0 & 0 & 0 & \cdots \\ uq^2xy & q^2f-1 & uq^2 & 0 & 0 & \cdots \\ 0 & uq^3xy & q^3f-1 & uq^3 & 0 & \cdots \\ 0 & 0 & uq^4xy & q^4f-1 & uq^4 & \cdots \\ 0 & 0 & 0 & uq^5xy & q^5f-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p^{(1)}(u) \\ p^{(2)}(u) \\ p^{(3)}(u) \\ p^{(4)}(u) \\ p^{(5)}(u) \\ \vdots \end{pmatrix} = \begin{pmatrix} -uqxy \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

The generating function $p_1(u)$ ($p(u)$ for short) for 1-source parallelogram polyominoes can be written as the ratio $n(u)/d(u)$, where $n(u)$ and $d(u)$ are infinite determinants. With the previous method, we deduce easily that we are in the context of Theorem 1, with $\lambda_{i,i} = q^{i+1}(x+y)$, $\lambda_{i+1,i} = q^{2i+3}xy$ for $i \geq 0$, and $\lambda_{i,j} = 0$ everywhere else. Thus we can express $p(u) = n(u)/d(u)$ as a Jacobi continued fraction (J-Fraction). From this continued fraction, we can extract the following q-equation system satisfied by $p(u)$:

$$p(u) = quxy + qu(x+y)p(u) + qup(u)p(uq).$$

With this equation, the symmetry in the variables x and y becomes apparent.

4 Other examples

4.1 Diagonally convex directed polyominoes

In this section, we work also with diagonals instead of columns. A directed polyomino P is a diagonally convex directed polyomino if its intersection with any diagonal is convex. We denote by $p^{(k)}(u, x, y, q)$ the generating function for the diagonally convex directed polyominoes having their left diagonal of height k. Note that in the case of diagonally convex directed polyominoes $d(P)$ is the site perimeter minus one . If $p^{(k)}(u,x,y,q)$ is such a series, we will often denote it, for short, by $p^{(k)}(u)$. There is $k+2-i$ ways of gluing a diagonal of height k on the left of a diagonally convex directed polyomino P whose first diagonal is of height i with $1 \leq i \leq k+1$. Thus this generating function verifies the following equation:

$$\begin{aligned} p^{(k)}(u) \;\; = \;\; & uq^k \Big((x+y+k-1)x^{k-1}y^{k-1}p^{(1)}(u) + (x+y+k-2)x^{k-2}y^{k-2}p^{(2)}(u) + \cdots + \\ & (x+y+1)xyp^{(k-1)}(u) + (x+y)p^{(k)}(u) + p^{(k+1)}(u) + x^k y^k\Big) \end{aligned}$$

for $k \geq 1$.

The generating function $p_1(u)$ for 1-source diagonally convex directed polyominoes can be written as $n(u)/d(u)$, where $n(u)$ and $d(u)$ are infinite determinants. With the same calculation as in section 2.2, we get a simpler form for $d(u)$ and the recurrences:

$$\begin{aligned} d_k^*(u) &= ud_{k-1}^*(u) - q^k(x+y-2xy)d_{k-1}^*(u) - q^k xy(2u + q^{k-1}(1+xy-x-y))d_{k-2}^*(u) \\ &\quad + uq^{2k-1}x^2y^2 d_{k-3}^*(u) \\ n_k^*(u) &= un_{k-1}^*(u) - q^k(x+y-2xy)n_{k-1}^*(u) - q^k xy(2u + q^{k-1}(1+xy-x-y))n_{k-2}^*(u) \\ &\quad + uq^{2k-1}x^2y^2 n_{k-3}^*(u), \end{aligned}$$

for $k > 3$. These recurrences show that we are in the context of Theorem 1, with $\lambda_{0,0} = q(x+y)$, $\lambda_{1,0} = q^2xy(\frac{1}{u}+q)$, $\lambda_{1,1} = q^2(x+y-xy)$, $\lambda_{i,i} = q^{i+1}(x+y-2xy) = h_{i+1}^*$, $\lambda_{i,i-1} = q^{i+1}xy(\frac{2}{u} + q^i(1+xy-x-y)) = g_{i+1}^*$, $\lambda_{i,i-2} = -\frac{1}{u}q^{2i+1}x^2y^2$ for $i \geq 2$ and $\lambda_{i,j} = 0$ everywhere else, if the polynomial $qxy(2+uq^k(1+xy-x-y))$ (resp. $uq^k(x+y-2xy)-1$) is denoted by g_k (resp. h_k). Thus we can express $p(u) = n(u)/d(u)$ as a Lukasiewicz multi-continued fraction (L-Fraction or d-Lukasiewicz fraction with $d=2$). For instance, the convergent $p_3(u) = n_3(u)/d_3(u)$ is equal to:

$$p_3(u) = \frac{qxyu}{1 - q\,(x+y)\,u - \dfrac{q^2xy\left(\frac{1}{u}+q\right)u^2}{1 - q^2\,(x+y-xy)\,u - \dfrac{g_3^* u^2}{1-h_3^* u}} + \dfrac{q^5x^2y^2u^2}{\left(1 - q^2\,(x+y-xy)\,u - \dfrac{g_3^* u^2}{1-h_3^* u}\right)(1-h_3^* u)}}.$$

From the previous continued fraction, we can extract the following q-equation system satisfied by $p(x)$:

$$\left\{\begin{aligned} & m(u) = uqxy + uq(x+y-2xy)m(u) + (2+uq(1-x)(1-y))m(u)m(uq) - \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m(u)m(uq)m(uq^2) \\ & p(u) = m(u) - m(u)m(uq) + 2m(u)p(u) - m(u)m(uq)p(u) \end{aligned}\right.$$

We can reduce this system to one functional equation and get the result

Lemma 1 *The generating function $p(u,x,y,q)$ ($p(u)$ for short) of diagonally convex directed polyominoes verify the following equation:*

$$uq^2\,(1+p(uq^2))\,\left((x+p(u))(y+p(u)) + (p(uq)-p(u))(xy+p(u))\right) = p(u)\,\left(q + qp(uq^2) - p(uq)\right) \quad (1)$$

To our knowledge, Equation (1) is a new result. Let us set q to 1. As shown by Bath, Bhan et Singh [1], Feretić and Svrtan [10], Fédou [9], Penaud [14], and Bousquet-Mélou [3], Equation (1) can be written:

$$p(u,x,y) = u\,(1+p(u,x,y))\,(x+p(u,x,y))\,(y+p(u,x,y))$$

This implies that the perimeter and site perimeter generating function is algebraic of degree 3. For the area generating function, see Privman and Švrakić [17], Feretić and Svrtan [10], and Bousquet-Mélou [3].

4.2 Directed and column convex polyominoes

In this section, we work with columns. Let $p^{(k)}(x,y,q)$ be the generating function for the directed and column convex polyominoes having their left column of height k. We will often denote it, for short, by $p^{(k)}(x)$.

There is k ways of gluing a column of height k on the left of a directed and column convex polyomino P such that $l(P)=i$ with $1 \le i$. Thus this generating function verifies the following equation:

$$p^{(k)}(x) = q^k x \left(y^k + \sum_{i=1}^{k-1} y^{k-i}([i]_y + k - i)p^{(i)}(x) + \sum_{j \ge k} [k]_y p^{(j)}(x) \right)$$

The generating function $p(x) = \sum_{k\ge 1} p^{(k)}(x)$ can be written as $n(x)/d(x)$, where $n(x)$ and $d(x)$ are infinite determinants. With the same calculation as in section 2.2, we get a simpler form for $d(x)$ and the recurrence:

$$d_k^*(x) \;=\; xd_{k-1}^*(x) - (q^k(1-y) - 2xqy)d_{k-1}^*(x) - x^2qy(2+qy)d_{k-2}^*(x) + x^3q^2y^2d_{k-3}^*(x)$$

for $k > 2$. This recurrence shows that we are in the context of Theorem 1, with $\lambda_{0,0} = q$, $\lambda_{1,0} = \frac{yq}{x}(\frac{1}{x}+q)$, $\lambda_{1,1} = q^2(1-y) - \frac{yq}{x}$, $\lambda_{i,i} = q^{i+1}(1-y) - \frac{2qy}{x}$, $\lambda_{i,i-1} = \frac{yq}{x^2}(2+yq)$, $\lambda_{i,i-2} = \frac{-y^2q^2}{x^3}$ for $i \ge 2$ and $\lambda_{i,j} = 0$ everywhere else. Thus we can express $p(x) = n(x)/d(x)$ as a Lukasiewicz multi-continued fraction (L-Fraction or d-Lukasiewicz fraction with $d = 2$). From this multi-continued fraction, we can extract the following q-equation system satisfied by $p(x)$:

$$\begin{cases} m(x) = 1 - 2yqm(x) + xq^2(1-y)m(x) + yq(2+yq)m(x)m(xq) + y^2q^2m(x)m(xq)m(xq^2) \\ p(x) = \dfrac{xyq(1-yqm(x))}{1 - yqm(x) - yqm(x)(1-yqm(xq)) - qx}. \end{cases}$$

Setting $x = 1$ and $y = 1$ in the multi-continued fraction, we obtain the area generating function (Klarner [12], Delest and Dulucq [7])

$$\frac{q(1-q)}{1-3q+q^2}.$$

When we set $q = 1$ in the q-equation system, we get the perimeter generating function (Delest and Dulucq [7]) that is algebraic

$$p^3(x,y) + 2(y-1)p^2(x,y) + (y-1)(y+x-1)p(x,y) + xy(y-1) = 0.$$

4.3 Column convex polyominoes

In this section, we work with columns. $p^{(k)}(x,y,q)$ is the generating function for the column convex polyominoes having their left column of height k. We will often denote it, for short, by $p^{(k)}(x)$.

There is $k+i-1$ ways of gluing a column of height k on the left of a column convex polyomino P such that $l(P)=i$ with $1\le i$. Thus this generating function verifies the following equation:

$$p^{(k)}(x) = q^k x \left(y^k + \sum_{i=1}^{k-1} y^{k-i}(2[i]_y + k - i - 1)p^{(i)}(x) + \sum_{j\ge k} (2[k]_y + j - k - 1)p^{(j)}(x) \right)$$

for $k\ge 1$. The generating function $p(x) = \sum_{k\ge 1} p^{(k)}(x)$ can be written as $n(x)/d(x)$, where $n(x)$ and $d(x)$ are infinite determinants. With the same calculation as in section 2.2, we get a simpler form for $d(x)$ and the recurrence:

$$\begin{aligned} d^*_k(x) &= x d^*_{k-1}(x) - ((1-y)^2 q^k - xy^2q^2 - 3xyq)d^*_{k-1}(x) \\ &\quad - xyq((1-y)^2q^{k-1} + 3x(1+yq)^2 - 2xyq)d^*_{k-2}(x) \\ &\quad - x^2y^2q^2(2xyq - (1-y)^2q^{k-1} - 3x(1+yq)^2)d^*_{k-3}(x) \\ &\quad - x^3y^3q^3(xyq + x(1+yq)^2 - (1-y)^2q^{k-2})d^*_{k-4}(x) + x^5y^5q^5 d^*_{k-5}(x) \end{aligned}$$

for $k>4$. This recurrence shows that we are in the context of Theorem 1. Hence we can express $p(x)=n(x)/d(x)$ as a Lukasiewicz multi-continued fraction (L-Fraction or d-Lukasiewicz fraction with $d=4$). From this multi-continued fraction, we can extract the following q-equation system satisfied by $p(x)$:

$$\begin{cases} p(x) = \dfrac{a(x)}{b(x)} \\ a(x) = y^4q^4m(x)m(xq)m(xq^2) - y^2q^2(1+xyq+2yq-xq)m(x)m(xq) \\ \qquad - yq(xyq - xq - yq - 2)m(x) - 1 \\ b(x) = q^3y^3(2xyq^2 + 3yq + 1 - xq^2)m(x)m(xq)m(xq^2) \\ \qquad + q^2y^2(xyq^2 + x^2yq^2 - xq^2 - xyq - 3 - 3yq)m(x)m(xq) \\ \qquad\qquad + qy(3 + xq + x^2q^2y + yq - xyq^2 - xyq)m(x) + xq - 1 \\ m(x) = 1/\left[1 - q^2x(1-y)^2 + y^2q^2 + 3yq - yq\left(q^2x(1-y)^2 + 3(1+yq)^2 - 2yq\right)m(xq)\right. \\ \qquad + y^2q^2\left(q^3x(1-y)^2 + 3(1+yq)^2 - 2yq\right)m(xq)m(xq^2) \\ \qquad - y^3q^3\left(yq + (1+yq)^2 - xq^3(1-y)^2\right)m(xq)m(xq^2)m(xq^3) \\ \qquad \left. + y^5q^5m(xq)m(xq^2)m(xq^3)m(xq^4)\right] \end{cases}$$

Setting $x=1$ and $y=1$ in the multi-continued fraction, we obtain the area generating function (Klarner [12])

$$\frac{q(1-q)^3}{1-5q+7q^2-4q^3}.$$

When we set $q=1$ and $y=x$ in the q-equation system, we get the perimeter generating function (Delest [6]) that is algebraic

$$\alpha = x^3 - 3x^2 - x - 1 \quad , \quad \beta = x^2 - 6x + 1 \quad , \quad \gamma = x^3 - 5x^2 - x + 1 \quad ,$$

$$\delta = 5x^3 - 35x^2 + 47x - 21 \quad , \quad \epsilon = 2x^2 - 9x - 2 \quad \text{et} \quad \zeta = \sqrt{(x-1)\gamma + \alpha\sqrt{\beta}}$$

$$p(x) = \frac{(1-x)\left(2x\delta - 6x(x^2-1)\sqrt{\beta} - [(x-1)\epsilon - (2x-1)(x-2)\sqrt{\beta}]\sqrt{2(x^2-1)\zeta}\right)}{8x^3(2x^3-23x^2+38x-18)}$$

4.4 Stack polyominoes

A stack polyomino is characterized, in the set of convex polyominoes, by the fact that two corners (the north-west corner and the south-west corner) of the minimal bounding rectangle of the polyomino must also belong to the polyomino itself. We can remark that a stack polyomino is directed.

In this section, we work with columns. $p^{(k)}(x,y,q)$ is the generating function for the stack polyominoes having their left column of height k. We will often denote it, for short, by $p^{(k)}(x)$.

There is $k-i+1$ ways of gluing a column of height k on the left of a stack polyomino P such that $l(P)=i$ with $k \leq i$. Thus this generating function verifies the following equation:

$$p^{(k)}(x) = q^k x \left(y^k + \sum_{i=1}^{k} y^{k-i}(k-i+1)p^{(i)}(x)\right)$$

for $k \geq 1$. Then we have to solve the following infinite linear system:

$$\begin{pmatrix} qx-1 & 0 & 0 & 0 & 0 & \cdots \\ 2q^2xy & q^2x-1 & 0 & 0 & 0 & \cdots \\ 3q^3xy^2 & 2q^3xy & q^3x-1 & 0 & 0 & \cdots \\ 4q^4xy^3 & 3q^4xy^2 & 2q^4xy & q^4x-1 & 0 & \cdots \\ 5q^5xy^4 & 4q^5xy^3 & 3q^5xy^2 & 2q^5xy & q^5x-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p^{(1)}(x) \\ p^{(2)}(x) \\ p^{(3)}(x) \\ p^{(4)}(x) \\ p^{(5)}(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} -qxy \\ -q^2xy^2 \\ -q^3xy^3 \\ -q^4xy^4 \\ -q^5xy^5 \\ \vdots \end{pmatrix}$$

4.5 Directed and convex polyominoes

A directed convex polyomino can be obtained by gluing a parallelogram polyomino on a stack polyomino.

Let S (resp. T) be the infinite matrix obtained for parallelogram polyominoes (resp. stack polyominoes). We denote by $\mathcal{T}$ the infinite matrix

$$\mathcal{T} = \begin{pmatrix} -1 & 0 \\ 0 & T \end{pmatrix}.$$

Thus the matrix for directed convex polyominoes is $\mathcal{T}\mathcal{S}$ [4]. Then we have to solve the following infinite linear system:

$$\mathcal{T}\mathcal{S} \begin{pmatrix} p^{(1)}(x) \\ p^{(2)}(x) \\ p^{(3)}(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} qxy \\ q^2xy^2 \\ q^3xy^3 \\ \vdots \end{pmatrix}.$$

With the same calculation as in section 2.2, we get a simpler form for $d(x)$ and the recurrences:

$$\begin{aligned} d_n^*(y) \quad = \quad & yd_{n-1}^*(y) - (xyq^{n-1}(1+q-xq^n) - q(2+q-xq^{n-1}))d_{n-1}^*(y) \\ & -q(1-xq^{n-2})(q(1+2q)+y(1-xq^{n-1})(2+q))d_{n-2}^*(y) \\ & -q^2(1-xq^{n-2})(1-xq^{n-3})(xyq^{n-1}-q^2-y-2qy)d_{n-3}^*(y) \\ & -q^4y(1-xq^{n-2})(1-xq^{n-3})(1-xq^{n-4})d_{n-4}^*(y) \end{aligned}$$

for $n > 4$. Hence we can express $p(x) = n(x)/d(x)$ as a Lukasiewicz multi-continued fraction (L-Fraction or d-Lukasiewicz fraction with $d = 3$). From this multi-continued fraction, we can extract the following q-equation system satisfied by $p(x)$:

$$\begin{cases} p(x) = \dfrac{xyq(1-yq(1+q-xq)m(x)+y^2q^3(1-xq))m(x)m(xq)}{(1-qx)(1-m(x)(yq(2+q)+m(xq)(y^2q^2(1+2q-xq^2)-y^3q^4(1-xq^2)m(xq^2))))} \\ 1/m(x) = 1-(xq(1+q-xq^2)-qy(2+q-xq)) \\ \qquad -yq(1-xq)\left(yq(1+2q)+(1-xq^2)(2+q)\right)m(xq) \\ \qquad\quad -y^2q^2(1-xq^2)(1-xq)\left(xq^3-yq^2-1-2q\right)m(xq)m(xq^2) \\ \qquad\qquad -y^3q^4(1-xq^3)(1-xq^2)(1-xq)m(xq)m(xq^2)m(xq^3) \end{cases}$$

Setting $x = 1$ and $y = 1$ in the multi-continued fraction, we obtain the area generating function (Lin and Chang [13])

$$p(q) = \frac{q^2}{\sqrt{1-4q}}.$$

5 Conclusion

In order to solve the algebraic q-equation $P(X(x,s,q)) = 0$, we can examine solutions of the form $X(x,s,q) = f(x,sq,q)/f(x,s,q)$. Then, in P, $X(x,s,q)X(x,sq,q)\cdots X(x,sq^{k-1},q)$ is replaced by $f(x,sq^k,q)/f(x,s,q)$. We therefore linearize the equation $P(X(x,s,q)) = 0$, as we get a linear q-equation expressing the generating function $f(x,s,q)$ in terms of a given formal power series in the variables x,s,q, and of the functions $f(x,sq^k,q)$, for a finite set of integers $k \geq 1$. In special cases, this equation can be solved by the Bousquet-Melou iterative method [2]. Note also that T. Prellberg and R. Brak present in [16] an explicit solution for quadratic q-equation.

Thus, in all the cases we consider here, the generating function $X(x,s,q)$ depends of a function $f(x,s,q)$ that verifies a linear q-equation expressing $f(x,s,q)$ in terms of the functions $f(x,sq^k,q)$, for a finite set of integers $k \geq 1$.

References

[1] V. K. Bhat, H. L. Bhan, and Y. Singh. Enumeration of directed compact site animals in two dimensional in two dimensions. *J. Phys. A: Math. Gen.*, 19:3261–3265, 1986.

[2] M. Bousquet-Mélou. A method for the enumeration of various classes of column-convex polygons. *Discrete Mathematics*, 154:1–25, 1996.

[3] M. Bousquet-Mélou. Rapport scientifique. Rapport d'habilitation à diriger des recherches, LaBRI, Université de Bordeaux 1, 1996.

[4] M. Bousquet-Mélou and J.-M. Fédou. The generating function of convex polyominoes: the resolution of a q-diferential system. *Discrete Math.*, 137:53–75, 1995.

[5] M. Bousquet-Mélou and X. G. Viennot. Empilements de segments et q-énumération de polyominos convexes dirigés. *J. Combin. Theory Ser. A*, 60:196–224, 1992.

[6] M.-P. Delest. Generating functions for column-convex polyominoes. *J. Combin. Theory A*, 48:12–31, 1988.

[7] M.-P. Delest and S. Dulucq. Enumeration of directed column-convex animals with a given perimeter and area. *Croat. Chem. Acta*, 66:59–80, 1993.

[8] I. Dutour. Grammaires d'objets : énumération, bijections et génération aléatoire. Thèse de doctorat, LaBRI, Université de Bordeaux 1, 1996.

[9] J.-M. Fédou. Grammaires et q-énumérations de polyominos. Thèse de doctorat, LaBRI, Université de Bordeaux 1, 1989.

[10] S. Feretić and D. Svrtan. Combinatorics of diagonally convex directed polyominies. *Discrete Mathematics*, 157:147–168, 1996.

[11] S. Feretić and D. Svrtan. On the number of column-convex poyominoes with given perimeter and number of columns. in: A. Barlotti, M. Delest and R. Pinzani eds., Proc. 5th Conf. Formal Power Series and Algebraic Combinatorics, Florence, Italy, June 1993.

[12] D. A. Klarner. Some results concerning polyominoes. *Fibonacci Quart.*, 3:9–20, 1965.

[13] K.Y. Lin and S.J. Chang. Rogorous results for the number of convex polygons on the square and honeycomb lattices. *J. Phys. A: Math. Gen.*, 21:2635–2642, 1988.

[14] J.-G. Penaud. Animaux dirigés diagonalement convexes et arbres ternaires. Technical report, LaBRI, Université de Bordeaux 1, 1990.

[15] T. Prellberg. Uniform q-series asymptotics for staircase polygons. *J. Phys. A: Math. Gen.*, 28:1289–1304, 1995.

[16] T. Prellberg and R. Brak. Critical exponents from nonlinear functional equations for partially directed cluster models. *J. Stat. Phys.*, 78:701–730, 1995.

[17] V. Privman and N. M. Švrakić. Exact generating function for fully directed compact lattice animals. *Phys. Rev. Lett.*, 60:1107–1109, 1988.

[18] E. Roblet. Une interprétation combinatoire des approximants de Padé. Thèse de doctorat, LaBRI, Université de Bordeaux 1, 1994.

[19] H.N.V. Temperley. Combinatorial problems suugested by the statistical mechanics of domains and rubber-like molecules. *Phys. Rev.*, 103:1–16, 1956.

[20] X. G. Viennot. Une théorie combinatoire des polynômes orthogonaux. Notes de conférences données à l'UQAM (1983); réedition : hors-série n^{0}1, Publications du LACIM, Université du Québec à Montréal, 1991.