Some e-positivity Results for the Chromatic Symmetric Function Obtained by Deletion-Contraction Techniques.

David D. Gebhard
Dr. Bruce Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824 U.S.A.

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Abstract

In his paper [6] R.P. Stanley introduces a symmetric function in commuting indeterminates for a graph G, which he calls X_G. This chromatic symmetric function is a generalization of the chromatic polynomial of a graph. In previous work [4] we have shown that an analogue of the chromatic symmetric function, Y_G, in noncommuting variables allows us to use deletion-contraction techniques to obtain results for Y_G and also for X_G. In this paper we continue using these techniques in order to show some new e-positivity results for X_G.

Résumé

Dans son article [6], R. P. Stanley a associé à chaque graphe, G, une fonction symmétrique, X_G, dans des variables commutatives qui généralise le polynôme chromatique de G. Dans [4] nous avons démontré qu'il y a un analogue, Y_G, de X_G dans des variables noncommutatives qui nous permet d'obtenir des résultats conçernant Y_G, et donc X_G, utilisant suppression-contraction. Dans ce travail nous continuons notre exploitation de cette méthode pour démontrer quelques nouveaux résultats sur le e-positivité de X_G.

1 Introduction

In "A Symmetric Function Generalization of the Chromatic Polynomial of a Graph" [7] (see also [6]), R. P. Stanley introduces a symmetric function, X_G, associated with a labeled (loopless) graph on d vertices as follows: Let G have vertex set $V(G) = \{v_1, v_2, \ldots, v_d\}$. Then for the commuting indeterminates $x_1, x_2, \ldots$ define a homogeneous function of degree d,

$$X_G = X_G(x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(v_1)} \ldots x_{\kappa(v_d)} = \sum_{\kappa} x_\kappa$$

where the sum ranges over all proper colorings, $\kappa : V(G) \to \mathbb{P}$. It is clear from the definition that X_G is a symmetric function. It is also a generalization of the chromatic polynomial,

$\mathcal{X}_G(n)$. In this paper we define an analogue of X_G which is a symmetric function in *non-commutative* variables. That is, for any labeled multigraph G (which may have loops) with $V(G) = \{v_1, v_2, \ldots, v_d\}$ we define the analogue of X_G as

$$Y_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)}$$

where again the sum is over all proper colorings of G, but the x_i are now *noncommuting* variables. We notice that if G has loops this sum is empty and so we would have $Y_G = 0$.

The reason for using *noncommuting* variables is so we can keep track of the color which κ assigns to each vertex. We wish to define an operation, $\uparrow$, on these non-commutative symmetric functions that will allow the use of deletion-contraction techniques on Y_G.

Before we begin with the results for Y_G, we need to have some basic background for the symmetric functions in noncommuting variables. In order to make this paper largely self-contained, some basic results from [4] will be repeated in this section.

The noncommutative symmetric functions will be indexed by elements of the partition lattice. We let Π_d denote the lattice of set partitions of $\{1, 2, \ldots, d\} = [d]$, ordered by refinement. The meet (greatest lower bound) of the elements σ and τ is denoted by $\sigma \wedge \tau$. The unique minimal element of Π_d is denoted by $\hat{0}$ and the unique maximal element is denoted by $\hat{1}$.

For $\pi \in \Pi_d$ we define $\lambda(\pi)$ to be the integer partition of d whose parts are the block sizes of π. So if $\pi = B_1/B_2/\ldots/B_k$, where the block sizes are in weakly decreasing order, we have $\lambda(\pi) = (|B_1|, |B_2|, \ldots, |B_k|)$. Further, if r_i is the number of blocks of π with i elements we may denote $\lambda(\pi) = (1^{r_1}, 2^{r_2}, \ldots, d^{r_d})$, and define $|\pi| = r_1! r_2! \cdots r_d!$ and $\pi! = 1!^{r_1} 2!^{r_2} \cdots d!^{r_d}$.

We define the *noncommutative monomial symmetric functions* m_π by:

$$m_\pi = \sum_{\{\{i_1, i_2, \ldots, i_d\}\}} x_{i_1} x_{i_2} \cdots x_{i_d} \tag{1}$$

where the sum is over all multisets $\{\{i_1, i_2, \ldots, i_d\}\}$ of the positive integers $\mathbb{P}$ such that $i_j = i_k$ if and only if j and k are in the same block of π, and where the x_i are noncommuting variables.

We notice, from [2] that letting the x_i commute transforms m_π into $|\pi| m_{\lambda(\pi)}$, where $m_{\lambda(\pi)}$ is the usual commutative monomial symmetric function. Since the set $\{m_\pi\}_{\substack{\pi \in \Pi_d \\ d \in \mathbb{N}}}$ is linearly independent over the rationals, they are a basis for their span, the *noncommutative symmetric functions.*

We can also obtain (see [2]) noncommutative versions of the power sum symmetric functions and the elementary symmetric functions given respectively by:

$$p_\pi = \sum_{\sigma \geq \pi} m_\sigma \quad \text{and} \quad e_\pi = \sum_{\sigma : \sigma \wedge \pi = \hat{0}} m_\sigma. \tag{2}$$

With these definitions one may derive the change of basis formulae found in the appendix of Doubilet's paper [2], and so these also form bases for the noncommutative symmetric functions. It is also not hard to see from these definitions that we have

$$p_\pi = \sum_{\{\{i_1,i_2,\ldots,i_d\}\}} x_{i_1}x_{i_2}\cdots x_{i_d} \tag{3}$$

where the sum is over all multisets $\{\{i_1, i_2, \ldots, i_d\}\}$ of $\mathbb{P}$ such that $i_j = i_k$ if j and k are in the same block of π. We also have a similar type of expression for e_π.

If we allow the variables to commute we transform p_π into $p_{\lambda(\pi)}$ and e_π into $\pi! e_{\lambda(\pi)}$. It should be clear that these noncommutative symmetric functions are symmetric in the usual sense.

2 On to Graphs

In this section we wish to notice that Y_G satisfies a deletion-contraction recurrence, and prove a version of Whitney's broken circuit theorem to demonstrate how it works. In order to get such a recurrence, however, it is necessary to have a distinguished edge, e.

For $\delta \in S_d$, we define $\delta \circ m_\pi = m_{\delta(\pi)}$, where the action of $\delta \in S_d$ on a set partition of $[d]$ is the obvious one which acts on the elements of the blocks. It follows that for any δ this action induces a vector space isomorphism, since it merely produces a permutation of the basis elements. To uniformly choose the distinguished edge, we also need to define the action of the symmetric group on a graph. For all $\delta \in S_d$ we let δ act on the vertices of G by $\delta(v_i) = v_{\delta(i)}$. This creates the action on graphs given by $\delta(G) = H$, where H is just a relabeling of G.

Proposition 1 (Relabeling Proposition) *Let G and H be two labelings for a given graph with $H = \delta(G)$. Then $Y_H = \delta \circ Y_G$.* ■

We now turn our attention to deletion-contraction techniques. Using the Relabeling Proposition allows us, without loss of generality, to choose a labeling of G such that we have our distinguished edge given by $e = v_{d-1}v_d$. We use this edge to derive the deletion-contraction recurrence Y_G satisfies.

To this end, we define the operation, $\uparrow$, on homogeneous noncommutative symmetric functions of degree $d-1$ which simply repeats the variable $x_{i_{d-1}}$ in the d^{th} position. More formally, for a monomial $x_{i_1}x_{i_2}\cdots x_{i_{d-2}}x_{i_{d-1}}$, define

$$(x_{i_1}x_{i_2}\cdots x_{i_{d-2}}x_{i_{d-1}})\!\uparrow = x_{i_1}x_{i_2}\cdots x_{i_{d-2}}x_{i_{d-1}}^2$$

and extend linearly.

This is a specific usage that we will only use for deletion-contraction techniques on the edge $e = v_d v_{d-1}$. We can extend the definition of $\uparrow$ to any edge $e = v_k v_l$: For $k < l$, let the operation $\uparrow_k^l$ be defined by

$$(x_{i_1}\cdots x_{i_k}\cdots x_{i_{l-1}}x_{i_l}\cdots x_{i_m})\!\uparrow_k^l = x_{i_1}\cdots x_{i_k}\cdots x_{i_{l-1}}x_{i_k}x_{i_l}\cdots x_{i_m}$$

and extend linearly.

Proposition 2 (Deletion-Contraction)[4] *For $e = v_{d-1}v_d$, we have $Y_G = Y_{G\setminus e} - Y_{G/e}\uparrow$* ∎

The Deletion-Contraction Proposition for Y_G gives us a tool for using induction. Now, for $S \subseteq E(G)$ let $\pi(S)$ denote the partition of $\{1, 2, \ldots, d\}$ with blocks corresponding to the connected components of S. We utilize the notation that for any fixed $\pi \in \Pi_d$ we let $\pi + (d+1)$ denote the partition of $[d+1]$ formed by inserting the element $d+1$ into the block of π which contains d. We will denote the block of π which contains d by B_π. We also let $\pi/d+1$ be the partition of $[d+1]$ formed by adding the block $\{d+1\}$ to π. From (1) and (3) it should be easy to see that for $\pi \in \Pi_{d-1}$, $m_\pi\uparrow = m_{\pi+(d)}$ and $p_\pi\uparrow = p_{\pi+(d)}$. With this notation we will be able to prove an analogue of Whitney's Broken Circuit Theorem [10]. Following the notation of [6] we will let B_G be the broken circuit complex of G.

Proposition 3 *We have*

$$Y_G = \sum_{S \in B_G} (-1)^{|S|} p_{\pi(S)}$$

where $\pi(S)$ is the partition whose parts are the connected components of the spanning subgraph induced by S.

Proof: We use the fact that there is a bijection between B_G and $B_{G\setminus e} \cup B_{G/e}$ given by

$$S \longrightarrow \begin{cases} \tilde{S} = S \setminus e \in B_{G/e} & \text{if } e \in S \\ \tilde{S} = S \in B_{G\setminus e} & \text{otherwise} \end{cases}$$

where we take e to be the first edge of G in the total order on the edges .

We now induct on the number of non-loops in$G$. If $n = 0$ the statement is easy to verify. For $n > 0$ we consider $Y_G = Y_{G\setminus e} - Y_{G/e}\uparrow$, where $G \setminus e$ and G/e both have one less edge than G, and so induction applies. We can easily see that we have the identification

$$\sum_{\substack{S \in B_G \\ e \notin S}} (-1)^S p_{\pi(S)} = \sum_{\tilde{S} \in B_{G\setminus e}} (-1)^S p_{\pi(S)}$$

and

$$\sum_{\substack{S \subseteq E(G) \\ e \in S}} (-1)^{|S|} p_{\pi(S)} = \sum_{\tilde{S} \in B_{G/e}} (-1)^{|\tilde{S}|} p_{\pi(\tilde{S})}\uparrow,$$

which gives the result. ∎

3 Elementary Symmetric Function Expansions

We now turn our attention to the expansion of Y_G in terms of the elementary symmetric function basis. An interesting question concerns the coefficients of Y_G. We will say that

Y_G (or just G) is e-positive, if the coefficients in the expansion of Y_G in the elementary symmetric function basis are all non-negative. In order obtain information about these coefficients using deletion-contraction techniques, it is necessary for us to have information about the coefficients of $e_\pi\uparrow$. It was noted before that for $\pi \in \Pi_{d-1}$, $p_\pi\uparrow = p_{\pi+(d+1)}$. We utilize the change of base formulae (see [2]) to obtain

$$e_\pi\uparrow = \sum_{\sigma\leq\pi} \frac{\mu(\hat{0},\sigma)}{\mu(\hat{0},\sigma+(d))} \sum_{\tau\leq\sigma+(d)} \mu(\tau,\sigma+(d))e_\tau. \qquad (4)$$

This expression does not immediately lend itself to convenient computations of the coefficients of $e_\pi\uparrow$. However, if the terms are grouped properly, many of the coefficients sum to zero. To motivate this result, we can use the following lemma.

Lemma 1 *For $\pi \in \Pi_d$, express $e_\pi\uparrow = \sum_{\tau\in\Pi_{d+1}} c_\tau e_\tau$. Then*

$$\sum_{\substack{\tau\in\Pi_{d+1}\\ \lambda(\tau)\text{ fixed}}} c_\tau = \begin{cases} \frac{1}{|B_\pi|} & \textit{if } \lambda(\tau) = \lambda(\pi/d+1) \\ \frac{-1}{|B_\pi|} & \textit{if } \lambda(\tau) = \lambda(\pi+(d+1)) \\ 0 & \textit{else.} \end{cases}$$

Proof. Let $\pi \in \Pi_n = B_1/B_2/\ldots/B_\pi/\ldots/B_k$. We also let b_i denote the size of the block B_i. We may now consider the graph on $d+1$ vertices given by

$$G_\pi = K_{B_1} \uplus K_{B_2} \uplus \cdots \uplus K_{B_\pi} + e \uplus \cdots \uplus K_{B_k},$$

where $K_{|B_i|}$ is the complete graph on b_i vertices labeled with the elements of B_i, and where $K_{B_\pi} + e$ is the complete graph on b_π vertices labeled with the elements of B_π which also contains an additional vertex labeled v_{n+1}, and has the additional edge $e = v_n v_{n+1}$.

By the recurrence relation for Y_G, we can see that $Y_{G_\pi} = Y_{G_\pi - e} - Y_{G_\pi/e}\uparrow$. Equivalently, this gives us that $Y_{G_\pi/e}\uparrow = Y_{G_\pi-e} - Y_{G_\pi}$. It is easy to see that $Y_{G_\pi/e} = e_\pi$, and so this gives us that $e_\pi\uparrow = Y_{G_\pi-e} - Y_{G_\pi}$. Finally, if we let C be the operator which allows the variables to commute, we get $C(e_\pi\uparrow) = X_{G_\pi-e} - X_{G_\pi}$.

We now proceed to calculate $X_{G_\pi-e}$ and X_{G_π}. Using the product rule for X_G [7], it is not hard to show that $X_{G_\pi-e} = \pi! e_{(\lambda(\pi),1)}$. To calculate X_{G_π} we again will use the product rule, together with the fact (see [6]) that

$$X_{K_{B_\pi}+e} = \sum_{\lambda\vdash d+1} a_\lambda |\lambda| m_\lambda,$$

where a_λ is the number of stable partitions of the vertex set of $K_{B_\pi}+e$ of type λ. Letting $\mu = B_1/B_2/\cdots/\hat{B}_d/\cdots/B_k$, and $q = |B_\pi|$, we can easily count the number of stable partitions of $K_{B_\pi+e}$ to obtain the equation

$$X_{G_\pi} = \mu! e_\mu \left[(q-1)(q-1)! m_{(1^{q-1},2)} + (q+1)! m_{(1^{q+1})}\right].$$

We may then use the fact that $m_{(1^{q-1},2)} = e_{(q,1)} - (q+1)e_{(q+1)}$ and $m_{(1^{q+1})} = e_{(q+1)}$ and combine this with the previous results to obtain:

$$C(e_\pi\uparrow) = \pi! e_{(\lambda(\pi),1)} - \mu! e_\mu \left[(q-1)(q-1)! \left[e_{(q,1)} - (q+1)e_{(q+1)}\right] + (q+1)! e_{(q+1)}\right].$$

Following simplification, this leads to :

$$C(e_\pi\uparrow) = \frac{(\pi/d+1)!}{q} e_{(\lambda(\pi),1)} - \frac{(\pi+(d+1))!}{q} e_{\lambda(\pi+(d+1))}. \tag{5}$$

This essentially completes the proof, for if $e_\pi\uparrow = \sum_{\tau\in\Pi_{d+1}} c_\tau e_\tau$, then $C(e_\pi\uparrow) = \sum_{\tau\in\Pi_{d+1}} c_\tau C(e_\tau)$. Hence $\tau! \sum_{\substack{\tau\in\Pi_{d+1} \\ \lambda(\tau)\ \text{fixed}}} c_\tau$ is the coefficient of $e_{\lambda(\tau)}$ in $C(e_\pi\uparrow)$. Putting this together with equation (5) gives the result. ■

This lemma show us that the terms can be combined in a nice way. We need to group the coefficients from partitions which are of the same type, and whose block containing d have the same size. We need to show, however, that there is a pattern here which will allow us to repeatedly use deletion-contraction techniques. We see that the contributing coefficients of $e_\pi\uparrow$ will have type $\lambda(\pi+(d+1))$ or $\lambda(\pi/d+1)$. If we want to be able to repeatedly use deletion-contraction, though, it will be necessary to know the **size** of the block of $e_\pi\uparrow$ which contains $d+1$. So we will let $P(\alpha) = P(\alpha_1, \alpha_2, \ldots, \alpha_l)$ be the set of all partitions of $[d+1]$ which are less than or equal to $\pi + (d+1)$, have blocks of size $\alpha_1, \alpha_2, \ldots, \alpha_l$, and for which $d+1$ is in a block of size α_1. That this is the proper grouping of the terms of $e_\pi\uparrow$ is shown by the following lemma.

Lemma 2 *If* $e_\pi \uparrow = \sum_{\tau\leq\pi+(d+1)} c_\tau e_\tau$, *then for any* $\alpha \vdash (d+1)$, *we have*

$$\sum_{\tau\in P(\alpha)} c_\tau = \begin{cases} 1/|B_\pi| & \text{if } P(\alpha) = \{\pi/d+1\} \\ -1/|B_\pi| & \text{if } P(\alpha) = \{\pi+(d+1)\} \\ 0 & \text{else.} \end{cases}$$

■

However, even for some of the simplest graphs, Y_G is not often $e-$positive. The only graphs which seem to be obviously $e-$positive are the complete graphs on n vertices, K_n, for which we have $Y_{K_n} = e_{[n]}$ and $\overline{K}_n$ which yields $Y_{\overline{K}_n} = e_{1/2/\cdots/n}$. Even paths, with the vertices labeled sequentially, are not $e-$positive, for we can compute that $Y_{P_3} = \frac{1}{2}e_{12/3} - \frac{1}{2}e_{13/2} + \frac{1}{2}e_{1/23} + \frac{1}{2}e_{123}$. However, in this example we can notice that while Y_{P_3} is not $e-$positive, if we group the terms according to their type and the size of the block which contains 3, the sum of all the coefficients of this type are always non-negative. This observation along with the proof of the previous lemma inspires us to introduce equivalence classes of this form. Hence, we define

$$\sigma \equiv_i \tau \text{ iff } \lambda(\sigma) = \lambda(\tau) \text{ and } |B_{\sigma,i}| = |B_{\tau,i}|$$

where $B_{\sigma,i}$ is the block of σ containing i and $B_{\tau,i}$ is the block of τ containing i. In a similar manner, we can define

$$e_\sigma \equiv_i e_\tau \quad \text{iff} \quad \sigma \equiv_i \tau.$$

We will let the notation (τ) denote the equivalence class of $\tau \in \Pi_d$ and we group all the terms in (τ) together, we have the formal sum

$$Y_G \equiv_i \sum_{(\tau)\subseteq \Pi_d} c_{(\tau)} e_{(\tau)} \quad \text{where} \quad c_{(\tau)} = \sum_{\sigma \in (\tau)} c_\sigma,$$

and $e_{(\tau)}$ is simply a placeholder for reference.

We need to note here that it can be shown using this notation and Lemma 2:

Corollary 1 *For $\pi \in \Pi_d$,*

$$e_\pi\uparrow \equiv_{d+1} \frac{1}{k} e_{(\pi/d+1)} - \frac{1}{k} e_{(\pi+(d+1))} \quad \textit{and} \quad e_\pi\uparrow \equiv_d \frac{1}{k} e_{(\tilde{\pi}/d)} - \frac{1}{k} e_{(\tilde{\pi}+(d))},$$

where $\tilde{\pi} = \pi$ with d replaced by $d+1$, $k = |B_\pi|$, and $\tilde{\pi} + (d)$ has the obvious meaning. ■

In addition, we can show that for $\tau \equiv_d \sigma$, $e_\tau\uparrow \equiv_{d+1} e_\sigma\uparrow$, and so the congruence respects induction.

Using this notation, we have $Y_{P_3} \equiv_3 \frac{1}{2} e_{(12/3)} + \frac{1}{2} e_{(123)}$, since $e_{13/2} \equiv_3 e_{1/23}$. We will say that a labeled graph G is $(e)-$*positive* if $c_{(\tau)}$ is positive for all $(\tau) \subseteq \Pi_n$ for some labeling of G and suitably chosen congruence. We notice that the (e)-positivity of a graph is not dependent on the labeling of the graph. If a different labeling is chosen, then we need only change the congruence class appropriately, and the graph will again be seen to be (e)-positive. This should be clear from the Relabeling Proposition.

In order to prove some of these results we need one more simple lemma.

Lemma 3 *Let H be a graph with $V(H) = V(G) \uplus \{d+1\}$ and $E(H) = E(G)$. If $Y_G = \sum_{\sigma \in \Pi_d} c_\sigma e_\sigma$, then $Y_H = \sum_{\sigma \in \Pi_d} c_\sigma e_{\sigma/d+1}$.* ■

Proposition 4 *For all $n \geq 1$, Y_{P_n} labeled so that $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$ is $(e)-$positive .*

Proof. We proceed by induction, using Lemma 2 and Lemma 3 above.

If $n = 1$, then we have $Y_{P_1} = e_1$ and the proposition is true.

We now assume the result for Y_{P_n}. We also note here that from Lemma 2, it should be clear that for $\tau \in \Pi_n$,

$$e_{(\tau)}\uparrow \equiv_{n+1} (1/|B_\tau|)\, e_{(\tau/n+1)} + (-1/|B_\tau|)\, e_{(\tau+(n+1))}.$$

So we assume by induction that

$$Y_{P_n} \equiv_n \sum_{(\tau)\in\Pi_n} c_{(\tau)} e_{(\tau)},$$

where $c_{(\tau)}$ is at least zero for all $(\tau) \in \Pi_n$. From our deletion-contraction recurrence, and Lemma 2, we see that if we let $e = v_n v_{n+1}$, then

$$\begin{aligned} Y_{P_{n+1}} &= Y_{P_n/n+1} - Y_{P_n}\uparrow \\ &\equiv_{n+1} \sum_{(\tau)\in\Pi_n} c_{(\tau)} e_{(\tau/n+1)} - \sum_{(\tau)\in\Pi_n} c_{(\tau)} e_{(\tau)}\uparrow \\ &\equiv_{n+1} \sum_{(\tau)\in\Pi_n} c_{(\tau)} e_{(\tau/n+1)} - \sum_{(\tau)\in\Pi_n} \frac{c_{(\tau)}}{|B_\tau|} e_{(\tau/n+1)} + \sum_{(\tau)\in\Pi_n} \frac{c_{(\tau)}}{|B_\tau|} e_{(\tau+(n+1))}. \end{aligned}$$

Since we know that all the $c_{(\tau)}$ are positive, and $|B_\tau| \geq 1$, this completes the induction step and the proof. ■

It is also easy to see that we can use (e)-positivity results for any Y_G and specialize them to the commutative case for X_G. Hence,

Corollary 2 *X_{P_n} is $e-$positive.* ■

It is then natural to ask if cycles will also be $(e)-$positive, for when we delete the edge of a cycle we obtain a path. While it is true that cycles will be $(e)-$positive, an even stronger relationship exists between paths and cycles.

Proposition 5 *If $Y_{P_n} \equiv_n \sum c_{(\tau)} e_{(\tau)}$, then $Y_{C_{n+1}} \equiv_{n+1} \sum c_{(\tau)} e_{(\tau+(n))}$, where the labelings satisfy $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$ and $E(C_{n+1}) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_{n+1}, v_{n+1}v_1\}$.*

Corollary 3 *For all $n \geq 1$, Y_{C_n} is $(e)-$positive, and so X_{C_n} is e-positive.* ■

The fact that paths can be shown to be (e)-positive in this manner gives us some hope for what is known as the (**3+1**)-free Conjecture of Stanley.and Stembridge. [8, 7, 1] If we let **a+b** be the poset which is a disjoint union of an a-element chain and a b-element chain, then the poset P is said to be (**a+b**)-free if it contains no induced subposet isomorphic to **a+b**. Let $G(P)$ denote the incomparability graph of P.

Conjecture 1 *If P is (**3+1**)-free, then $X_{G(P)}$ is e-positive.* ■

A special case of this conjecture is the class of *indifference graphs*. These graphs can be characterized (see [7]) as graphs whose vertex set is $V = [d]$, and whose edge set is $E = \{uv : u, v \text{ belong to some } I \in \mathcal{C}\}$, where $\mathcal{C}$ is just a collection of intervals $[i, j] = \{i, i+1, \ldots, j\}$ in $[d]$.

This characterization gives us a very nice structure for our deletion-contraction techniques. Our method does show that indifference graphs of a certain type, which we will call $K_\lambda - chains$, are (e)-positive. For any composition of n, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, let $\tilde{\lambda_i} = \sum_{j \leq i} \lambda j$. A $K_\lambda - chain$ is the graph on n vertices which is an indifference graph using the collection of intervals $\{[1, \tilde{\lambda_1}], [\tilde{\lambda_1}, \tilde{\lambda_2}], \ldots, [\tilde{\lambda_{k-1}}, \tilde{\lambda_k}]\}$. So this is really just a string of complete graphs, whose sizes are given by the parts of λ attached to one another sequentially by sharing a vertex. We notice that the K_λ-chain for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ can be obtained from the K_τ-chain for $\tau = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1})$ by attaching the graph K_{λ_k} to it at its last vertex.

Using this attachment idea, define $G + K_m$ to have $V(G + K_m) = \{v_1, \ldots, v_{d+m-1}\}$ and $E(G + K_m) = E(G) \cup \{e = v_i v_j : d \leq i < j \leq d + m - 1\}$. With deletion-contraction techniques, we are able to obtain information about the (e)-expansion of $G + K_m$ in terms of the coefficients in the (e)-expansion of G. For $\pi \in \Pi_d$ we will use $\pi + i$ to denote the partition given by π with an additional i elements added to B_π, We also let $\pi + (i)$ denote the partition given by π with the element i inserted into B_π. The following two lemmas show how the coefficients of Y_G and Y_{G+K_m} are related.

Lemma 4 *If G is a graph on d vertices with*

$$Y_G \equiv_d \sum_{(\pi) \in \Pi_d} c_{(\pi)} e_{(\pi)},$$

then

$$Y_{G+K_m} \uparrow_{d,d+m} \equiv_{d+m} \sum_{(\pi)} \sum_{i=0}^{m-1} \frac{c_{(\pi)} < m-1 >_i \left(e_{(\pi+i/d+i+1,\ldots,d+m)} - e_{(\pi+i+(d+m)/d+i+1,\ldots,d+m-1)} \right)}{(k)_{i+1}},$$

where $k = |B_\pi|$. ■

Lemma 5 *If $Y_G \equiv_d \sum_{(\pi) \in \Pi_d} c_{(\pi)} e_{(\pi)}$, then*

$$Y_{G+K_{m+1}} \equiv_{d+m} \sum_{(\pi) \subseteq \Pi_d} \sum_{i=0}^{m-1} \frac{c_{(\pi)} < m-1 >_i}{(k)_{i+1}} \left((i+1) e_{(\overline{\pi})} + (k - m + i) e_{(\tilde{\pi})} \right)$$

where

$$\overline{\pi} = \pi + i + (d+m)/d + i + 1, \ldots, d + m - 1),$$
$$\tilde{\pi} = \pi + i/d + i + 1, \ldots, d + m \text{ and where } k = |B_\pi|.$$

■

Examining this lemma, we can see that in $Y_{G+K_{m+1}}$ we have the same sign on all coefficients as we have in Y_G, with the possible exception of the terms where $k < m - i$ in the sum.

With little effort, can see that these coefficients are actually zero. This means that all the coefficients in $Y_{G+K_{m+1}}$ have the same sign as the corresponding coefficients in Y_G. So clearly, if Y_G is (e)-positive, then $Y_{G+K_{m+1}}$ is also (e)-positive. This is our main result.

Theorem 1 *If Y_G is (e)-positive, then Y_{G+K_m} is also (e)-positive.* ■

In particular, we can show:

Corollary 4 *If G is a K_λ-chain, then Y_G is (e)-positive and so X_G is e-positive.* ■

References

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