THE ROGERS-RAMANUJAN IDENTITIES, THE FINITE GENERAL LINEAR GROUPS, AND THE HALL-LITTLEWOOD POLYNOMIALS

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Abstract: The Rogers-Ramanujan identities have been studied from the viewpoints of combinatorics, number theory, affine Lie algebras, statistical mechanics, and conformal field theory. This note connects the Rogers-Ramanujan identities with the finite general linear groups and the Hall-Littlewood symmetric polynomials. The $n \to \infty$ probability that an element of $GL(n,q)$ or $Mat(n,q)$ is semisimple is also computed.

Résumé: Les équations de Rogers-Ramanujan ont été étudiées du point de vue de la combinatoire, de la théorie des nombres, de l'algèbre affine de Lie, de la physique statisque, et de la théorie des champs conformels. Cet article relie les équations de Rogers-Ramanujan aves les groupes linéaires finis généraux et les polynômes symétriques de Hall-Littlewood. On y calcule aussi la probabilité $n \to \infty$ qu 'un élément de $GL(n,q)$ ou de $Mat(n,q)$ soit semi-simple.

1 Introduction

The Rogers-Ramanujan identities are among the most famous partition identities in number theory and combinatorics. This note will be concerned with the following generalization of the Rogers-Ramanujan identities, due to Gordon. Let $(x)_n$ denote $(1-x)(1-x^2)\cdots(1-x^n)$.

Theorem 1 *(Andrews [1], page 111) For $1 \leq i \leq k, k \geq 2$, and $|x| < 1$*

$$\sum_{n_1,\cdots,n_{k-1} \geq 0} \frac{x^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(x)_{n_1}\cdots(x)_{n_{k-1}}} = \prod_{\substack{r=1 \\ r \neq 0, \pm i (mod\ 2k+1)}}^{\infty} \frac{1}{1-x^r}$$

where $N_j = n_j + \cdots n_{k-1}$.

Gordon's generalization of the Rogers-Ramanujan identities has been widely studied and appears in many places in mathematics and physics. Andrews [1] discusses combinatorial aspects of these identities. Berndt [4] describes some number theoretic connections. Feigen and Frenkel [6] interpret the Gordon identities as a character formula for the Virasoro algebra. Andrews, Baxter, and Forrester [2] and Warnaar [15] relate Gordon's generalization to statistical mechanics. Berkovich, McCoy, and Orrick [3] and Kirillov [9] discuss the connection with conformal field theory.

We use the following standard notation from the theory of partitions. Call λ a partition of $n = |\lambda|$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\sum_i \lambda_i = n$. The λ_i are referred to as the parts of λ. Let $m_i(\lambda)$ be the number of parts of λ of size i, and set $\lambda_i' = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$. Define $n(\lambda)$ by $\sum_{i \geq 1}(i-1)\lambda_i$. Let $[u^n]f(u)$ denote the coefficient of u^n in a power series $f(u)$.

2 Main Results

To begin recall the Hall-Littlewood polynomials (page 208 of Macdonald [12]). Let the permutation w act on variables $x_1, x_2, \cdots$ by sending x_i to $x_{w(i)}$. There is also a coordinate-wise action of w on $\lambda = (\lambda_1, \cdots, \lambda_n)$ and S_n^λ is defined as the subgroup of S_n stabilizing λ in this action. Recall that $m_i(\lambda)$ is the number of parts of λ of size i. For a partition $\lambda = (\lambda_1, \cdots, \lambda_n)$ of length $\leq n$, two definitions of the Hall-Littlewood polynomials are

$$\begin{aligned} P_\lambda(x_1, \cdots, x_n; t) &= \frac{1}{\prod_{i \geq 0} \prod_{r=1}^{m_i(\lambda)} \frac{1-t^r}{1-t}} \sum_{w \in S_n} w(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j}) \\ P_\lambda(x_1, \cdots, x_n; t) &= \sum_{w \in S_n / S_n^\lambda} w(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{\lambda_i > \lambda_j} \frac{x_i - t x_j}{x_i - x_j}). \end{aligned}$$

At first glance it is not obvious that these are polynomials, but the denominators cancel out after the symmetrization. The Hall-Littlewood polynomials interpolate between the Schur functions ($t = 0$) and the monomial symmetric functions ($t = 1$).

Theorem 2 connects the Hall-Littlewood polynomials with Gordon's generalization of the Rogers-Ramanujan identities.

Theorem 2 *For $k \geq 2$,*

$$\sum_{\lambda: \lambda_1 < k} \frac{P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \cdots; \ frac1q)}{q^{n(\lambda)}} = \prod_{\substack{r=1 \\ r \neq 0, \pm k (mod \ \ 2k+1)}}^{\infty} (\frac{1}{1 - \frac{1}{q^r}})$$

$$\sum_{\lambda: \lambda_1 < k} \frac{P_\lambda(\frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \frac{1}{q^5}, \cdots; \frac{1}{q})}{q^{n(\lambda)}} = \prod_{\substack{r=1 \\ r \neq 0, \pm 1 (mod \ \ 2k+1)}}^{\infty} (\frac{1}{1 - \frac{1}{q^r}}).$$

PROOF: Macdonald's principal specialization formula (page 337 of Macdonald [12]) states that

$$P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \cdots; \frac{1}{q}) = \frac{1}{q^{|\lambda| + n(\lambda)}} \prod_i \frac{1}{(\frac{1}{q})_{m_i(\lambda)}}.$$

Combining this with the elementary fact that $n(\lambda) = \sum_i \binom{\lambda_i'}{2}$ shows that

$$\frac{P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \cdots; \frac{1}{q})}{q^{n(\lambda)}} = \frac{1}{q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}}.$$

The first equality of the theorem follows by applying Theorem 1 with $i = k, x = \frac{1}{q}, n_j = \lambda_j$ and $N_j = \lambda_j'$.

For the second equality, observe that

$$\begin{aligned} \frac{P_\lambda(\frac{u}{q}, \frac{u}{q^2}, \frac{u}{q^3}, \frac{u}{q^4}, \cdots; \frac{1}{q})}{q^{n(\lambda)}} &= u^{|\lambda|} \frac{P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \cdots; \ frac1q)}{q^{n(\lambda)}} \\ &= \frac{u^{|\lambda|}}{q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}}. \end{aligned}$$

Now set $u = \frac{1}{q}$ and apply Theorem 1 with $i = 1, x = \frac{1}{q}, n_j = \lambda_j$ and $N_j = \lambda'_j$. $\square$

Remark: Stembridge [13] used properties of the Hall-Littlewood polynomials as a tool in giving proofs of the Rogers-Ramanujan identities. However the statement of Theorem 2 seems to be a more direct connection between the Hall-Littlewood polynomials and the Rogers-Ramanujan identities.

Recall from elementary linear algebra (for instance from Chapter 6 of Herstein [7]) that the conjugacy classes of $GL(n,q)$ are parameterized by rational canonical form. This form corresponds to the following combinatorial data. To each monic non-constant irreducible polynomial ϕ over a field of q elements, associate a partition (perhaps the trivial partition) λ_ϕ of some non-negative integer $|\lambda_\phi|$. Let $deg(\phi)$ denote the degree of ϕ. The only restrictions necessary for this data to represent a conjugacy class are that $|\lambda_z| = 0$ and $\sum_\phi |\lambda_\phi| deg(\phi) = n$.

Definition For $\alpha \in GL(n,q)$ and ϕ a monic, irreducible polynomial over F_q, a field of q elements, define $\lambda_\phi(\alpha)$ to be the partition corresponding to the polynomial ϕ in the rational canonical form of α.

The following elementary lemmas will be of use.

Lemma 1 *If the Taylor series of $f(u)$ around 0 converges at $u = 1$, then*

$$lim_{n\rightarrow\infty}[u^n]\frac{f(u)}{1-u} = f(1).$$

PROOF: Write the Taylor expansion $f(u) = \sum_{n=0}^{\infty} a_n u^n$. Then observe that $[u^n]\frac{f(u)}{1-u} = \sum_{i=0}^{n} a_i$. $\square$

Lemma 2 *For $t \geq 1$ and q a prime power,*

$$\prod_{\phi \ irred.} (1 - \frac{u^{deg(\phi)}}{q^{deg(\phi)t}}) = 1 - \frac{u}{q^{t-1}}.$$

PROOF: Assume that $t = 1$, the general case following by replacing u by $\frac{u}{q^{t-1}}$. Expanding $\frac{1}{1-\frac{u^{deg(\phi)}}{q^{deg(\phi)}}}$ as a geometric series, unique factorization in $F_q[x]$ implies that the coefficient of u^d in the reciprocal of the left hand side is $\frac{1}{q^d}$ times the number of monic polynomials of degree d, hence 1. Comparing with the reciprocal of the right hand side proves the claim. $\square$

Lemma 3 *For $t \geq 1$ and q a prime power,*

$$1 - u = \prod_{\substack{\phi \neq z \\ \phi \ irred.}} \prod_{r=1}^{\infty} (1 - \frac{u^{deg(\phi)}}{q^{r \cdot deg(\phi)}}).$$

PROOF: By Lemma 2,

$$\prod_{r=1}^{\infty}(1 - \frac{u}{q^{r-1}}) = \prod_{\phi \ irred.} \prod_{r=1}^{\infty}(1 - \frac{u^{deg(\phi)}}{q^{r \cdot deg(\phi)}}).$$

The lemma follows by cancelling the terms corresponding to $\phi = z$. $\square$

Lemma 4

$$\sum_{\lambda} \frac{u^{|\lambda|}}{q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}} = \prod_{r=1}^{\infty} (\frac{1}{1-\frac{u}{q^r}})$$

PROOF: On page 225 of Macdonald [12] it is proved that

$$\prod_{r=1}^{\infty} (1-x_i) \sum_{\lambda} t^{n(\lambda)} P_{\lambda}(x_1, x_2, x_3, \cdots; t) = 1.$$

Applying Macdonald's principal specialization formula with $t = \frac{1}{q}, x_i = \frac{u}{q^i}$ as in Theorem 2 proves the result. □

Lemma 5 *(Euler) For $|q| > 1$ and $0 < u < 1$,*

1. $\prod_{r=1}^{\infty} (1 - \frac{u}{q^r}) = \sum_{n=0}^{\infty} \frac{(-u)^n}{(q^n-1)\cdots(q-1)}$
2. $\prod_{r=1}^{\infty} (\frac{1}{1-\frac{u}{q^r}}) = \sum_{n=0}^{\infty} \frac{u^n q^{\binom{n}{2}}}{(q^n-1)\cdots(q-1)}$.

Theorem 3 relates the Gordon identities with the finite general linear groups.

Theorem 3 *Let ϕ be a monic, irreducible polynomial over F_q. Let $k \geq 2$ be an integer. Then the $n \rightarrow \infty$ limit of the chance that a uniformly chosen element α of $GL(n,q)$ has the largest part of the partition $\lambda_{\phi}(\alpha)$ less than k is equal to*

$$\prod_{\substack{r=1 \\ r=0,\pm k (mod\ 2k+1)}}^{\infty} (1 - \frac{1}{q^{r \cdot deg(\phi)}}).$$

PROOF: Assume for simplicity of notation that $\phi = z - 1$. From the proof it will be clear that the general case follows.

Stong [14] used Kung's [10] formula for the sizes of the conjugacy classes of $GL(n,q)$ to find a "cycle index" for the general linear groups. Using the notation

$$d_i(\lambda) = 1m_1(\lambda) + 2m_2(\lambda) + \cdots + (i-1)m_{i-1}(\lambda) + i(m_i(\lambda) + m_{i+1}(\lambda) + \cdots + m_j(\lambda)),$$

he obtained the equality

$$1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{\alpha \in GL(n,q)} \prod_{\substack{\phi \neq z \\ \phi\ irred.}} x_{\phi, \lambda_{\phi}(\alpha)} = \prod_{\substack{\phi \neq z \\ \phi\ irred.}} \left[\sum_{\lambda} x_{\phi,\lambda} \frac{u^{|\lambda| deg(\phi)}}{\prod_i \prod_{k=1}^{m_i(\lambda)} (q^{deg(\phi) d_i} - q^{deg(\phi)(d_i - k)})} \right].$$

Observe that

$$\begin{aligned} \prod_i \prod_{k=1}^{m_i} (q^{d_i} - q^{(d_i-k)}) &= q^{\sum_i m_i(\lambda) d_i(\lambda)} \prod_i (\frac{1}{q})_{m_i(\lambda)} \\ &= q^{\sum_i m_i(\lambda)[(\sum_{h<i} h m_h(\lambda)) + i m_i(\lambda) + \sum_{i<k} i m_k(\lambda)]} \prod_i (\frac{1}{q})_{m_i(\lambda)} \end{aligned}$$

$$
\begin{aligned}
&= q^{\sum_i [i m_i(\lambda)^2 + 2 m_i(\lambda) \sum_{h<i} h m_h(\lambda)]} \prod_i (\frac{1}{q})_{m_i(\lambda)} \\
&= q^{\sum_i [\sum_{h<i} m_h(\lambda)]^2} \prod_i (\frac{1}{q})_{m_i(\lambda)} \\
&= q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}.
\end{aligned}
$$

Combining this observation with Lemma 3 shows that

$$
\begin{aligned}
&1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{\alpha \in GL(n,q)} \prod_{\substack{\phi \neq z \\ \phi\ irred.}} x_{\phi,\lambda_\phi(\alpha)} \\
= &\frac{1}{1-u} \prod_{\substack{\phi \neq z \\ \phi\ irred.}} \left[\prod_{r=1}^{\infty} (1 - \frac{u^{deg(\phi)}}{q^{r \cdot deg(\phi)}}) \right] \left[\sum_\lambda x_{\phi,\lambda} \frac{u^{|\lambda| deg(\phi)}}{q^{deg(\phi) \sum_i (\lambda_i')^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}} \right].
\end{aligned}
$$

Setting $x_{z-1,\lambda} = 0$ if the largest part of λ is greater than equal to k, and all $x_{\phi,\lambda} = 1$ otherwise shows by Lemma 4 that the sought probability is

$$
\lim_{n \to \infty} [u^n] \frac{1}{1-u} \left(\prod_{r=1}^{\infty} (1 - \frac{u}{q^r}) \right) \left(\sum_{\lambda : \lambda_1 < k} \frac{u^{|\lambda|}}{q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}} \right).
$$

By part 1 of Lemma 5, the Taylor series expansion in u around 0 of

$$
\prod_{r=1}^{\infty} (1 - \frac{u}{q^r})
$$

converges at $u = 1$. By Lemma 4 and part 2 of Lemma 5 the Taylor series expansion in u around 0 of

$$
\sum_{\lambda : \lambda_1 < k} \frac{u^{|\lambda|}}{q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}}
$$

converges at $u = 1$. Thus the product of these two Taylor series converges at $u = 1$ and Lemma 1 implies that

$$
\begin{aligned}
&\lim_{n \to \infty} [u^n] \frac{1}{1-u} \left(\prod_{r=1}^{\infty} (1 - \frac{u}{q^r}) \right) \left(\sum_{\lambda : \lambda_1 < k} \frac{u^{|\lambda|}}{q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}} \right) \\
= &\left(\prod_{r=1}^{\infty} (1 - \frac{1}{q^r}) \right) \left(\sum_{\lambda : \lambda_1 < k} \frac{1}{q^{\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q})_{m_i(\lambda)}} \right).
\end{aligned}
$$

The theorem follows by using Gordon's generalization of the Rogers-Ramanujans identities with $i = k, x = \frac{1}{q}, n_j = m_j(\lambda)$, and $N_j = \lambda_j'$. $\square$

Let $Mat(n,q)$ denote the set of $n \times n$ matrices with entries in the finite field F_q. Recall that $\alpha \in Mat(n,q)$ is said to be semisimple if it is diagonalizable over the algebraic closure $\bar{F}_q$. Theorems 4 and 5 compute the $n \to \infty$ limiting probability that an element of $GL(n,q)$ or $Mat(n,q)$ is semisimple.

Theorem 4 *The $n \to \infty$ limiting probability that an element of $GL(n,q)$ is semisimple is*

$$\prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} \frac{(1-\frac{1}{q^{r-1}})}{(1-\frac{1}{q^r})}.$$

PROOF: From the theory of Jordan canonical forms (Chapter 6 of Herstein [7]), an element α of $GL(n,q)$ is semisimple if and only if the largest part of $\lambda_{\phi}(\alpha)$ is less than two for all ϕ. Thus by the cycle index for $GL(n,q)$ (see the proof of Theorem 3) and Lemma 3, the sought probability is

$$\begin{aligned} & lim_{n\to\infty}[u^n] \prod_{\substack{\phi \neq z \\ \phi\ irred.}} \sum_{\lambda:\lambda_1<2} \frac{u^{deg(\phi)|\lambda|}}{q^{deg(\phi)\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}} \\ = \; & lim_{n\to\infty}[u^n] \frac{1}{1-u} \prod_{\substack{\phi \neq z \\ \phi\ irred.}} \left(\prod_{r=1}^{\infty} (1-\frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}}) \right) \left(\sum_{\lambda:\lambda_1<2} \frac{u^{deg(\phi)|\lambda|}}{q^{deg(\phi)\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}} \right). \end{aligned}$$

By Lemma 3 with $i=1, k=2, x=\frac{1}{q^{deg(\phi)}}, n_j = \lambda_j, N_j = \lambda_j'$ and Lemma 2, the desired probability becomes

$$\begin{aligned} & \prod_{\substack{\phi \neq z \\ \phi\ irred.}} \prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} (1-\frac{1}{q^{r\cdot deg(\phi)}}) \\ = \; & \prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} (\frac{1}{1-\frac{1}{q^r}}) \prod_{\phi\ irred.} \prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} (1-\frac{1}{q^{r\cdot deg(\phi)}}) \\ = \; & \prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} \frac{(1-\frac{1}{q^{r-1}})}{(1-\frac{1}{q^r})}. \end{aligned}$$

□

Theorem 5 *The $n \to \infty$ limiting probability that an element of $Mat(n,q)$ is semisimple is*

$$\prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} (1-\frac{1}{q^{r-1}}).$$

PROOF: The orbits of $GL(n,q)$ on $Mat(n,q)$ under conjugation are also parameterized by the data λ_{ϕ}. However the polynomial z may appear with non-zero multiplicity, so the restriction that $|\lambda_z| = 0$ does not apply. Stong [14] obtained the generating function equality

$$1+\sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{\alpha \in Mat(n,q)} \prod_{\phi\ irred.} x_{\phi,\lambda_{\phi}(\alpha)} = \prod_{\phi\ irred.} \left[\sum_{\lambda} x_{\phi,\lambda} \frac{u^{|\lambda| deg(\phi)}}{\prod_i \prod_{k=1}^{m_i(\lambda)} (q^{deg(\phi) d_i} - q^{deg(\phi)(d_i-k)})} \right],$$

where $d_i(\lambda)$ is defined by

$$d_i(\lambda) = 1m_1(\lambda) + 2m_2(\lambda) + \cdots + (i-1)m_{i-1}(\lambda) + i(m_i(\lambda) + m_{i+1}(\lambda) + \cdots + m_j(\lambda)).$$

Manipulations identical to those in Theorem 3 show that

$$
\begin{aligned}
& 1+\sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{\alpha \in Mat(n,q)} \prod_{\phi\ irred.} x_{\phi,\lambda_{\phi}(\alpha)} \\
= & \frac{1}{1-u} \prod_{\phi\ irred.} \left[\prod_{r=1}^{\infty}(1-\frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}})\right] \left[\sum_{\lambda} x_{\phi,\lambda} \frac{u^{|\lambda| deg(\phi)}}{q^{deg(\phi)\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}}\right].
\end{aligned}
$$

Arguing as in Theorem 4 shows that the $n \rightarrow \infty$ limit of the chance that an element of $Mat(n,q)$ is semisimple is

$$
\begin{aligned}
& lim_{n \rightarrow \infty} \frac{|GL(n,q)|}{q^{n^2}} [u^n] \prod_{\phi\ irred.} \sum_{\lambda:\lambda_1<2} \frac{u^{deg(\phi)|\lambda|}}{q^{deg(\phi)\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}} \\
= & lim_{n \rightarrow \infty} \frac{|GL(n,q)|}{q^{n^2}} [u^n] \frac{1}{1-u} \frac{1}{\prod_{r=1}^{\infty}(1-\frac{u}{q^r})} \prod_{\phi\ irred.} \\
& [\prod_{r=1}^{\infty}(1-\frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}}) \sum_{\lambda:\lambda_1<2} \frac{u^{deg(\phi)|\lambda|}}{q^{deg(\phi)\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}}] \\
= & \prod_{\phi\ irred.} [\prod_{r=1}^{\infty}(1-\frac{1}{q^{r\cdot deg(\phi)}})][\sum_{\lambda:\lambda_1<2} \frac{1}{q^{deg(\phi)\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}}] \\
= & \prod_{\phi\ irred.} \prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} (1-\frac{1}{q^{r\cdot deg(\phi)}}) \\
= & \prod_{\substack{r=1 \\ r=0,\pm 2 (mod\ 5)}}^{\infty} (1-\frac{1}{q^{r-1}}).
\end{aligned}
$$

□

Theorems 2 and 3 suggest a connection between the Hall-Littlewood polynomials and the finite general linear groups. Theorem 6 makes this connection precise.

Theorem 6

$$
\begin{aligned}
& (1-u)[1+\sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{\alpha \in GL(n,q)} \prod_{\substack{\phi \neq z \\ \phi\ irred.}} x_{\phi,\lambda_{\phi}(\alpha)}] \\
= & \prod_{\substack{\phi \neq z \\ \phi\ irred.}} \prod_{r=1}^{\infty}(1-\frac{u}{q^r}) \left[\sum_{\lambda} \frac{P_{\lambda}\left((\frac{u}{q})^{deg(\phi)}, (\frac{u}{q})^{2deg(\phi)}, (\frac{u}{q})^{3deg(\phi)}, \cdots; \frac{1}{q}\right)}{q^{n(\lambda)}}\right]
\end{aligned}
$$

Proof: The proof of Theorem 3 contained the equality

$$
\begin{aligned}
& (1-u)[1+\sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{\alpha \in GL(n,q)} \prod_{\substack{\phi \neq z \\ \phi\ irred.}} x_{\phi,\lambda_{\phi}(\alpha)}] \\
= & \prod_{\substack{\phi \neq z \\ \phi\ irred.}} \left[\prod_{r=1}^{\infty}(1-\frac{u^{deg(\phi)}}{q^{r\cdot deg(\phi)}})\right] \left[\sum_{\lambda} x_{\phi,\lambda} \frac{u^{|\lambda| deg(\phi)}}{q^{deg(\phi)\sum_i (\lambda_i')^2} \prod_i (\frac{1}{q^{deg(\phi)}})_{m_i(\lambda)}}\right].
\end{aligned}
$$

The theorem now follows from the use of Macdonald's principal specialization formula (as in Theorem 2) to conclude that

$$\frac{P_\lambda(\frac{u}{q},\frac{u}{q^2},\frac{u}{q^3},\frac{u}{q^4},\cdots;\frac{1}{q})}{q^{n(\lambda)}} = \frac{u^{|\lambda|}}{q^{\sum_i (\lambda_i')^2}\prod_i(\frac{1}{q})_{m_i(\lambda)}}.$$

$\square$

Remark: Although Theorem 6 followed easily from techniques in Theorems 2 and 3, its formulation admits an attractive probabilistic interpretation. Fix u such that $0 < u < 1$ and fix an irreducible monic polynomial $\phi \neq z$. Then pick a natural number with probability of getting n equal to $(1-u)u^n$. Finally, choose α uniformly at random in $GL(n,q)$ and let $\lambda_\phi(\alpha)$ be the random partition so defined. This procedure induces a probability measure on the set of all partitions of all integers. Theorem 6 implies that measures obtained for different ϕ are independent, and that the mass the measure for a given ϕ assigns to a partition λ is equal to

$$\prod_{r=1}^{\infty}(1-\frac{u}{q^r})\left[\frac{P_\lambda\left((\frac{u}{q})^{deg(\phi)},(\frac{u}{q})^{2deg(\phi)},(\frac{u}{q})^{3deg(\phi)},\cdots;\frac{1}{q}\right)}{q^{n(\lambda)}}\right].$$

This connection with symmetric functions leads to a probabilistic algorithm for growing the random partitions λ_ϕ. This viewpoint is used in [5] to give probabilistic proofs of some theorems in group theory.

3 Conclusion

This note has provided connections between Gordon's generalization of the Rogers-Ramanujan identities, the Hall-Littlewood polynomials, and generating functions which arise in the study of the finite general linear groups. These connections suggest some ideas for future work. We consider the following possibilities to hold the most promise.

- Find analogs of the Rogers-Ramanujan identities for the finite unitary, symplectic, and orthogonal groups.
- It is known (e.g. Kac [8]) that the product side of the Rogers-Ramanujan identities has interesting modular properties. Can this phenomenon be understood group theoretically in terms of the conjugacy classes of the general linear groups–i.e. is there "general linear group moonshine"?
- Connect the Hall-Littlewood polynomials with the representation theory of the Virasoro algebra. The motivation for this is the paper of Feigen and Frenkel [6], which connects the Rogers-Ramanujan identities with representations of the Virasoro algebra.

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