Interval Reductions and Extensions of Orders: Bijections to Chains in Lattices*

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Abstract. We discuss bijections that relate families of chains in lattices associated to an order P and families of interval orders defined on the ground set of P. Two bijections of this type have been known:

(1) The bijection between maximal chains in the antichain lattice $\mathcal{A}(P)$ and the linear extensions of P.

(2) A bijection between maximal chains in the lattice of maximal antichains $\mathcal{A}_M(P)$ and minimal interval extensions of P.

We discuss two approaches to associate interval orders to chains in $\mathcal{A}(P)$. This leads to new bijections generalizing Bijections 1 and 2. As a consequence we characterize the chains corresponding to weak-order extensions and minimal weak-order extensions of P.

Seeking for a way of representing interval reductions of P by chains we came up with the separation lattice $\mathcal{S}(P)$. Chains in this lattice encode an interesting subclass of interval reductions of P. Let $\mathcal{S}_M(P)$ be the lattice of maximal separations in the separation lattice. Restricted to maximal separations the above bijection specializes to a bijection which nicely complements 1 and 2.

(3) A bijection between maximal chains in the lattice of maximal separations $\mathcal{S}_M(P)$ and minimal interval reductions of P.

Résumé. Nous nous intéressons à certaines bijections qui relient des familles de chaînes de certains treillis associés à un ordre P et des familles d'ordres d'intervalles définies sur le même ensemble que P. Deux bijections de ce type sont déjà connues :

(1) La bijection entre les chaînes maximales du treillis des antichaînes $\mathcal{A}(P)$ et les extensions linéaires de P.

(2) Une bijection entre chaînes du treillis des antichaînes maximales $\mathcal{AM}(P)$ et les extensions intervallaires minimales de P.

Nous étudions deux approches permettant d'associer des ordres d'intervalles à des chaînes de $\mathcal{A}(P)$. Ceci nous permet de mettre en évidence de nouvelles bijections généralisant les bijections (1) et (2). Nous en déduisons une caractérisation des chaînes qui correspondent aux extensions de P en ordre faible ou aux extensions minimales en ordre faible.

Afin de de proposer une représentation des réductions intervallaires de P, nous introduisons ensuite le treillis des séparations $\mathcal{S}(P)$. Les chaînes de ce treillis codent une sous classe intéressante des réductions intervallaires de P. Si l'on restreint cette bijection aux seules séparations maximales, on obtient une nouvelle bijection qui complète les bijections (1) et (2). Notons $\mathcal{S}_M(P)$ le treillis engendré par les seules séparations maximales.

(3) Une bijection entre les chaînes maximales de $\mathcal{S}_M(P)$ et les réductions intervallaires maximales de P.

*Extended Abstract

1 Introduction

1.1 Overview

Several lattices can be associated to a partial order P and reflect different aspects of the structure of P. We consider families of chains in such lattices related to P, namely families of chains that bijectively correspond to certain families of orders related to P.

In the sequel we give several such bijections. We begin with a classical result appearing in early work of Bonnet, Monjardet, Pouzet and Stanley [13]. It connects chains in the lattice of antichains $\mathcal{A}(P)$ of an order P with linear extensions.

Bijection 1 *There is a bijection between the linear extensions of P and maximal chains in $\mathcal{A}(P)$.*

Some years ago Habib et. al. [7] gave a theorem which is similar in spirit. It connects chains in the lattice of maximal antichains $\mathcal{A}_M(P)$ of an order P with minimal interval extensions.

Bijection 2 *There is a bijection between the minimal interval extensions of P and maximal chains in $\mathcal{A}_M(P)$.*

In Section 1.3 we define what it means for an interval representation or interval order to be essential for P or strongly-essential for P. We relate these notions to extensions and reductions of P. In Section 2 we characterize classes of interval orders corresponding bijectively to chains in $\mathcal{A}(P)$. First, Bijection 3 relates to essential weak-orders. This extends to a characterization of those chains corresponding to weak-order extensions, Proposition 1, and minimal weak-order extensions, Proposition 2. Bijection 4 is a second bijection involving chains in $\mathcal{A}(P)$. It maps the chains to strongly-essential representations of interval extensions of suborders of P. This extends to a characterization of those chains corresponding to strongly-essential interval extensions of P in Proposition 3.

In Section 3 we show that Bijection 4 specializes to Bijection 2 when restricted to maximal chains in $\mathcal{A}_M(P)$. In particular we give a new and more transparent proof for Bijection 2.

In Section 4 we define the lattice of separations $\mathcal{S}(P)$ and the lattice of maximal separations $\mathcal{S}_M(P)$ of an order P. In Section 5, Bijection 5 relates chains in $\mathcal{S}(P)$ and essential representations of interval reductions of P. In Proposition 4 we characterize a class of chains generating each essential interval reduction exactly once. In Section 6 we restrict Bijection 5 to maximal chains of maximal separations and obtain Bijection 6 that relates maximal chains in $\mathcal{S}_M(P)$ to maximal interval reductions of P.

1.2 Basics

An *order* $P = (V, <_P)$ consists of a finite set V of distinguishable *elements of* P and the *relations of* P which are pairs (x, y) of elements with $x <_P y$. The relation $<_P$ is transitive and irreflexive. In some cases we will see a reflexive order relation $\leq_P$.

An order Q is an *interval order* if there is a pair (l,r) of functions assigning to each element $x \in V_Q$ real numbers l_x, r_x on the real line so that $l_x < r_x$ for all $x \in V_Q$, i.e., (l_x, r_x) is an open interval, and $x <_Q y$ if and only if $r_x \leq l_y$ for all $x, y \in V_Q$. The pair (l,r) is called a *representation* of the interval order Q. (See Möhring [10] for a good introduction to interval orders.)

An *interval reduction* of an order P is an interval order Q on the same ground set such that $x <_Q y$ implies $x <_P y$ for all x, y. An interval reduction Q of P *is a maximal interval reduction* if there is no interval reduction R with more relations, i.e., with $<_Q \subset <_R \subseteq <_P$.

An *interval extension* of P is an interval order Q on the same ground set such that $x <_P y$ implies $x <_Q y$ for all x, y. An interval extension Q of P *is a minimal interval extension* if there is no interval extension R with less relations, more formally, with $<_P \subset <_R \subseteq <_Q$.

1.3 Interval Representations

Throughout the paper we only work with *integer representations* of interval orders, i.e., with representations in which all interval end-points are non-negative integers. An integer representation (l,r) such that there is a positive integer K with $l(V) \cup r(V) = \{0, 1, \ldots, K\}$ is called a *dense representation*. The number K is the *magnitude of the representation*.

Every interval order has a unique dense representation of minimal magnitude the *canonical representation* (see [10, 4]). The *magnitude of the interval order* is defined as the magnitude of the canonical representation. The canonical representation is characterized by $l(V) = \{0, 1, \ldots, K-1\}$ and $r(V) = \{1, \ldots, K-1, K\}$ with K being the magnitude.

Given a dense representation of an interval order Q let $D_i = \{x \in V_Q : r_x \leq i\}$. Each D_i for $i = 0..K$ is an ideal (downward closed set) of Q, hence we obtain a chain

$$\emptyset = D_0 \subseteq D_1 \subseteq D_2 \subseteq \ldots \subseteq D_K = V_Q \tag{1}$$

of ideals. Symmetrically, $U_i = \{x \in V_Q : l_x \geq i\}$ is a filter (upward closed set) of Q and

$$V_Q = U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots \supseteq U_K = \emptyset \tag{2}$$

is a chain of filters. Let M_i be the set of elements whose intervals contain $(i-1, i)$. Obviously M_i is an antichain of Q and D_{i-1}, M_i, U_i is a partition of V_Q.

Definition 1 *Let $P = (V, <_P)$ be an order and $Q = (V, <_Q)$ be an interval order on the same ground set. A dense representation of Q is called an* essential representation for P *if D_i is an ideal of P and U_i is a filter of P for all $i = 0, .., K$. An interval order Q is called* essential for P *if Q admits an essential representation for P.*

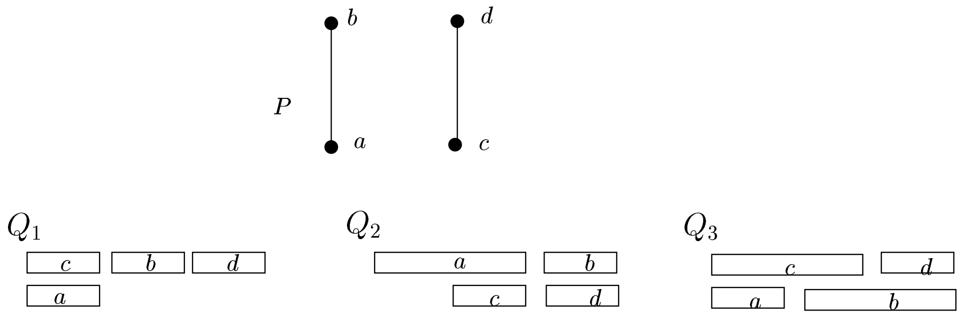


Figure 1: Representations of Interval Extensions

Note that if Q is an essential interval order for P then Q can be an extension of P, it can be a reduction of P or it can be neither extension nor reduction. Note also that the conditions on the ideals and on the filters in the definition of essential representations are independent of each other.

For $X \subset V$ let $X^{\downarrow}$ be the smallest ideal containing X, i.e., $X^{\downarrow} = \{y : y \leq_P x$ for some $x \in X\}$, and let $X^{\uparrow} = \{y : x \leq_P y$ for some $x \in X\}$ be the smallest filter containing X. With this additional notation we can characterize essential representations of interval orders as those with $D_i = D_i^{\downarrow}$ and $U_i = U_i^{\uparrow}$ for all i.

Definition 2 *A dense representation of an interval order Q is called a* strongly-essential representation for P *if the ideals of M_i taken in Q and P equal each other for all i. Formally noted this condition is $M_i^{\downarrow} = D_{i-1} \cup M_i$. Interval order Q is called* strongly-essential for P *if Q admits a strongly-essential representation for P.*

Observe that the definition of strongly-essential is somewhat asymmetric. It only imposes conditions on ideals and not on filters.

Lemma 1 *A representation (l, r) of an interval order Q which is strongly-essential for P is also essential for P.*

1.4 Essential Representations for Interval Extensions and Reductions

We prove basic connections between essential interval representations and maximality and minimality of interval reductions and extension.

Lemma 2 *Every representation of an interval extension Q of P is essential.*

With respect to the property *strongly-essential* representations of interval extensions of P behave less nicely. This is exemplified in Example 1.

Example 1 Let $P = II$, the parallel composition of two 2 element chains. Figure 1 displays three represented interval extensions of P. The representation

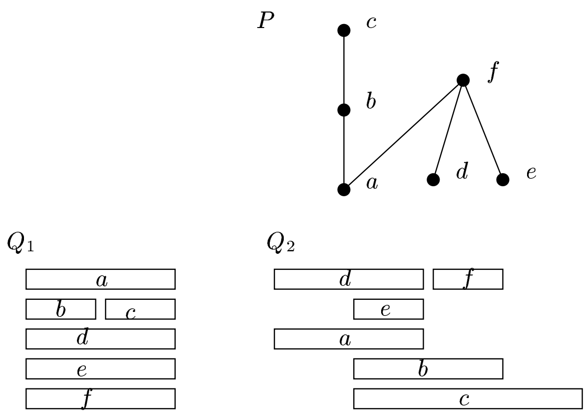


Figure 2: Representations of Interval Reductions

of Q_1 is canonical but Q_1 has no strongly-essential representation. The representation of Q_2 is strongly-essential but not canonical. Note that, because of the asymmetry of the definition of strongly-essential the reflection of the representation Q_2 is not strongly-essential for the dual of P. Q_3 is a minimal extension of P, the representation is canonical and hence, by Lemma 4 strongly-essential.

The next lemma shows that a strongly-essential representation fulfills one side of the defining conditions for canonical representations.

Lemma 3 *If (l,r) is a dense representation of magnitude K and (l,r) is strongly-essential for some P then $l(V) = \{0,\ldots,K-1\}$.*

Lemma 4 *The canonical representation of a minimal interval extension Q of P is strongly-essential for P.*

We now turn to interval reductions and their relation to the notion of essential representations.

Lemma 5 *The canonical representation (l,r) of a maximal interval reduction Q of P is essential for P.*

Example 2 Figure 2 shows an order P with three represented interval reductions. The representation of Q_1 is canonical but Q_1 has no essential representation for P. The representation of Q_2 is essential but not canonical. Interchanging r_b and r_c gives a non-essential representation of Q_2. Q_3 is a maximal reduction of P, the representation is canonical and hence, by Lemma 5 essential.

Consider interval orders as elements of the *extension lattice* $\mathrm{Ext}(V)$. This lattice has as elements all orders on ground set V with relations $P <_{\mathrm{Ext}} Q$ iff $<_P \subset <_Q$. The 0 of the lattice is the antichain on V and there is an artificial 1 above the linear orders on V. $\mathrm{Ext}(V)$ is a ranked lattice where the rank of an element P of $\mathrm{Ext}(V)$, i.e., of an order P, is the number of relations of P. We show that the set of interval orders is in a certain sense dense in $\mathrm{Ext}(V)$.

Theorem 1 *Let Q_1 and Q_2 be interval orders with $Q_1 <_{\mathrm{Ext}} Q_2$. If Q_2 has at least two relations more then Q_1 then there is an interval order R in between, i.e., with $Q_1 <_{\mathrm{Ext}} R <_{\mathrm{Ext}} Q_2$.*

2 Chains in the Lattice of Antichains

In this section we characterize two classes of orders corresponding to chains in the lattice of antichains $\mathcal{A}(P)$ of an order P. We begin with a brief review of some basic facts about $\mathcal{A}(P)$.

The lattice $\mathcal{A}(P)$ is most conveniently described as the set of all ideals of P ordered by inclusion. We take this as definition. Since unions and intersections of ideals are ideals the lattice $\mathcal{A}(P)$ is distributive. The fundamental theorem of finite distributive lattices states that for every finite distributive lattice L there is an order P such that L is isomorphic to $\mathcal{A}(P)$ (see *e.g.* [14], Chapter 3).

For our naming of $\mathcal{A}(P)$ as the lattice of antichains of P recall the one-to-one correspondence between ideals and antichains: With an ideal I of P associate the antichain $A_I = \textsc{Max}(I)$ of maximal elements of I. Conversely, with antichain A associate the ideal $A^{\downarrow}$.

Taking complements in the ground set of P bijectively maps ideals to filters and vice versa, hence, $\mathcal{A}(P)$ is the set of filters of P ordered by reverse inclusion. Filters have their own 'natural' one-to-one correspondence with antichains. With a filter F of P associate the antichain $A_F = \textsc{Min}(F)$ of minimal elements of F. Conversely, with antichain A associate the filter $A^{\uparrow}$. We have mentioned these 'dual' representations of $\mathcal{A}(P)$ since they will help understand the reasons for the asymmetry in the property strongly-essential.

Minimum and maximum of $\mathcal{A}(P)$ are given by $\hat{0} = \emptyset$ and $\hat{1} = V$. A chain is called a *closed chain* if it contains $\hat{0}$ and $\hat{1}$.

With the next two bijections we characterize two classes of orders corresponding to chains in $\mathcal{A}(P)$.

Bijection 3 *There is a bijection between closed chains in $\mathcal{A}(P)$ and weak-orders which are essential for P.*

Bijection 4 *There is a bijection between chains in $\mathcal{A}(P)$ and strongly-essential representations of interval extensions of induced suborders of P.*

We give the two mappings from chains in $\mathcal{A}(P)$ to representations of interval orders right away. The proofs of the theorems are given in subsequent subsections.

Bij. 3 A closed chain $\emptyset = I_0, I_1, \ldots, I_{K-1}, I_K = V$ in $\mathcal{A}(P)$ induces an ordered partition $(I_1, I_2 \setminus I_1, \ldots, I_K \setminus I_{K-1})$ of V. Let $B_i = I_i \setminus I_{i-1}$ be the ith block of this partition. Define the order $W = (V, <_W)$ by $x <_W y$ if and only if $x \in B_i$, $y \in B_j$ and $i < j$. If $*$ denotes serial composition W can be written as $W = B_1 * B_2 * \ldots * B_K$.

In case the chain is a maximal chain W will be a linear extension. In this sense Bijection 3 extends Bijection 1.

Bij. 4 Transform a chain $I_1, I_2, \ldots, I_K$ in $\mathcal{A}(P)$ into $A_1, A_2, \ldots, A_K$ with $A_i = \textsc{Max}(I_i)$. From this chain of antichains define $V_Q = \bigcup_{i=1}^{K} A_i$. The representation (l, r) of an interval order Q on V_Q is given by $l_x = -1 + \min\{i : x \in A_i\}$ and $r_x = \max\{i : x \in A_i\}$ for all $x \in V_Q$.

In case all antichains A_i in the chain are maximal antichains of P and the chain is maximal with this property, i.e., if the chain is a maximal chain in $\mathcal{A}_M(P)$, the representation (l, r) is the canonical representation of a minimal interval extension of P. In this sense Bijection 4 extends Bijection 2.

2.1 Bijection 3 and Weak-Orders Related to P

Weak-orders can be defined as the interval orders admitting an integer representation with all intervals of unit length. Such a representation for the order W corresponding to the chain in $\mathcal{A}(P)$ is given by $l_x = i - 1$ and $r_x = i$ for all $x \in B_i$. This representation is essential since $D_i = I_i$ which is an ideal by definition and $F_i = V \setminus I_i$ which clearly is a filter. Hence W is a weak-order and essential for P.

Let W be a weak-order which is essential for P and let D_i be defined by the canonical representation of W. From canonical we obtain that $D_i \subset D_j$ for $1 \leq i < j \leq K$ and from essential we obtain that D_i is an ideal in P. Hence $D_1, D_2, \ldots, D_K$ is a chain in $\mathcal{A}(P)$.

The above considerations show that the image of the mapping is the set of weak-orders W which are essential for P. Since the mapping is obviously injective and all sets are finite it is a bijection. This completes the proof of Bijection 3. $\square$

Let W be a weak-order which is essential for P. Each block B_i is an antichain of W and can thus be seen as a reduction of the suborder induced by P on this set. On the other hand W contains all relations in $B_i \times B_j$ for $i < j$. Since W is essential for P every pair x, y with $x \in B_i$ and $y \in B_j$ is either incomparable in P or $x <_P y$. Therefore, between blocks W behaves as an extension of P.

This observation enables us to characterize weak-order extensions of an order P. A detailed treatment of weak-order extensions has recently been given by Bertet et. al [2]. Therefore, we will confine us to the indication of the main ideas.

An essential weak-order is an extension of P if and only if every 'reduction class' B_i is already an antichain of P. In terms of the chain in $\mathcal{A}(P)$ corresponding to W this condition translates to the condition that $I_i \setminus I_{i-1}$ is an antichain in P. Call a pair (I_{i-1}, I_i) of ideals *legal* if $I_i \setminus I_{i-1}$ is an antichain in P.

Proposition 1 *The weak-order extensions of P correspond bijectively to closed chains in $\mathcal{A}(P)$ such that every pair (I_{i-1}, I_i) of the chain is legal.*

Two weak-orders W_1 and W_2 on the same ground set have $W_1 <_{\mathrm{Ext}} W_2$ if and only if the partition $(B_1, \ldots, B_{K_2})$ induced by W_2 is a refinement of the partition $(A_1, \ldots, A_{K_1})$ induced by W_1. From this we conclude a characterization of minimal weak-order extensions of P.

Proposition 2 *The minimal weak-order extensions of P correspond bijectively to closed chains in $\mathcal{A}(P)$ such that every pair (I_{i-1}, I_i) but no pair (I_i, I_j) with $j - i > 1$ of the chain is legal.*

2.2 Proof of Bijection 4

Recall the mapping Bij. 4 from chains in $\mathcal{A}(P)$ to representations of interval orders: For chain $C = (A_1, A_2, \ldots, A_K)$ and $x \in \bigcup_{i=1}^{K} A_i$ define $l_x = -1 + \min\{i : x \in A_i\}$ and $r_x = \max\{i : x \in A_i\}$. Let ρ be this mapping, i.e., $\rho(C) = (l, r)$.

The following observations are immediate:

- $\rho(C)$ is a representation of an interval order Q on $V_Q = \bigcup_{i=1}^{K} A_i$.
- ρ is injective.

Lemma 6 *Let $C = (A_1, A_2, \ldots, A_K)$ be a chain in $\mathcal{A}(P)$. The interval order Q represented by $\rho(C)$ is an interval extension of the order P' induced by P on $\bigcup_{i=1}^{K} A_i$, moreover, $\rho(C)$ is strongly-essential for P'.*

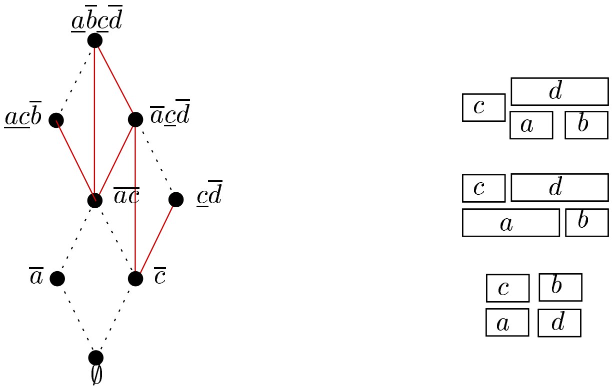
Lemma 7 *Let (l, r) be a representation of an interval extension Q of a suborder P' of P such that (l, r) is strongly-essential for P' then there is a chain C in $\mathcal{A}(P)$ with $(l, r) = \rho(C)$.*

The two lemmas show that the image of ρ is precisely the set of representations of interval extensions of suborders P' of P which are strongly-essential for P'. Since as noted before ρ is injective and all sets are finite ρ is a bijection. This completes the proof of Bijection 4.

The above bijection may give an abundance of representations for the same interval extension and it gives interval extensions of suborders P' of P. We might, however, be interested in generating every interval extensions of P itself without too much overhead. We come to a result similar in flavor to the results about weak-order extensions (Proposition 1).

Call a pair (A, A') of of antichains with $A <_{\mathcal{A}} A'$ *legal* if $A_{i+1} \setminus A_i \neq \emptyset$ and $A_i \setminus A_{i+1} \neq \emptyset$ and $A_{i+1}^{\downarrow} \setminus A_i^{\downarrow} \subset A_{i+1}$. It can be shown that $\rho(C)$ is a canonical representation of an extension of P only if every consecutive pair of antichains in C is legal. Together with Bijection 4 we obtain the proposition.

Proposition 3 *The strongly-essential interval extensions of P correspond bijectively to chains in $\mathcal{A}(P)$ such that every pair (A_i, A_{i+1}) of the chain is legal and $\bigcup_i A_i = V$.*



(a) The lattice $\mathcal{A}(N)$, legal arcs are drawn solid.

(b) Representations of strongly-essential extensions of N.

Figure 3: The strongly essential extensions of N

Example 3 Consider the four element poset N with comparabilities $a < b, a < d, c < d$. As seen from Fig. 3 $\mathcal{A}(N)$ has 6 legal arcs, but only four of them are used in paths that collect all vertices in V. In Fig. 3(a) the maximal (minimal) elements are overlined (underlined). In fact there are three strongly essential interval extensions of N, one such extension being N itself. The canonical interval models are given in Fig. 3(b).

3 Bijection 2 and Minimal Interval Extensions

Let $C = (A_1, A_2, \ldots, A_K)$ be a maximal chain in $\mathcal{A}_M(P)$. Recall the representation $\rho(C) = (l, r)$ given by $l_x + 1 = \min\{i : x \in A_i\}$ and $r_x = \max\{i : x \in A_i\}$, and let $Q(C)$ be the interval order corresponding to this representation. In Lemma 8 we show that $Q(C)$ ins a minimal interval extension of P and C is the unique chain in $\mathcal{A}_M(Q)$. Conversely, given a minimal interval extension Q of P Lemma 9 will show that the unique chain in $\mathcal{A}_M(Q)$ is a chain in $\mathcal{A}_M(P)$. Together the two lemmas readily establish Bijection 2.

Lemma 8 *Let $C = (A_1, A_2, \ldots, A_K)$ be a maximal chain in $\mathcal{A}_M(P)$ the corresponding interval order Q is a minimal interval extension of P. Moreover, C is the unique chain in $\mathcal{A}_M(Q)$.*

Lemma 9 *Let Q be a minimal interval extension of P. The unique chain $M_1, M_2, \ldots, M_K$ of maximal antichains in $\mathcal{A}(Q)$ is a maximal chain $\mathcal{A}(P)$.*

4 The Separation Lattice

In this section we define the separation lattices and provide some material about them that will be needed in our discussion of interval reductions. A *separation* of an order $P = (V, <_P)$ is a pair (I, F) of subsets of V such that

(1) I and F are disjoint,

(2) $x <_P y$ for all $x \in I$ and $y \in F$ and

(3) I is an ideal and F is a filter of P.

The name separation was motivated by the observation that $V \setminus (I \cup F)$ is a separator of the cocomparability graph of P whenever (I, F) is a separation of P. This mapping $(I, F) \to V \setminus (I \cup F)$ from separations of P to separators of the cocomparability graph is onto but not one-to-one. In general a separation is not characterized by the supporting set $I \cup F$, *e.g.* if $I \cup F$ is a series composition of more than two components it is not.

It is natural to define an order relation on the set $\mathcal{S}(P)$ of separations of P by

$$(I, F) \leq_{\mathcal{S}} (I', F') \iff I \subseteq I' \text{ and } F \supseteq F' \tag{3}$$

For examples for $\mathcal{S}(P)$ see Figure 4. Subfigure 4(a) shows it for $P = N$ and 4(b) for $P = II$, see Figure 1.

Theorem 2 *$\mathcal{S}(P)$ with the above defined order relation is a distributive lattice.*

By the fundamental theorem of finite distributive lattices every such lattice is the lattice of antichains of some order. More precisely, a distributive lattice L is isomorphic to the lattice of antichains $\mathcal{A}(J_L)$ where J_L is the set of join irreducibles of L with the order relation induced by L.

In Lemma 10 we characterize the join-irreducible elements of $\mathcal{S}(P)$. Let $\mathrm{Pr}[x] = \{y : y \leq x\}$ and $\mathrm{Su}(x) = \{y : x < y\}$.

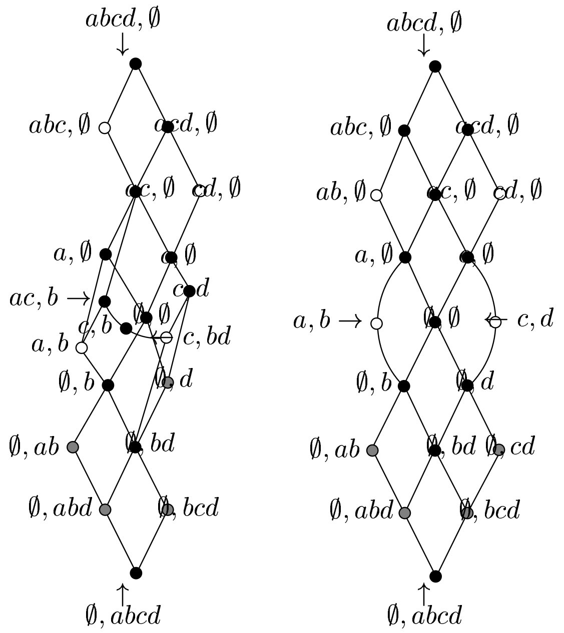
Lemma 10 *Besides $\hat{0} = (\emptyset, V)$ the join irreducible elements of the separation lattice $\mathcal{S}(P)$ are the pairs $(\emptyset, V \setminus \mathrm{Pr}[x])$ for all $x \in V$ together with the pairs $(\mathrm{Pr}[x], \mathrm{Su}(x))$ for all $x \in V$.*

In Figure 4 the copy of P in $\mathcal{S}(P)$ induced by the separations of type (a) in the lemma is indicated by grey circles. A second copy of P in $\mathcal{S}(P)$ induced by the separations of type (b) in the lemma is indicated by white circles.

Lemma 11 *The order relation of $J_{\mathcal{S}(P)}$ is isomorphic to the order Q on two copies V' and V'' of V with relations:*

$$x' <_Q y' \text{ iff } x <_P y, \quad x'' <_Q y'' \text{ iff } x <_P y, \quad x' <_Q y'' \text{ iff } y \not<_P x.$$

For $X \subseteq V$ let $\mathrm{Pr}(X) = \{y \in V \mid y <_P x \text{ for all } x \in X\}$, and $\mathrm{Su}(X) = \{y \in V \mid y >_P x \text{ for all } x \in X\}$. Clearly, $\mathrm{Pr}(X)$ is an ideal and $\mathrm{Su}(X)$ is a filter in P.



(a) $\mathcal{S}(N)$ (b) $\mathcal{S}(II)$

Figure 4: Examples for the Separation Lattice

It is easyly verified that $\mathrm{Pr} \circ \mathrm{Su}$ and $\mathrm{Su} \circ \mathrm{Pr}$ are closure operators. Observe further that for any $X \subseteq V$ the pairs $\alpha(X) = (\mathrm{Pr}(X), \mathrm{Su}(\mathrm{Pr}(X)))$ and $\omega(X) = (\mathrm{Pr}(\mathrm{Su}(X)), \mathrm{Su}(X))$ are separations.

A separation (I, F) is *maximal* if there is no separation (I', F') with $I \subseteq I'$, $F \subseteq F'$ and $I \cup F \subset I' \cup F'$. Let $\mathcal{S}_M(P)$ denotes the set of maximal separations of P with the order induced by $\mathcal{S}(P)$. In Figure 4 the elements of $\mathcal{S}_M(P)$ are marked with arrows.

Lemma 12 *A separation (I, F) is maximal iff $I = \mathrm{Pr}(F)$ and $F = \mathrm{Su}(I)$.*

As special case consider the case of an interval order Q. Given the canonical representation of Q the maximal separations are exactly the pairs (D_i, U_i) for $i = 0, \ldots, K$. In particular the order induced by $\mathcal{S}(Q)$ on the maximal separations is a chain.

In general $\mathcal{S}_M(P)$ is only an suborder of $\mathcal{S}(P)$, but not a sublattice, i.e., for $S, S' \in \mathcal{S}_M(P)$ the join $S \sqcup S'$ need not be in $\mathcal{S}_M(P)$. Nevertheless, $\mathcal{S}_M(P)$ is itself a lattice, this will be shown in Theorem 3.

Lemma 13 *For every $X \subseteq V$, $\alpha(X)$ and $\omega(X)$ are maximal separations. Moreover, $\alpha(X)$ is the unique $(<_S)$–maximal separation in $\mathcal{S}_M(P)$ with X contained in the filter. Symmetrically, $\omega(X)$ is the unique $(<_S)$ minimal separation in $\mathcal{S}_M(P)$ with X contained in the ideal.*

Theorem 3 *$\mathcal{S}_M(P)$ is a lattice with lattice operations $S \vee S' = \omega(I \cup I')$ and $S \wedge S' = \alpha(F \cup F')$ for $S = (I, F)$ and $S' = (I', F')$.*

5 Chains in the Separation Lattice

Bijection 5 *There is a bijection between closed chains in $\mathcal{S}(P)$ and dense, essential representations of interval reductions of P.*

Bij. 5 Given a closed chain $C = (\hat{0} = S_0 <_{\mathcal{S}} S_1 <_{\mathcal{S}} \ldots <_{\mathcal{S}} S_K = \hat{1})$ in $\mathcal{S}(P)$ with $S_i = (I_i, F_i)$ let $r_x = \min\{i : x \in I_i\}$ and $l_x = \max\{i : x \in F_i\}$. This defines a representation $(l, r) = \sigma(C)$ of an interval order of magnitude K.

In Lemma 14 we show that $\sigma(C)$ is a dense representation of an essential interval reduction of P. Conversely, Lemma 15 shows that every such representation is the image of a closed chain in $\mathcal{S}(P)$. Since σ is easily seen to be injective this establishes the bijection of the theorem.

Lemma 14 *Let C be a closed chain in $\mathcal{S}(P)$. The representation $\sigma(C)$ is a dense representation of an essential interval reduction of P.*

Lemma 15 *Let (l,r) be a dense representation of an essential interval reduction of P then there is chain C in $\mathcal{S}(P)$ such that $(l,r) = \sigma(C)$.*

As result of this section we have so far obtained a nice bijection. This bijection, however, gives an abundance of representations for the same interval reduction. We now discuss a possibility of avoiding multiple generation. The result will be similar in flavor to the results about weak-order extensions (Proposition 1) and strongly-essential interval extensions (Proposition 3). Again we specify legal edges in the lattice such that the objects we look for correspond to chains with every consecutive pair in the chain being a legal pair.

Let $0 < i \leq K$ and (l, r) be a canonical representation of an interval order Q of magnitude K. Then there is an x with $r_x = i$ and a y with $l_y = i - 1$. If Q is a reduction of P and $C = (S_0, S_1, \ldots, S_K)$ with $S_i = (I_i, F_i)$ is the chain in $\mathcal{S}(P)$ with $\sigma(C) = (l, r)$ then $x \in I_i \setminus I_{i-1}$ and $y \in F_{i-1} \setminus F_i$. It follows that in the chain corresponding to the canonical representation of an interval reduction any two consecutive separations differ in both components. It is easily seen that the converse is true as well. If in chain C in $\mathcal{S}(P)$ any two consecutive separations differ in both components then $\sigma(C)$ is canonical. Call a pair (S_{i-1}, S_i) of separations *legal* if $I_i \setminus I_{i-1} \neq \emptyset$ and $F_{i-1} \setminus F_i \neq \emptyset$. We summarize these considerations with a proposition.

Proposition 4 *The essential interval reductions of P correspond bijectively to closed chains in $\mathcal{S}(P)$ such that every pair (S_{i-1}, S_i) of the chain is legal.*

6 Maximal Interval Reductions

We show that the mapping σ from Section 5 specializes to a bijection between maximal chains in $\mathcal{S}_M(P)$ and maximal interval reductions of P.

Bijection 6 *The mapping σ is a bijection between maximal chains in the lattice of maximal separations $\mathcal{S}_M(P)$ and maximal interval reductions of P.*

The mapping is the same as in the previous section. This mapping is injective, Lemma 16 shows that it maps maximal chains in $\mathcal{S}_M(P)$ to canonical representations of maximal interval reduction and Lemma 17 shows that every such representation is in the image of mapping σ. Together this proves the theorem.

Lemma 16 *Let $C = (\hat{0} = S_0, S_1, \ldots, S_K = \hat{1})$ be a maximal chain in $\mathcal{S}_M(P)$. Then $(l, r) = \sigma(C)$ is the canonical representation of a maximal interval reduction Q of P.*

Lemma 17 *Let Q be a maximal interval reduction of P and let (l, r) be the canonical representation of Q. Then there is a maximal closed chain $C = (S_0, S_1, \ldots, S_K)$ in $\mathcal{S}_M(P)$ with $(l, r) = \sigma(C)$.*

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