Forestal Algebras and Algebraic Forests

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SUMMARY In this paper we introduce and study a new class of cellular algebras, *forestal algebras*, defined inductively from one point algebras by taking direct sums and wreath products. Apart from this we introduce *algebraic forests* which are exactly the graphs the cellular algebras of which are forestal. It turns out that the automorphism groups of both of them are compact, i.e. can be considered as permutation groups for which a natural analog of Birkhoff's theorem on doubly stochastic matrices holds. This follows from the observation that each weak isomorphism of two forestal algebras is induced by a strong isomorphism. We give a complete description of cellular algebras of disconnected graphs, which implies that the class of algebraic forests includes cographs, forests and interval graphs, and even a joint superclass of the latter two, namely, rooted directed path graphs. We also present efficient membership and isomorphism tests for the classes of forestal algebras and algebraic forests.

SOMMAIRE Dans cette étude nous introduisons et étudions une nouvelle classe des algèbres cellulaires denotées *algèbres forestales* et definies inductivement en utilisant les deux operations *somme directe* et *produit de couronne*. De plus, nous introduisons des *forêts algébriques* qui sont exactement les graphes dont les algèbres associées sont des algèbres forestales. Il se trouve que les groupes des automorphismes de tous les deux sont compacts, i.e. peuvent être considérés comme les groupes des permutations satisfaisant à une sorte du théorème de Birkhoff sur des matrices bistochastiques. Cela résulte de l'observation que tout faible isomorphisme entre deux algèbres forestales est une isomorphisme forte. Nous donnons une caractérisation complète des algèbres cellulaires associées aux graphes disconnectés et en déduisons que la classe des forêts algébriques contient des cographes, des forêts, des graphes représentatifs d'une famille d'intervalles, et de plus, des rooted directed paths graphs. Outre cela, nous présentons des algorithmes efficaces pour le problème de reconnaître ainsi que le problème de l'isomorphisme dans les classes des algèbres forestales et des forêts algébriques.

*Partially supported by DFG visit grant to Munich 436 RUS 17/64/97

†Partially supported by DFG visit grant to Munich 436 RUS 17/64/97

‡Partially supported by the German Israel Foundation for Scientific Research and Development under contract # I-0333-263.06/93

1 Introduction

This paper deals with compact cellular algebras and weakly compact graphs. The term *compact* used here denotes a *combinatorial* property rather than a topological one. This property has nothing to do with the compactness of topological spaces.

Combinatorial compactness has a short history. Compact graphs (or Birkhoff's ones) were introduced by G. TINHOFER in [14] in order to create a non-standard approach to the graph isomorphism problem. A graph Γ is called compact if

$$\mathrm{DS}(\Gamma) = \mathrm{Conv}(\mathrm{Aut}(\Gamma)), \tag{1}$$

where $\mathrm{DS}(\Gamma)$ is the set of all doubly stochastic matrices commuting with the adjacency matrix of Γ and $\mathrm{Conv}(\mathrm{Aut}(\Gamma))$ is the convex hull of the permutation matrices corresponding to the automorphisms of Γ. We note that Birkhoff's theorem on doubly stochastic matrices establishes the compactness of a complete graph. In the subsequent papers [15], [13] several classes of compact graphs have been introduced and studied in some detail. In [16] it was proved that compact graphs allow a very simple isomorphism test. In all these papers no deep algebraic tools have been involved into the treatment of the topic.

The notation *compactness* is due to R. A. BRUALDI and appeared first in the paper [3]. BRUALDI introduced the even stronger graph property of *super-compactness* and presented various interesting examples of super-compact or at least compact graphs. In a more recent contribution [11] C. D. GODSIL discussed the role of *equitable partitions* of the vertex set of a graph in the context of compactness. He also showed that a regular compact graph has a *generously transitive* automorphism group whose rank equals the number of distinct eigenvalues of the graph. A short treatment of compact graphs appeared also in [4].

Very recently S. EVDOKIMOV, M. KARPINSKI and I. PONOMARENKO extended the notion of compactness to cellular algebras. They called a *cellular algebra W compact* iff every doubly stochastic matrix X which commutes with all elements $A \in W$ is a convex combination of permutation matrices corresponding to the automorphisms of W. Further, they introduced a new class of graphs by calling a graph *weakly compact* if it generates a compact cellular algebra. In [6] a comprehensive discussion of compactness is given. Perhaps the most important result of this paper is that compact algebras are Schurian, i.e. coincide with the centralizer algebras of permutation groups. Further, the authors showed that the compactness of two coherent algebras W_1 and W_2 is equivalent to the compactness of the direct sum $W_1 \boxplus W_2$, the wreath product $W_1 \wr W_2$ and, if one of the algebras is semi-regular, also to the tensor product $W_1 \otimes W_2$. This enabled them to present several series of examples of compact and non-compact cellular algebras, respectively graphs. Weakly compact graphs form a larger class than compact ones and still admit an efficient isomorphism test. Namely, the authors showed that for compact cellular algebras, and hence for weakly compact graphs, canonical labelings can be computed in polynomial time by a repeated application of the *Weisfeiler-Lehman* algorithm.

As mentioned above combinatorial compactness is connected with the graph isomorphism problem which is one of the famous unsettled problems of computational complexity theory. Up to the present many classes of graphs have been found for the members of which isomorphism testing can be done by polynomial-time algorithms (see [1]). In this paper we introduce and investigate a new class of graphs for which isomorphism testing can be done very efficiently by exploiting the structure of the cellular algebras generated by these graphs. We call them *algebraic* *forests*. Algebraic forests are weakly compact, they include cographs, trees and interval graphs, and even a joint superclass of the latter two, namely *rooted directed path graphs* (briefly, rdp-graphs). In fact, our class is much larger than these classes, since any graph is an induced subgraph of some algebraic forest. The class of algebraic forests admits also an efficient membership test.

Let us describe the results of the paper more formally. The key notion of our technique is a cellular (coherent) algebra introduced in [17] as a subalgebra of Mat_V closed under the Hadamard (componentwise) multiplication, the Hermitian conjugation and containing the identity matrix and the all one matrix. (Here Mat_V is the set of all complex matrices whose rows and columns are indexed by the elements of a finite set V.) The automorphism group $\mathrm{Aut}(W)$ of the cellular algebra W consists by definition of all permutations g of V such that the permutation matrix corresponding to g centralizes W. As already mentioned above, W is called compact if

$$\mathrm{DS}(W) = \mathrm{Conv}(\mathrm{Aut}(W))$$

where $\mathrm{DS}(W)$ is the polytope of all doubly stochastic matrices centralizing W (cf. (1)). Let us associate to each graph Γ the smallest cellular algebra $W(\Gamma)$ containing its adjacency matrix. The importance of this algebra lies in the fact that $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(W(\Gamma))$. The graph Γ is called weakly compact if $W(\Gamma)$ is a compact cellular algebra. Thus all compact graphs are obviously weakly compact.

In Section 4 we define the class of *forestal* cellular algebras playing the central role in the paper. The term is explained in Proposition 6.4 which states that each forestal algebra coincides with some restriction of the cellular algebra of some forest. Our definition is constructive in the sense that each forestal algebra can be obtained from one-point cellular algebras by means of taking direct sums and wreath products by symmetric groups (these operations are defined and studied in Section 3). Similar operations are well-known in permutation group theory and we prove (Theorem 4.4) that forestal algebras are just the centralizer algebras of permutation groups which can be obtained from the identity group by means of taking direct sums and wreath products by symmetric groups. This fact is the consequence of the following very strong property of forestal algebras: each weak isomorphism of them is induced by a bijection of their point sets (Theorem 4.2). (By a weak isomorphism of cellular algebras we mean an algebra isomorphism preserving the Hadamard multiplication.) Another consequence of this fact is that forestal algebras are compact (Corollary 4.3).

In Section 6 we define the class of algebraic forests consisting by definition of all graphs the cellular algebras of which are forestal. Our definition implies that each algebraic forest is a weakly compact graph (Theorem 6.1). We prove that forests (and so trees), cographs and rooted directed path graphs (and so interval graphs) are algebraic forests (Theorems 6.3, 6.5 and 6.6 respectively). The proof of these facts is based on the characterization of cellular algebras of disconnected graphs as direct sums or wreath products by symmetric groups (Theorem 5.1).

In the full version of the paper we consider algorithmic problems concerning forestal algebras and algebraic forests. First we present an $O(n^2 \log n)$ algorithm which, given a cellular algebra on n points, tests whether this algebra is forestal or not. This enables us to describe an $O(n^3 \log n)$ membership and isomorphism tests for n-vertex algebraic forests. As a consequence we improve the upper bound of the isomorphism test complexity for rdp-graphs given in [2]. Another interesting consequence of our results is an $O(n^3 \log n)$ upper bound for the cardinality of a complete set of invariants of an n-vertex algebraic forest. We complete the paper by discussing an algorithm (of the same complexity) which given a semisimple algebra with a selected linear base tests whether this algebra has a faithful cellular representation respecting the base.

In this extended abstract we omit some proofs and all algorithms. The preprint [9] is the preliminary version of the full paper.

Notation. As usual by $\mathbb{C}$ we denote the complex field.

Throughout the paper V denotes a finite set with $n = |V|$ elements. A subset of $V \times V$ is called a relation on V. For a relation R on V we define its support V_R to be the smallest set $U \subset V$ such that $R \subset U \times U$.

Under an equivalence E on V we always mean an ordinary equivalence on a subset of V (coinciding with V_E), the set of its equivalence classes is denoted by V/E.

The algebra of all complex matrices whose rows and columns are indexed by the elements of V is denoted by Mat_V, its unit element (the identity matrix) by I_V and the all one matrix by J_V. For $U \subset V$ the algebra Mat_U can be considered in a natural way as a subalgebra of Mat_V.

The adjacency matrix of a relation R is denoted by $A(R)$. For $U, U' \subset V$ let $J_{U,U'}$ denote the $\{0,1\}$-matrix with 1's exactly on the places belonging to $U \times U'$. The transpose of a matrix A is denoted by A^T, its Hermitian conjugate by A^*.

Each bijection $g : V \to V'$ defines a natural algebra isomorphism from Mat_V onto $\mathrm{Mat}_{V'}$. The image of a matrix A under g will be denoted by A^g.

The group of all permutations of V is denoted by $\mathrm{Sym}(V)$.

For an integer s we denote the set $\{1, \ldots, s\}$ by $[s]$ and $\mathrm{Sym}([s])$ by $\mathrm{Sym}(s)$.

2 Cellular algebras

2.1 By a *cellular algebra* on V we mean a subalgebra W of Mat_V for which the following conditions are satisfied:

(C1) $I_V, J_V \in W$;

(C2) $\forall A \in W : \quad A^* \in W$;

(C3) $\forall A, B \in W : \quad A \circ B \in W$,

where $A \circ B$ is the Hadamard (componentwise) product of the matrices A and B. It follows from (C2) that W is a semisimple algebra over $\mathbb{C}$. The elements of V are called the *points*, the set V is called the point set of W.

Each cellular algebra W on V has a uniquely determined linear basis $\mathcal{R} = \mathcal{R}(W)$ (the *standard basis* of W) consisting of $\{0,1\}$-matrices such that

$$\sum_{R \in \mathcal{R}} R = J_V \quad \text{and} \quad R \in \mathcal{R} \Leftrightarrow R^T \in \mathcal{R}. \tag{2}$$

The nonnegative integers $c_{R,S}^T$ defined by $RS = \sum_{T \in \mathcal{R}} c_{R,S}^T T$ where $R, S \in \mathcal{R}$, are called the *structure constants* of W.

Set $\mathrm{Cel}(W) = \{U \subset V : I_U \in \mathcal{R}\}$ and $\mathrm{Cel}^*(W) = \{U \subset V : I_U \in W\}$. Each element of $\mathrm{Cel}(W)$ (resp. $\mathrm{Cel}^*(W)$) is called a *cell* of W (resp. a *cellular set* of W). Obviously,

$$V = \bigcup_{U \in \mathrm{Cel}(W)} U \qquad \text{(disjoint union)}.$$

The algebra W is called *homogeneous* if $|\mathrm{Cel}(W)| = 1$.

For $U, U' \in \mathrm{Cel}^*(W)$ set $\mathcal{R}_{U,U'} = \{R \in \mathcal{R} : \ R \circ J_{U,U'} = R\}$. Then

$$\mathcal{R} = \bigcup_{U,U' \in \mathrm{Cel}(W)} \mathcal{R}_{U,U'} \qquad \text{(disjoint union).}$$

Moreover, for two cells U, U' the number of 1's in the uth row (resp. vth column) of the matrix $R \in \mathcal{R}_{U,U'}$ does not depend on the choice of $u \in U$ (resp. $v \in U'$).

For each $U \in \mathrm{Cel}^*(W)$ we view the subalgebra $I_U W I_U$ of W as a cellular algebra on U and denote it by W_U. The basis matrices of W_U are in 1-1 correspondence to the matrices of $\mathcal{R}_{U,U}$. If $U \in \mathrm{Cel}(W)$ we call W_U the *homogeneous component* of W corresponding to U.

Each matrix $R \in \mathcal{R}$ being a $\{0,1\}$-matrix is the adjacency matrix of some relation on V called a *basis relation* of W. By (2) the set of all of them forms a partition of $V \times V$ which can be interpreted as a coherent configuration on V (see [12]). We use all the notations introduced for basis matrices also for basis relations.

2.2 A large class of cellular algebras comes from permutation groups as follows (see [17]). Let $G \le \mathrm{Sym}(V)$ be a permutation group and

$$\mathcal{Z}(G) = \{A \in \mathrm{Mat}_V : \ A^g = A, \ g \in G\}$$

be its centralizer algebra. Then $\mathcal{Z}(G)$ is a cellular algebra on V. Its basis relations are exactly the 2-orbits of G i.e. the orbits of the coordinatewise action of G on $V \times V$. In particular, $\mathrm{Cel}(\mathcal{Z}(G)) = \mathrm{Orb}(G)$ where $\mathrm{Orb}(G)$ is the set of G-orbits of V.

Let W and W' be cellular algebras on V and V', respectively, and let $g : V \to V'$ be a bijection such that $W' = \{A^g : A \in W\}$. Then W and W' are called *strongly isomorphic* (briefly, isomorphic) and g is called a *strong isomorphism* (briefly, isomorphism) from W to W'. Clearly, g induces a bijection between the sets $\mathcal{R}(W)$ and $\mathcal{R}(W')$. The group of all isomorphisms from W to itself contains a normal subgroup

$$\mathrm{Aut}(W) = \{g \in \mathrm{Sym}(V) : \ A^g = A, \ A \in W\}$$

called the *automorphism group* of W. We stress that each $g \in \mathrm{Aut}(W)$ induces the identical map of W and so not every possible isomorphism from W to itself belongs to $\mathrm{Aut}(W)$.

If $W = \mathcal{Z}(\mathrm{Aut}(W))$, then W is called *Schurian.* In this case the basis relations and cells of W are exactly the 2-orbits and orbits of its automorphism group.

2.3 The set of all cellular algebras on V is ordered by inclusion. The largest and the smallest elements of this set are respectively the full matrix algebra Mat_V and the simplex $S(V) = \mathcal{Z}(\mathrm{Sym}(V))$, i.e. the algebra with the linear base $\{I_V, J_V\}$. For cellular algebras W and W' we write $W \le W'$ if W is a subalgebra of W'.

Given $X \subset \mathrm{Mat}_V$, we denote by $[X]$ the cellular closure of X, i.e. the smallest cellular algebra containing X. For $A \in \mathrm{Mat}_V$ we write $[A]$ instead of $[\{A\}]$.

3 Weak isomorphisms, direct sums and wreath products

3.1 Along with the notion of a strong isomorphism for cellular algebras we consider also weak isomorphisms (see [17, 5]). Cellular algebras W on V and W' on V' are called *weakly isomorphic* if there exists an algebra isomorphism $\varphi : W \to W'$ such that

(WI) $\varphi(A \circ B) = \varphi(A) \circ \varphi(B)$ for all $A, B \in W$.

Any such φ is called a *weak isomorphism* from W to W'. The set of all of them is denoted by $\mathrm{Isow}(W, W')$. If $W = W'$ we write $\mathrm{Isow}(W)$ instead of $\mathrm{Isow}(W, W)$. Clearly, $\mathrm{Isow}(W)$ forms a group which is isomorphic to a subgroup of $\mathrm{Sym}(\mathcal{R}(W))$. Evidently, each strong isomorphism from W to W' induces a weak isomorphism between these algebras in a natural way.

The following statement establishes the simplest properties of weak isomorphisms.

Lemma 3.1 *Let $W \le \mathrm{Mat}_V$ and $W' \le \mathrm{Mat}_{V'}$ be cellular algebras and $\varphi \in \mathrm{Isow}(W, W')$ a weak isomorphism. Then*

(1) *$\varphi(\mathcal{R}) = \mathcal{R}'$ where $\mathcal{R} = \mathcal{R}(W)$ and $\mathcal{R}' = \mathcal{R}(W')$. Moreover, $\varphi(R^T) = \varphi(R)^T$ for all $R \in \mathcal{R}$.*

(2) *φ induces a natural bijection $U \mapsto U^\varphi$ from $\mathrm{Cel}^*(W)$ onto $\mathrm{Cel}^*(W')$ preserving cells such that $\varphi(I_U) = I_{U^\varphi}$. Moreover, $|U| = |U^\varphi|$ and, in particular, $|V| = |V'|$.*

(3) *$\varphi(\mathcal{R}_{U_1,U_2}) = \mathcal{R}'_{U_1^\varphi,U_2^\varphi}$ for all $U_1, U_2 \in \mathrm{Cel}^*(W)$.*■

It follows from statement (1) of the lemma that the structure constants of weakly isomorphic algebras coincide. More exactly, if $\varphi \in \mathrm{Isow}(W, W')$, then $c_{R,S}^T = c_{\varphi(R),\varphi(S)}^{\varphi(T)}$ for all $R, S, T \in \mathcal{R}(W)$.

3.2 Let W be a cellular algebra on V and E be an equivalence on V. The following definitions were introduced in [5]. We say that E is an *equivalence* of W if it is the union of basis relations of W. A nonempty equivalence E of W is called *indecomposable* (in W) if E is not a disjoint union of two nonempty equivalences of W (equivalently, the smallest cellular set of W containing a class of E coincides with V_E). Otherwise, the equivalence is called *decomposable*. Each equivalence of W can uniquely be represented as a disjoint union of indecomposable ones called *indecomposable components* of E.

Let E be an equivalence of W with classes $V_1, \ldots, V_s$. For each $i \in [s]$ the set $W_{E,V_i} = I_{V_i} W I_{V_i}$ can be viewed as a cellular algebra on V_i with the standard basis

$$\mathcal{R}(W_{E,V_i}) = \{I_{V_i} R I_{V_i} \;:\; R \in \mathcal{R},\; R \circ A(E) = R,\; I_{V_i} R I_{V_i} \neq 0\} \tag{3}$$

where $A(E)$ is the adjacency matrix of E. Clearly, each basis matrix of W_{E,V_i} can uniquely be represented in the form $I_{V_i} R I_{V_i}$ with $R \in \mathcal{R}$.

Lemma 3.2 ([5]) *If E is an indecomposable equivalence of W then*

(1) *the mapping $\varphi_{V_i,V_j} : W_{E,V_i} \to W_{E,V_j}$ such that $\varphi_{V_i,V_j}(I_{V_i} A I_{V_i}) = I_{V_j} A I_{V_j}$ for all $A \in W$ is a weak isomorphism from W_{E,V_i} to W_{E,V_j} for all $i, j \in [s]$.*

(2) *$|V_i \cap U| = |V_j \cap U| > 0$ for all cell $U \subset V_E$ and $i, j \in [s]$.*■

Set $W_E = W \circ A(E)$. Then W_E is a subalgebra of W closed with respect to the Hadamard multiplication and the Hermitian conjugation. In fact the lemma shows that the mapping

$$\varphi_{V_i} : W_E \to W_{E,V_i}, \quad A \mapsto I_{V_i} A I_{V_i}$$

is an algebra isomorphism satisfying (WI), and also $\varphi_{V_i,V_j} = \varphi_{V_i}^{-1} \varphi_{V_j}$ for all $i, j \in [s]$.

The following statement shows that equivalences of a cellular algebra are preserved by weak isomorphisms.

Lemma 3.3 *Let $W \leq \mathrm{Mat}_V$ and $W' \leq \mathrm{Mat}_{V'}$ be cellular algebras and $\varphi \in \mathrm{Isow}(W, W')$. Then φ induces a natural bijection $E \mapsto E^\varphi$ from the set of all equivalences of W on that of W' such that $\varphi(A(E)) = A(E^\varphi)$. Moreover, $|V_E| = |V'_{E^\varphi}|$ and $|V/E| = |V'/E^\varphi|$.* ■

3.3 Let $W_1, \ldots, W_s$ be cellular algebras on sets $V_1, \ldots, V_s$ respectively. Then the set of cellular algebras W on the disjoint union of $V_1, \ldots, V_s$ such that V_i is a cellular set of W and $W_{V_i} = W_i$ for all i, has the smallest element. It is called the *direct sum* of the cellular algebras $W_1, \ldots, W_s$ (see [17]) and is denoted by $W_1 \boxplus \cdots \boxplus W_s$. The basis $\mathcal{R}$ of the direct sum satisfies $|\mathcal{R}_{U,U'}| = 1$ for all of its cells $U \subset V_i$, $U' \subset V_j$ with $i \neq j$.

3.4 Let W be a cellular algebra on V and Ψ be a group of its weak isomorphisms. Following [7] we define a cellular algebra W^Ψ as follows. For each $R \in \mathcal{R}$ denote by Ψ_R the subgroup of Ψ leaving R fixed and set

$$R^\Psi = \frac{1}{|\Psi_R|} \sum_{\psi \in \Psi} \psi(R)$$

where $\mathcal{R} = \mathcal{R}(W)$. Then clearly R^Ψ is a {0,1}-matrix and any two such matrices either coincide or orthogonal with respect to the Hadamard multiplication. Set

$$W^\Psi = [\mathcal{R}^\Psi] \quad \text{where} \quad \mathcal{R}^\Psi = \{R^\Psi : \ R \in \mathcal{R}\}.$$

It was proved in [7] that the set $\mathcal{R}^\Psi$ is the standard basis of the cellular algebra W^Ψ. Thus the basis relations of W^Ψ are in 1-1 correspondence with the orbits of the action of the group Ψ on $\mathcal{R}$.

Let $\mathcal{W} = \{W_i\}_{i=1}^s$ be a family of cellular algebras $W_i \leq \mathrm{Mat}_{V_i}$ and $\Phi = \{\varphi_{i,j}\}_{i,j=1}^s$ be a family of weak isomorphisms $\varphi_{i,j} \in \mathrm{Isow}(W_i, W_j)$ such that

$$\varphi_{i,j} \circ \varphi_{j,k} = \varphi_{i,k}, \qquad i, j, k \in [s] \tag{4}$$

where $\circ$ denotes the composition of maps. Obviously, $\varphi_{i,i} = \mathrm{id}_{W_i}$ and $\varphi_{j,i} = \varphi_{i,j}^{-1}$ for all i, j. Set $W = \boxplus_{i=1}^s W_i$. Any permutation $g \in \mathrm{Sym}(s)$ induces a weak isomorphism $\varphi_g : W \to W$ by means of s weak isomorphisms $\varphi_{i,i^g} : W_i \to W_{i^g}$, $i \in [s]$. For each group $G \leq \mathrm{Sym}(s)$ the set $\Phi_G = \{\varphi_g : \ g \in G\}$ is a subgroup of the group $\mathrm{Isow}(W)$.

Definition 3.4 ([8]) *The cellular algebra W^{Φ_G} is called the wreath product of the family $\mathcal{W}$ by the group G with respect to Φ and is denoted by $\mathcal{W} \wr_\Phi G$.*

The equivalence E with classes $V_1, \ldots, V_s$ is called the *standard equivalence* of $\mathcal{W} \wr_\Phi G$. In homogeneous case it is uniquely determined by this algebra. (However, this can be false if $\mathcal{W} \wr_\Phi G$ is not homogeneous and $G = \mathrm{Sym}(2)$.) It follows from the definition that the adjacency matrix of E belongs to $\mathcal{W} \wr_\Phi G$ and

$$[\mathcal{W} \wr_\Phi G, I_{V_1}, \ldots, I_{V_s}] = W_1 \boxplus \cdots \boxplus W_s. \tag{5}$$

Remark 3.5 *If G is transitive and W_i is homogeneous for all i, then the wreath product defined above is a special case of the corresponding construction of [17]. Besides, if $\varphi_{i,j}$ is induced by a strong isomorphism for all i, j, then $\mathcal{W} \wr_\Phi G$ is strongly isomorphic to the wreath product $W_1 \wr \mathcal{Z}(G)$ as defined in [6].*

The form of the basis matrices of the wreath product $\mathcal{W} \wr_\Phi G$ with $\mathcal{W} = \{W_i\}_{i=1}^s$ and $\Phi = \{\varphi_{i,j}\}_{i,j=1}^s$ is especially simple for a transitive group $G \leq \mathrm{Sym}(s)$. In this case they are divided into two parts. The first part consists of the block-diagonal matrices $\sum_{i=1}^s \varphi_i(R)$ where $R \in \mathcal{R}(W_1)$ and $\varphi_i = \varphi_{1,i}$, $i \in [s]$. The matrices from the second part are of the form $\sum_{(i,j) \in O} J_{U^{\varphi_i}, U'^{\varphi_j}}$ where

$U,U' \in \mathrm{Cel}(W_1)$ and O is a nonreflexive 2-orbit of the group G. For example, if $G = \mathrm{Sym}(s)$ we get the matrices $\sum_{i,j\in[s],\ i\neq j} J_{U^{\varphi_i},U'^{\varphi_j}}$.

Let $W \le \mathrm{Mat}_V$ be a cellular algebra and E be an indecomposable equivalence of W. Set $\mathcal{W}_E = \{W_{E,V_i}\}_{i=1}^s$ and $\Phi_E = \{\varphi_{V_i,V_j}\}_{i,j=1}^s$ where $V_1,\dots,V_s$ are the classes of E and W_{E,V_i} and φ_{V_i,V_j} are defined in Subsection 3.2. In this case the family Φ_E satisfies (4) by Lemma 3.2 and so for any $G \le \mathrm{Sym}(s)$ the wreath product $\mathcal{W}_E \wr_{\Phi_E} G$ can be considered.

Lemma 3.6 *The following statements hold:*

(1) *If G is a transitive group, then the standard equivalence E of the wreath product $\mathcal{W} \wr_\Phi G$ is indecomposable. Moreover, $\mathcal{W} = \mathcal{W}_E$ and $\Phi = \Phi_E$.*

(2) *If E is an arbitrary indecomposable equivalence of a cellular algebra $W \le \mathrm{Mat}_V$ with $V_E = V$, then $W \ge \mathcal{W}_E \wr_{\Phi_E} \mathrm{Sym}(s)$ where s is the number of classes of E.*■

3.5 Now we are ready to prove the main result of this section which states that the property of a cellular algebra to be direct sum or wreath product is preserved under weak isomorphisms.

Theorem 3.7 *Let $W \le \mathrm{Mat}_V$, $W' \le \mathrm{Mat}_{V'}$ be cellular algebras and $\varphi \in \mathrm{Isow}(W,W')$. Then*

(1) *if $W = W_1 \boxplus W_2$, then $W' = W_1' \boxplus W_2'$ where $W_i' = \varphi(W_i)$, $i = 1,2$;*

(2) *if $W = \mathcal{W} \wr_\Phi \mathrm{Sym}(s)$, then $W' = \mathcal{W}' \wr_{\Phi'} \mathrm{Sym}(s)$ for some $\mathcal{W}'$ and Φ'. More exactly, if $\mathcal{W} = \mathcal{W}_E$ and $\Phi = \Phi_E$ where E is the standard equivalence of $\mathcal{W} \wr_\Phi \mathrm{Sym}(s)$, then $\mathcal{W}' = \mathcal{W}_{E^\varphi}$ and $\Phi' = \Phi_{E^\varphi}$. Besides, $\varphi(W_E) = W_{E^\varphi}$.*

Proof. To prove the first statement we note that $W_i = I_{V_i} W I_{V_i}$ for some cellular set V_i of W, $i = 1,2$. So by statement (2) of Lemma 3.1 we conclude that $W_i' = I_{V_i^\varphi} W' I_{V_i^\varphi}$. It follows from the definition of the direct sum that $|\mathcal{R}_{U_1,U_2}| = 1$ for all $U_1,U_2 \in \mathrm{Cel}(W)$ such that $U_i \subset V_i$, $i = 1,2$. So by statement (3) of Lemma 3.1 we obtain that $|\mathcal{R}'_{U_1',U_2'}| = 1$ for all $U_1',U_2' \in \mathrm{Cel}(W')$ such that $U_i' \subset V_i^\varphi$, $i = 1,2$. This proves statement (1).

Let us prove the second statement. Since the equivalence E is indecomposable by statement (1) of Lemma 3.6, so is E^φ. Lemma 3.3 implies that $|V'/E^\varphi| = |V/E| = s$. Thus, using statement (2) of Lemma 3.6 we find that $W' \ge \mathcal{W}_{E^\varphi} \wr_{\Phi_E} \mathrm{Sym}(s)$. Since both algebras are obviously of the same dimension we see that $W' = \mathcal{W}_{E^\varphi} \wr_{\Phi_{E^\varphi}} \mathrm{Sym}(s)$. The equality $\varphi(W_E) = W_{E^\varphi}$ follows from the definition of W_E.■

4 Forestal cellular algebras

4.1 Let us recursively define a class $\mathcal{F}$ of cellular algebras as follows:

(F1) Any cellular algebra on a one-point set belongs to $\mathcal{F}$.

(F2) If $W, W' \in \mathcal{F}$, then $W \boxplus W' \in \mathcal{F}$.

(F3) If $\mathcal{W} = \{W_i\}_{i=1}^s$ with $W_i \in \mathcal{F}$, then $\mathcal{W} \wr_\Phi \mathrm{Sym}(s) \in \mathcal{F}$ for all Φ.

Obviously, the class $\mathcal{F}$ is closed under strong isomorphisms. A member of $\mathcal{F}$ is called a *forestal* cellular algebra. The term will be explained in Section 6 (Proposition 6.4). Let us describe the simplest properties of forestal slgebras.

Lemma 4.1 *The following statements hold:*

(1) $W \in \mathcal{F} \;\Rightarrow\; [W, I_U] \in \mathcal{F}$ *for all* $U \subset V$*;*

(2) $W \in \mathcal{F} \;\Rightarrow\; W_U \in \mathcal{F}$ *for all* $U \in \mathrm{Cel}^*(W)$*;*

(3) $W \boxplus W' \in \mathcal{F} \;\Leftrightarrow\; W, W' \in \mathcal{F}$*;*

(4) $\mathcal{W} \wr_\Phi \mathrm{Sym}(s) \in \mathcal{F} \;\Leftrightarrow\; W_i \in \mathcal{F}$ *for all* $i \in [s]$.■

Forestal cellular algebras form a nice class from the point of view of isomorphism testing. The following statement which can be proved by using Theorem 3.7 provides a theoretical background.

Theorem 4.2 *The class* $\mathcal{F}$ *is closed under weak isomorphisms. Each weak isomorphism between cellular algebras belonging to* $\mathcal{F}$ *is induced by a strong isomorphism.*■

Corollary 4.3 *Each forestal algebra is compact.*

Proof. Since the direct sum of compact algebras is compact (see [6], Theorem 4.1), it suffices to check that if $\mathcal{W} = \{W_i\}_{i=1}^s$ is a family of compact forestal algebras and $\Phi = \{\varphi_{i,j}\}_{i,j=1}^s$ is a family of weak isomorphisms satisfying (4), then $\mathcal{W} \wr_\Phi \mathrm{Sym}(s)$ is also compact. By Theorem 4.2 the weak isomorphism $\varphi_{i,j}$ is induced by a strong isomorphism for all i, j. By Remark 3.5 this implies that the algebra $\mathcal{W} \wr_\Phi \mathrm{Sym}(s)$ is isomorphic to $W_1 \wr \mathcal{Z}(\mathrm{Sym}(s))$. However according to Theorem 4.2 of [6], the last algebra is compact, since so are W_1 and $\mathcal{Z}(\mathrm{Sym}(s))$ (the first one by the assumption and the second one by Birkhoff's theorem).■

In [6] the following question was formulated: is it true that two weakly isomorphic compact cellular algebras are isomorphic? It follows from Theorem 4.2 and Corollary 4.3 that the answer is "yes" at least in the case of the subclass $\mathcal{F}$ of the class of all compact algebras.

4.2 Now we turn into the decsription of the class $\mathcal{F}$ in permutation group terms. To do this we recursively define a class $\mathcal{G}$ of permutation groups as follows:

(G1) Any permutation group on a one-point set belongs to $\mathcal{G}$.

(G2) If $G, G' \in \mathcal{G}$, then $G \boxplus G' \in \mathcal{G}$.

(G3) If $G \in \mathcal{G}$, then $G \wr \mathrm{Sym}(s) \in \mathcal{G}$ for all s.

where for $G \le \mathrm{Sym}(V)$ and $G' \le \mathrm{Sym}(V')$ the permutation group $G \boxplus G'$ is defined by the induced action of $G \times G'$ on the disjoint union of V and V' and $G \wr \mathrm{Sym}(s)$ denotes the imprimitive action of the wreath product of G and $\mathrm{Sym}(s)$ on the set $V \times [s]$. We note that by (G1) and (G3) the class $\mathcal{G}$ contains any symmetric group $\mathrm{Sym}(s)$ in its natural representation. As the following statement shows the maps $\mathrm{Aut} : \mathcal{F} \to \mathcal{G}$ and $\mathcal{Z} : \mathcal{G} \to \mathcal{F}$ are inverse to each other.

Theorem 4.4 *All forestal algebras are Schurian. Moreover, a cellular algebra is forestal iff it coincides with the centralizer algebra of a permutation group from the class* $\mathcal{G}$*. Besides, the automorphism group of a forestal algebra belongs to* $\mathcal{G}$.■

5 Cellular algebras of disconnected graphs

5.1 Throughout this section the word "graph" means an undirected graph without loops and multiple edges. For a graph $\Gamma = (V, E)$ with vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$ we set $\Gamma(v) = \{u \in V : (v, u) \in E\}$. For a set $U \subset V$ we set $\Gamma_U = (U, (U \times U) \cap E)$ and call it a subgraph of Γ induced by U.

The adjacency matrix of the graph Γ is denoted by $A(\Gamma)$. By the cellular algebra of Γ we mean $W(\Gamma) = [A(\Gamma)]$. It is well-known (see [17]) that $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(W(\Gamma))$ where $\mathrm{Aut}(\Gamma)$ is the automorphism group of Γ.

5.2 In this subsection we characterize cellular algebras of disconnected graphs. The next theorem in fact states that they are direct sums or wreath products by symmetric groups.

Theorem 5.1 *Let Γ be a disconnected graph with components $\Gamma_1, \ldots, \Gamma_s$. Then exactly one of the following statements holds:*

(1) $W(\Gamma) = W(\Gamma') \boxplus W(\Gamma'')$ *where Γ' and Γ'' are nonempty subgraphs of Γ the disjoint union of which equals Γ;*

(2) $W = \mathcal{W} \wr_\Phi \mathrm{Sym}(s)$ *for $\mathcal{W} = \{W(\Gamma_i)\}_{i=1}^s$ and some $\Phi = \{\varphi_{i,j}\}_{i,j=1}^s$ such that $\varphi_{i,j}(A(\Gamma_i)) = A(\Gamma_j)$ for all i, j.*

Proof. Let $W = W(\Gamma)$ and $E = \cup_{i=1}^s V_i \times V_i$ where $V_i = V(\Gamma_i)$, $i \in [s]$. Evidently, E is an equivalence of W. Suppose first that E is decomposable. Then there exist equivalences E' and E'' of W such that $V' = V'_{E'}$ and $V'' = V''_{E''}$ are disjoint cellular sets of W with $V' \cup V'' = V(\Gamma)$. It may be easily checked directly that $A(\Gamma') \in W_{V'}$ and $A(\Gamma'') \in W_{V''}$ where Γ' and Γ'' are the subgraphs of Γ induced by V' and V'', respectively. So $W' \boxplus W'' \le W$ where $W' = W(\Gamma')$ and $W'' = W(\Gamma'')$. On the other hand, by the definition of V' and V'' we conclude that $A(\Gamma) \in W' \boxplus W''$. This implies that $W' \boxplus W'' \ge W$, whence the first statement follows.

Let E be an indecomposable equivalence of W. Note that $W(\Gamma_i)$ is a subalgebra of W_{E,V_i} for all i. By statement (1) of Lemma 3.2 the mapping $\varphi_{i,j} = \varphi_{V_i,V_j}$ is a weak isomorphism from $W(\Gamma_i)$ to $W(\Gamma_j)$ for all i, j. We show that $W = \mathcal{W} \wr_\Phi \mathrm{Sym}(s)$ where $\Phi = \{\varphi_{i,j}\}_{i,j=1}^s$. From statement (2) of Lemma 3.6 it follows that $W \ge \mathcal{W} \wr_\Phi \mathrm{Sym}(s)$. The converse inclusion is trivial, since $A(\Gamma)$ belongs to $\mathcal{W} \wr_\Phi \mathrm{Sym}(s)$. ∎

6 Algebraic forests

6.1 A graph Γ is called an *algebraic forest* if $W(\Gamma)$ is a forestal cellular algebra. It follows from the definition that the class of algebraic forests includes all graphs whose cellular algebras are full matrix algebras. Less trivial examples of algebraic forests will be described below. Our interest in algebraic forests is explained by the following fundamental statement which immediately follows from Corollary 4.3.

Theorem 6.1 *Each algebraic forest is weakly compact.*∎

The following statement shows a simple way for constructing algebraic forests from smaller ones.

Lemma 6.2 *The class of algebraic forests is closed under taking the complement of a graph and taking the disjoint union of graphs.*

Proof. The first statement immediately follows from the fact that the cellular algebras of a graph and its complement coincide. The second one can be proved inductively by using Theorem 5.1.■

Note that the class of algebraic forests can not be characterized in terms of forbidden subgraphs. Indeed, any graph is an induced subgraph of a graph the cellular algebra of which is a full matrix algebra. However, the latter is obviously an algebraic forest.

6.2 Our first example of algebraic forests is the class of trees. It was proved in [14] that trees are compact graphs. Due to a result in [6] this implies that the cellular algebra of a tree is Schurian, i.e. coincides with the centralizer algebra of its automorphism group. However, it follows from [10] that the automorphism group of a tree belongs to the class $\mathcal{G}$ (see section 4). So by Theorem 4.4 we conclude that the cellular algebra of a tree is forestal. With the help of Theorem 5.1 and Lemma 6.2 this gives a proof of the following statement.

Theorem 6.3 *Each forest (in particular, each tree) is an algebraic forest.*■

The converse statement explaining the term "forestal algebras" is also true in the following sense.

Proposition 6.4 *Let* $W \le \mathrm{Mat}_V$ *be a forestal algebra. Then* $W = W(\Gamma)_U$ *for some forest* Γ *and some cellular set* U *of the algebra* $W(\Gamma)$.■

6.3 Let us consider the well-known class of *cographs* which are defined as exactly those graphs which contain no induced path on four vertices. It is a well-known fact that cographs can be constructed from one-vertex graphs by means of the following two operations: taking the complement of a graph and taking the disjoint union of graphs. With the help of Lemma 6.2, this observation leads to the following statement.

Theorem 6.5 *Each cograph is an algebraic forest.*■

It follows from Theorems 6.1 and 6.5 that cographs are weakly compact. However, the following example shows that not all of them is compact. Let $\Gamma_{m,n}$ be a disjoint union of m undirected cycles with three vertices and n undirected cycles with four vertices. Then it is easy to see that $\Gamma_{m,n}$ is a cograph and so is weakly compact. On the other hand, it is not compact for $n \ge 1$ and $m \ge 1$, since it is a regular graph (of degree two) with intransitive automorphism group.

6.4 A graph Γ is called a *rooted directed path graph* (rdp-graph) if there exists a rooted tree T and a map $f : V(\Gamma) \to 2^{V(T)}$ such that the subgraph $T_{f(v)}$ is a directed path of T for all $v \in V(\Gamma)$ and $(u, v) \in E(\Gamma)$ iff $f(u) \cap f(v) \ne \emptyset$. Since each tree is obviously an rdp-graph, the following statement (which can be proved by using the results of paper [2]) generalizes Theorem 6.3.

Theorem 6.6 *Each rdp-graph is an algebraic forest.*■

Observing that interval graphs (the intersection graphs of line segments) are rdp-graphs, we obtain the following consequence of Theorem 6.6.

Theorem 6.7 *Each interval graph is an algebraic forest.*■

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