A Generalization of the Terwilliger Algebra

Eric S. Egge
Department of Mathematics
University of Wisconsin
Madison, WI 53706 USA

0 Introduction

In [5, 6, 7], Paul Terwilliger introduced the subconstituent algebra of a commutative association scheme, which later became known as the Terwilliger algebra. This algebra is constructed from the Bose Mesner and dual Bose Mesner algebras of the scheme in a natural way. Terwilliger showed in [5] that each vanishing intersection number and krein parameter of the scheme gives rise to a relation in the Terwilliger algebra. Many results concerning Terwilliger algebras follow from these relations in a purely algebraic way, independent of the underlying association scheme; this paper begins an exploration of these results. Towards this end, we first replace the Bose Mesner and dual Bose Mesner algebras with a pair of dual character algebras (see Definitions 1.1 and 2.1). We then use this pair to define an analogue $\boldsymbol{T}$ of the Terwilliger algebra. We conclude by obtaining an irreducible module for $\boldsymbol{T}$, which we call the primary module, and using the primary module to discern some of the structure of $\boldsymbol{T}$.

Throughout we write $\mathbf{C}$ to denote the field of complex numbers and $\mathbf{R}$ to denote the field of real numbers.

1 Character Algebras

We begin by recalling the notion of a character algebra.

Definition 1.1 *Suppose D is a positive integer. A* ***character algebra*** *$C = < X_0, \ldots, X_D >$ is a finite dimensional associative $\mathbf{C}$-algebra together with a distinguished basis $X_0, \ldots, X_D$ having the following properties.*

1. *C is commutative.*
2. *$X_0 = 1$ is the multiplicative identity element of C.*
3. *Let $p^h_{ij} (0 \leq h, i, j \leq D)$ denote complex numbers such that*

$$X_i X_j = \sum_{h=0}^{D} p^h_{ij} X_h. \qquad (0 \leq i, j \leq D) \tag{1}$$

   Then $p^h_{ij} \in \mathbf{R}$, $0 \leq h, i, j \leq D$.

4. *There exist an involution $i \mapsto i'$ of the set $\{0, \ldots, D\}$ and positive real numbers $k_0, k_1, \ldots, k_D$ such that*

$$p^0_{ij} = \delta_{ji'} k_i. \qquad (0 \leq i, j \leq D) \tag{2}$$

5. *The linear map $\tau : C \to C$ satisfying $\tau(X_i) = X_{i'}$ for $0 \leq i \leq D$ is a $\mathbf{C}$-algebra isomorphism.*
6. *The linear map $\pi_0 : C \to \mathbf{C}$ satisfying $\pi_0(X_i) = k_i$ for $0 \leq i \leq D$ is a $\mathbf{C}$-algebra homomorphism.*

We present two examples of character algebras.

Example 1.2 *Suppose G is a finite group and let $\mathbf{C}G$ denote the group algebra. Let $C_0 = \{1_G\}, C_1, \ldots, C_D$ denote the conjugacy classes of G and write*

$$X_i = \sum_{x \in C_i} x. \qquad (0 \leq i \leq D)$$

Then $C = < X_0, \ldots, X_D >$ is a character algebra, with $C_{i'} = \{x^{-1} \mid x \in C_i\}$ and $k_i = |C_i|$ for $0 \le i \le D$. We observe that C is the center of $\mathbf{C}G$.

Example 1.3 *Suppose G is a finite group. Let χ_0 denote the trivial character of G, and let $\chi_1, \ldots, \chi_D$ denote the remaining irreducible characters of G. Write*

$$X_i = d_i \chi_i, \qquad\qquad (0 \le i \le D)$$

where $d_i = deg\ \chi_i$ for $0 \le i \le D$. Then $C = < X_0, \ldots, X_D >$ is a character algebra, with $\chi_{i'} = \overline{\chi_i}$ and $k_i = d_i^2$ for $0 \le i \le D$.

2 Duality

For a given finite group, the character algebra of Example 1.2 and the character algebra of Example 1.3 are closely related. This relationship generalizes to arbitrary character algebras in the following way.

Suppose $C = < X_0, \ldots, X_D >$ is a character algebra; then it's known that C is semisimple. Since C is also commutative, it follows that C has a basis $E_0, \ldots, E_D$ of mutually orthogonal primitive idempotents. In addition, we may always take $E_0 = N^{-1} \sum_{h=0}^{D} X_h$, where $N = \sum_{i=0}^{D} k_i$. Let P denote the matrix in $M_{D+1}(\mathbf{C})$ which satisfies

$$X_i = \sum_{j=0}^{D} P_{ji} E_j. \qquad\qquad (0 \le i \le D) \tag{3}$$

We refer to P as the ***matrix of eigenvalues*** of C (with respect to the given ordering of the primitive idempotents) and we refer to N as the ***size*** of C.

Definition 2.1 *Suppose $C = < X_0, \ldots, X_D >$ and $C^* = < X_0^*, \ldots, X_D^* >$ are character algebras with primitive idempotents $E_0, \ldots, E_D$ and $E_0^*, \ldots, E_D^*$. Let P and P^* denote the matrices of eigenvalues of C and C^* respectively. We say C and C^* are **dual** (with respect to the given orderings of the primitive idempotents) whenever $PP^* \in Span(I)$. In this case we have $N = N^*$ and $PP^* = NI$. More generally, we say C and C^* are **dual** whenever they are dual with respect to some orderings of their primitive idempotents.*

Definition 2.1 is related to Examples 1.2 and 1.3 as follows. Suppose G is a finite group, let C denote the corresponding character algebra of Example 1.2, and let C^* denote the corresponding character algebra of Example 1.3. Then C and C^* are dual in the sense of Definition 2.1.

3 The Terwilliger Algebra

We now construct the Terwilliger algebra from a pair of dual character algebras.

Definition 3.1 *Suppose $C = < X_0, \ldots, X_D >$ and $C^* = < X_0^*, \ldots, X_D^* >$ are dual character algebras with respect to the orderings $E_0, \ldots, E_D$ and $E_0^*, \ldots, E_D^*$ of their primitive idempotents. Let $\boldsymbol{T}$ denote the associative $\mathbf{C}$-algebra generated by the symbols $x_0, \ldots, x_D, x_0^*, \ldots, x_D^*$ subject to the relations*

1. $x_0 = x_0^* = 1$,

2. $x_i x_j = \sum_{s=0}^{D} p_{ij}^s x_s, \qquad (0 \le i, j \le D)$

2*. $x_i^* x_j^* = \sum_{s=0}^{D} p_{ij}^{s*} x_s^*, \qquad (0 \le i, j \le D)$

3. $e_i^* x_j e_h^* = 0$ *if* $p_{ij}^h = 0, \qquad (0 \le h, i, j \le D)$

3*. $e_i x_j^* e_h = 0$ *if* $p_{ij}^{h*} = 0, \qquad (0 \le h, i, j \le D)$

where $e_i = N^{-1}\sum_{j=0}^{D} P^*_{ji} x_j$ *and* $e^*_i = N^{-1}\sum_{j=0}^{D} P_{ji} x^*_j$ *for* $0 \leq i \leq D$. *Here* P *is the matrix of eigenvalues of* C *with respect to the ordering* $E_0, \ldots, E_D$ *and* P^* *is the matrix of eigenvalues of* C^* *with respect to the ordering* $E^*_0, \ldots, E^*_D$. *We refer to* $\boldsymbol{T}$ *as the* ***Terwilliger algebra*** *associated with* C, C^* *and the orderings* $E_0, \ldots, E_D$ *and* $E^*_0, \ldots, E^*_D$.

Suppose $C = < X_0, \ldots, X_D >$ and $C^* = < X^*_0, \ldots, X^*_D >$ are dual character algebras. The orderings of the primitive idempotents of C and C^* with respect to which these character algebras are dual are not necessarily unique. However, the isomorphism class of the Terwilliger algebra associated with C and C^* is independent of the orderings of the primitive idempotents of C and C^*. With this in mind, we will suppress the orderings of the primitive idempotents of C and C^* when discussing the associated Terwilliger algebra.

4 The Primary Module

Suppose $C = < X_0, \ldots, X_D >$ and $C^* = < X^*_0, \ldots, X^*_D >$ are dual character algebras and let $\boldsymbol{T}$ denote the associated Terwilliger algebra. Then for any $\boldsymbol{T}$-module V,

$$V = \sum_i e_i V, \qquad \text{(direct sum)} \tag{4}$$

where the sum is over all $i \in \{0, \ldots, D\}$ for which $e_i V \neq 0$. Similarly,

$$V = \sum_i e^*_i V, \qquad \text{(direct sum)} \tag{5}$$

where the sum is over all $i \in \{0, \ldots, D\}$ for which $e^*_i V \neq 0$. We now describe a certain finite dimensional irreducible module for $\boldsymbol{T}$ for which the decompositions in (4) and (5) are especially nice.

Proposition 4.1 *Suppose* $C = < X_0, \ldots, X_D >$ *and* $C^* = < X^*_0, \ldots, X^*_D >$ *are dual character algebras and* $\boldsymbol{T}$ *is the associated Terwilliger algebra. Then there exists an irreducible* $\boldsymbol{T}$*-module* V_0 *such that*

(i) *dim* $e_i V_0 = 1$, $\quad (0 \leq i \leq D)$

(ii) *dim* $e^*_i V_0 = 1$, $\quad (0 \leq i \leq D)$

where $e_0, \ldots, e_D$ *and* $e^*_0, \ldots, e^*_D$ *are as in Definition 3.1. Moreover,* V_0 *is unique up to isomorphism of* $\boldsymbol{T}$*-modules. We refer to* V_0 *as the* ***primary module*** *for* $\boldsymbol{T}$.

As the next proposition explains, the primary module posseses a basis with respect to which the action of the generators $x_0, \ldots, x_D$ is especially nice.

Proposition 4.2 *Suppose* $C = < X_0, \ldots, X_D >$ *and* $C^* = < X^*_0, \ldots, X^*_D >$ *are dual character algebras,* $\boldsymbol{T}$ *is the associated Terwilliger algebra, and* V_0 *is the primary module for* $\boldsymbol{T}$. *Then there exists a basis* $v_0, \ldots, v_D$ *of* V_0 *such that*

(i) $v_i \in e_i V_0$ $\quad (0 \leq i \leq D)$

(ii) $x_i v_j = P_{ji} v_j$, $\quad (0 \leq i, j \leq D)$

(iii) $x^*_i v_j = \sum_{r=0}^{D} p^{j*}_{ir} v_r$, $\quad (0 \leq i, j \leq D)$

where P *is the matrix of eigenvalues of* C.

Reversing the roles of C and C^* in Proposition 4.2, we obtain a second basis for V_0.

Proposition 4.3 *Suppose* $C = < X_0, \ldots, X_D >$ *and* $C^* = < X^*_0, \ldots, X^*_D >$ *are dual character algebras,* $\boldsymbol{T}$ *is the associated Terwilliger algebra, and* V_0 *is the primary module for* $\boldsymbol{T}$. *Then there exists a basis* $v^*_0, \ldots, v^*_D$ *of* V_0 *such that*

(i) $v^*_i \in e^*_i V_0$ $\quad (0 \leq i \leq D)$

(ii) $x^*_i v^*_j = P^*_{ji} v^*_j$, $\quad (0 \leq i, j \leq D)$

(iii) $x_i v^*_j = \sum_{r=0}^{D} p^{j}_{ir} v^*_r$, $\quad (0 \leq i, j \leq D)$

where P^* is the matrix of eigenvalues of C^*.

In the next proposition we describe how the elements $e_0, \ldots, e_D$ and $e_0^*, \ldots, e_D^*$ act on the primary module by considering their action on the bases of Propositions 4.2 and 4.3.

Proposition 4.4 *Suppose $C = < X_0, \ldots, X_D >$ and $C^* = < X_0^*, \ldots, X_D^* >$ are dual character algebras, $\boldsymbol{T}$ is the associated Terwilliger algebra, V_0 is the primary module for $\boldsymbol{T}$, and $v_0, \ldots, v_D$ and $v_0^*, \ldots, v_D^*$ are the bases of V_0 described in Propositions 4.2 and 4.3. Then*

(i) $e_i v_j = \delta_{ij} v_j, \qquad (0 \leq i, j \leq D)$

(ii) $e_i v_j^* = N^{-1} P_{ji}^* \sum_{r=0}^{D} P_{ir} v_r^*, \qquad (0 \leq i, j \leq D)$

(iii) $e_i^* v_j = N^{-1} P_{ji} \sum_{r=0}^{D} P_{ir}^* v_r, \qquad (0 \leq i, j \leq D)$

(iv) $e_i^* v_j^* = \delta_{ij} v_j^*, \qquad (0 \leq i, j \leq D)$

where P is the matrix of eigenvalues of C, P^* is the matrix of eigenvalues of C^*, and $e_0, \ldots, e_D$ and $e_0^*, \ldots, e_D^*$ are as in Definition 3.1.

Suppose $C = < X_0, \ldots, X_D >$ and $C^* = < X_0^*, \ldots, X_D^* >$ are dual character algebras and $\boldsymbol{T}$ is the associated Terwilliger algebra. Using the primary module, we obtain the following result.

Proposition 4.5 *Suppose $C = < X_0, \ldots, X_D >$ and $C^* = < X_0^*, \ldots, X_D^* >$ are dual character algebras and $\boldsymbol{T}$ is the associated Terwilliger algebra. Comparing (1) with relations 3 and 3^* of Definition 3.1, we see that the linear maps $\phi : C \rightarrow \boldsymbol{T}$ and $\phi^* : C^* \rightarrow \boldsymbol{T}$ which have $\phi(X_i) = x_i$ and $\phi(X_i^*) = x_i^*$ for $0 \leq i \leq D$ are $\mathbf{C}$-algebra homomorphisms. These maps satisfy (i)-(iii) below.*

(i) *ϕ is an injection.*

(ii) *ϕ^* is an injection.*

(iii) *$\phi(C) \cap \phi^*(C^*) = Span(I)$.*

References

[1] E. Bannai and T. Ito. *Algebraic Combinatorics I: Association Schemes.* Lecture Note 58. The Benjamin/Cummings Publishing Company, Inc., Menlo Park, 1984.

[2] C. W. Curtis and I. Reiner. *Representation Theory of Finite Groups and Associative Algebras.* Interscience, New York, 1962.

[3] E. Egge. Character Algebras and Duality, in preparation.

[4] P. Terwilliger. Algebraic Graph Theory. *Course Notes*, 1996 (unpublished).

[5] P. Terwilliger. The subconstituent algebra of an association scheme (part I). *Journal of Algebraic Combinatorics*, 1(4):363-388, 1992.

[6] P. Terwilliger. The subconstituent algebra of an association scheme (part II). *Journal of Algebraic Combinatorics*, 2(1):73-103, 1993.

[7] P. Terwilliger. The subconstituent algebra of an association scheme (part III). *Journal of Algebraic Combinatorics*, 2(2):177-210, 1993.