Algebraic Shifting Increases Relative Homology

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Extended Abstract

Summary.

We show that algebraically shifting a pair of simplicial complexes (weakly) increases their relative homology Betti numbers in every dimension.

More precisely, let $\Delta(K)$ denote the algebraically shifted complex of simplicial complex K, and let $\beta_j(K,L) = \dim_{\boldsymbol{k}} H_j(K,L;\boldsymbol{k})$ be the dimension of the jth relative homology group over a field $\boldsymbol{k}$ of a pair of simplicial complexes $L \subseteq K$. Then $\beta_j(K,L) \leq \beta_j(\Delta(K),\Delta(L))$ for all j.

Résumé.

On montre que le shift algebrique d'une paire de simplexe complexe (faiblement) augmente le nombre d'homologie relatif de Betti dans toute direction.

Plus precisement, soit $\Delta(K)$ le shift algebrique du simplexe complexe K, et soit $\beta_j(K,L) = \dim_{\boldsymbol{k}} H_j(K,L;\boldsymbol{k})$ la dimension du $j$eme groupe homologique sur le corps $\boldsymbol{k}$ de la paire de simplexe complexe $L \subseteq K$. Alors, $\beta_j(K,L) \leq \beta_j(\Delta(K),\Delta(L))$ pour tout j.

Introduction.

Algebraic shifting is a remarkable procedure that finds, for any simplicial complex K, a shifted (and hence combinatorially simpler) simplicial complex $\Delta(K)$ with many of the same properties as K. For instance, the f-vector and homology Betti numbers are preserved; Björner and Kalai [BK1] used this fact to characterize the f-vectors and Betti numbers of simplicial complexes.

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However, the situation for pairs of complexes and relative homology is different. In a simple example on three vertices (given later in this abstract), algebraically shifting a pair of complexes increases their relative homology in dimensions 0 and 1. Upon seeing this one example, Keith Pardue (private communication) conjectured that algebraic shifting always (weakly) increases relative homology in every dimension. Our main result is that this conjecture is true.

Pardue's conjecture was grounded in more than just this one simple example. Algebraic shifting is similar to using Gröbner bases on generic initial ideals; and quantities such as free resolution Betti numbers weakly increase upon taking generic initial ideals (see, *e.g.*, [Bi] and [Hu]). Pardue's insight was that these results would carry over to algebraic shifting.

It would be ideal, then, to prove his conjecture by translating the algebraic shifting problem to a generic initial ideal problem, and then invoking the existing results. However, this approach has been unsuccessful, so far. The proof here instead directly refines Björner and Kalai's correspondence between the homology of the original complexes and the combinatorics of the algebraically shifted complexes. The hope is that this result will serve as a first step towards, or further evidence of, the deeper connection between algebraic shifting and generic initial ideals.

Preliminaries.

For basic definitions of simplicial complexes and their homology and relative homology, see, *e.g.*, [Mu, Chapter 1] or [St2, Section 0.3]. We allow the empty simplicial complex $\emptyset$ consisting of no faces; all other complexes must contain the empty set as a (-1)-dimensional face. We also allow the complex $\{\emptyset\}$ consisting of only the empty face, but we do distinguish between the two complexes $\emptyset$ and $\{\emptyset\}$. Let K_j denote the set of j-dimensional faces of a simplicial complex K. The **f-vector** of K is the sequence $(f_0, \ldots, f_{d-1})$, where $f_j = f_j(K) = |K_j|$ and $d - 1 = \dim K$. The same notion of f-vector will apply in this paper to every finite collection of sets.

Let $\boldsymbol{k}$ be a field, fixed throughout the paper. The jth **Betti number** of a simplicial complex K is $\beta_j = \beta_j(K) = \dim_{\boldsymbol{k}} \widetilde{H}_j(K)$, where $\widetilde{H}_j(K)$ is the jth reduced homology group of K (with respect to $\boldsymbol{k}$). Similarly, the jth **relative Betti number** of a pair of simplicial complexes $L \subseteq K$ is $\beta_j = \beta_j(K, L) = \dim_{\boldsymbol{k}} \widetilde{H}_j(K, L)$, where $\widetilde{H}_j(K, L)$ is the jth reduced **relative homology group** of the pair (K, L) (with respect to $\boldsymbol{k}$).

"Reduced" homology means precisely to treat the empty set as a face of any non-empty complex, so β_0 is one less than the number of connected components of Δ, and hence one less than the "unreduced" β_0. Furthermore, $\beta_{-1} = 0$, unless $\Delta = \{\emptyset\}$, in which case $\beta_{-1} = 1$. Reduced relative homology, which also treats the empty set as a face of any non-empty complex, is the same as unreduced relative homology, except that $\beta_{-1}(\{\emptyset\}, \emptyset) = 1$; for any other pair of complexes, $\beta_{-1} = 0$.

Recall that over a field $\boldsymbol{k}$, $\dim_{\boldsymbol{k}} \widetilde{H}^j(K; \boldsymbol{k}) = \dim_{\boldsymbol{k}} \widetilde{H}_j(K; \boldsymbol{k})$, so that Betti numbers measure reduced homology as well as reduced cohomology.

Shifted complexes.

Definition. If $S = \{s_1 < \cdots < s_j\}$ and $T = \{t_1 < \cdots < t_j\}$ are j-subsets of integers, then:

- $S \leq_P T$ under the standard **partial order** if $s_p \leq t_p$ for all p; and
- $S <_L T$ under the **lexicographic order** if there is a q such that $s_q < t_q$ and $s_p = t_p$ for $p < q$.

Lexicographic order is a total order which refines the partial order.

Definition. A collection $\mathcal{C}$ of k-subsets is **shifted** if $S \leq_P T$ and $T \in \mathcal{C}$ together imply that $S \in \mathcal{C}$. A simplicial complex Δ is **shifted** if the set of j-dimensional faces of Δ is shifted for every j.

Björner and Kalai showed in [BK1] that shifted complexes are near-cones, which we now define.

Definition. A **near-cone** with **apex** v_0 is a simplicial complex Δ satisfying the following property: For all $F \in \Delta$, if $v_0 \notin F$ and $w \in F$, then

$$(F - \{w\}) \cup \{v_0\} \in \Delta.$$

For Δ a near-cone with apex v_0, let $B(\Delta) = \{F \in \Delta\colon F \cup \{v_0\} \notin \Delta\}$ and $\Delta' = \{F \in \Delta\colon v_0 \notin F, F \cup \{v_0\} \in \Delta\}$; then

$$\Delta = (v_0 * \Delta') \dot\cup B(\Delta),$$

where $*$ denotes topological join (so $v_0 * \Delta' = \Delta' \dot\cup \{\{v_0\} \dot\cup F\colon F \in \Delta'\}$). Both Δ' and $\Delta' \dot\cup B(\Delta)$ are subcomplexes of Δ. If $B(\Delta) = \emptyset$, then Δ is simply a **cone**.

Note, in particular, that $\emptyset$ and $\{\emptyset\}$ are near-cones (the condition in the definition is vacuous in this case) and that $\emptyset = v_0 * \emptyset$ and $\{\emptyset\} = (v_0 * \emptyset) \dot\cup \{\emptyset\}$. If Δ is a near-cone with apex v_0, then v_0 is one of the vertices of Δ, unless $\Delta = \emptyset$ or $\{\emptyset\}$.

It is not hard to see that shifted simplicial complexes are near-cones with apex 1.

Every $F \in B(\Delta)$ is maximal in Δ, so the collection of faces in $B(\Delta)$ forms an antichain. Further, $f(B(\Delta)) = \beta(\Delta)$, which follows by contracting $v_0 * \Delta'$ to v_0, leaving a sphere for every face in $B(\Delta)$ [BK1, Theorem 4.3]. In other words, if Δ is a near-cone, then

$$\beta_j(\Delta) = |\{F \in \Delta_j\colon v_0 \notin F,\ v_0 \dot\cup F \notin \Delta\}|. \tag{1}$$

This observation is generalized by [Du, Lemma 5.1]: If $\Gamma \subseteq \Delta$ is a pair of near-cones with common apex v_0, then

$$\beta_j(\Delta, \Gamma) = |\{F \in (\Delta - \Gamma)_j\colon v_0 \notin F,\ v_0 \dot\cup F \notin \Delta\}| + |\{G \in (\Delta - \Gamma)_j\colon v_0 \in G,\ G - \{v_0\} \in \Gamma\}|. \tag{2}$$

Algebraic shifting.

Algebraic shifting transforms a simplicial complex into a shifted simplicial complex with the same f-vector and Betti numbers. It also preserves many algebraic properties of the original complex. Algebraic shifting was introduced by Kalai in [Ka1]; our exposition is summarized from [BK1] and included for completeness (see also [BK2, Ka2]). We start with the exterior face ring.

Definition. Let K be a $(d-1)$-dimensional simplicial complex with vertices $V = \{e_1, \ldots, e_n\}$ linearly ordered $e_1 < \cdots < e_n$. Let $\Lambda(\boldsymbol{k}V)$ denote the exterior algebra of the vector space $\boldsymbol{k}V$; it has a $\boldsymbol{k}$-vector space basis consisting of all the monomials $e_S := e_{i_1} \wedge \cdots \wedge e_{i_j}$, where $S = \{e_{i_1} < \cdots < e_{i_j}\} \subseteq V$ (and $e_\emptyset = 1$). Note that $\Lambda(\boldsymbol{k}V) = \oplus_{j=0}^{n} \Lambda^j(\boldsymbol{k}V)$ is a graded $\boldsymbol{k}$-algebra, and that $\Lambda^j(\boldsymbol{k}V)$ has basis $\{e_S\colon |S| = j\}$. Let $(I_K)_j$ be the subspace of $\Lambda^{j+1}(\boldsymbol{k}V)$ generated by the basis $\{e_S\colon |S| = j+1,\ S \notin K\}$. Then $I_K := \oplus_{j=-1}^{d-1}(I_K)_j$ is the homogeneous graded ideal of $\Lambda(\boldsymbol{k}V)$ generated by $\{e_S\colon S \notin K\}$. Let $\Lambda_j[K] := \Lambda^{j+1}(\boldsymbol{k}V)/(I_K)_j$. Then the graded quotient algebra $\Lambda[K] := \oplus_{j=-1}^{d-1}\Lambda_j[K] = \Lambda(\boldsymbol{k}V)/I_K$ is called the **exterior face ring** of K (over $\boldsymbol{k}$).

The exterior face ring is the exterior algebra analogue to the Stanley-Reisner face ring of a simplicial complex [St2]. For $x \in \boldsymbol{k}V$, let $\widetilde{x}$ denote the image of x in $\Lambda[K]$.

Definition. Let $\{f_1, \ldots, f_n\}$ be a "generic" basis of $\boldsymbol{k}V$, *i.e.*, $f_i = \sum_{j=1}^{n} \alpha_{ij} e_j$, where the α_{ij}'s are n^2 transcendentals, algebraically independent over $\boldsymbol{k}$. Define $f_S := f_{i_1} \wedge \cdots \wedge f_{i_j}$ for $S = \{i_1 < \cdots < i_j\}$ (and set $f_\emptyset = 1$). Let

$$\Delta(K, \boldsymbol{k}) := \{S \subseteq [n]\colon \widetilde{f}_S \notin \operatorname{span}\{\widetilde{f}_R\colon R <_L S\}\}$$

be the **algebraically shifted complex** obtained from K. We will write $\Delta(K)$ instead of $\Delta(K, \boldsymbol{k})$ when the field is understood to be $\boldsymbol{k}$.

The j-subsets of $\Delta(K)$ can be chosen by listing all the j-subsets of $[n]$ in lexicographic order and omitting those that are in the span of earlier subsets on the list, modulo I_K and with respect to the f-basis.

It is easy to see that algebraic shifting preserves the f-vector, *i.e.*, $f_j(K) = f_j(\Delta(K))$. Björner and Kalai [BK1] showed that algebraic shifting also preserves Betti numbers, *i.e.*, $\beta_j(K) = \beta_j(\Delta(K))$. The reason lies in the relation between algebraic shifting and coboundaries. Define the **weighted coboundary operator** $\delta\colon \Lambda[K] \to \Lambda[K]$ by $\delta(x) = \widetilde{f}_1 \wedge x$, so

$$\delta(\widetilde{e}_S) = \widetilde{f}_1 \wedge \widetilde{e}_S = \sum_{j=1}^{n} \alpha_{1j}\widetilde{e}_j \wedge \widetilde{e}_S = \sum_{\substack{j \notin S \\ S \cup \{j\} \in K}} \pm\alpha_{1j}\widetilde{e}_{S\cup\{j\}}$$

(hence the name weighted coboundary operator). Betti numbers may be computed using this δ, *i.e.*, $\beta_j(K) = \dim_{\boldsymbol{k}}(\ker\delta)_j/(\operatorname{im}\delta)_j$ [BK1, pp. 289–290]. Furthermore, the action of δ

is easy to describe in the $\widetilde{f}$-basis: $\delta(\widetilde{f}_F)$ equals $\widetilde{f}_{1 \dot\cup F}$ if $1 \notin F$ and $1 \dot\cup F \in \Delta(K)$, but is zero otherwise.

What about relative homology? First, we note a result of Kalai's [Ka2, Theorem 2.2] that if $L \subseteq K$ is a pair of simplicial complexes, then $\Delta(L) \subseteq \Delta(K)$. For every pair (K, L), then, there is a corresponding pair $(\Delta(K), \Delta(L))$. In contrast to the single complex case, however, the homology of (K, L) and $(\Delta(K), \Delta(L))$ need not coincide, as the following example shows.

Example. Let K be the simplicial complex on vertices $\{1, 2, 3\}$ whose maximal faces are $\{1\}$ and $\{2, 3\}$, and L be the subcomplex consisting of just the two vertices $\{1\}$ and $\{2\}$. The only shifted complex with three vertices and one edge has maximal faces $\{1, 2\}$ and $\{3\}$, so this must be $\Delta(K)$. Furthermore, L is the only simplicial complex with two vertices and no edges, so $\Delta(L) = L$. But then it is easy to see that $(\Delta(K), \Delta(L))$ has non-trivial relative homology in dimensions 0 and 1, while (K, L) has no non-trivial relative homology.

Relative homology.

In order to say anything about $(\Delta(K), \Delta(L))$, we must first consider (K, L). Let Q be the set difference $K - L$ (the "Q" is for "quotient"), and define $\Delta(Q) = \Delta(K) - \Delta(L)$. This is primarily a combinatorial definition, with the algebra packaged in the computation of $\Delta(K)$ and $\Delta(L)$. We now examine how to interpret $\Delta(Q)$ algebraically.

Let

$$\widetilde{Q} = \text{span}\{\widetilde{e}_F \colon F \in Q\}.$$

It is not hard to see, then, that we may compute $\Delta(L)$ using $\Lambda[K]$ instead of $\Lambda[L]$, by modding out by $\widetilde{Q}$ on $\Lambda[K]$ instead of by I_L on $\Lambda[L]$, since $\widetilde{Q} = \widetilde{I_L}$ (see [Du, Section 3]). Lemmas 1 and 2 show how $\Delta(Q)$ is related to $\widetilde{Q}$, namely that $\Delta(Q)$ indexes a nice basis of $\widetilde{Q}$.

Lemma 1. *If $F \in \Delta(Q)$, then there is a unique linear combination $\sum_{\substack{G <_L F \\ G \in \Delta(L)}} a_G \widetilde{f}_G$, such that $\widetilde{f}_F - \sum_{\substack{G <_L F \\ G \in \Delta(L)}} a_G \widetilde{f}_G \in \widetilde{Q}$.*

Proof. The proof is omitted from this extended abstract. □

Definition. By Lemma 1, we may define, for any $F \in \Delta(Q)$, $\widetilde{\gamma}_F$ supported on $\Delta(L)$ such that $\widetilde{f}_F - \widetilde{\gamma}_F \in \widetilde{Q}$.

Lemma 2. *$\{\widetilde{f}_F - \widetilde{\gamma}_F \colon F \in \Delta(Q)\}$ is a basis of $\widetilde{Q}$.*

Proof. The proof is omitted from this extended abstract. □

To compute relative homology we rely upon earlier results involving $\widetilde{Q}$. We adopt the shorthand

$$\delta^{-1}\widetilde{Q} = \{\widetilde{x} \in \Lambda[K] \colon \delta\widetilde{x} \in \widetilde{Q}\}.$$

Lemma 3. *For any pair of simplicial complexes $L \subseteq K$,*

(a) $\beta_j(L) = \dim((\delta^{-1}\widetilde{Q})/(\operatorname{im}\delta + \widetilde{Q}))_j$; *and*

(b) $\beta_j(K,L) = \dim((\ker\delta \cap \widetilde{Q})/(\delta\widetilde{Q}))_j$.

Proof. This is [Du, Lemmas 3.3 and 3.4]. □

Now we compare $\operatorname{im}\delta \cap \widetilde{Q}$ and $\delta\widetilde{Q}$ to combinatorially defined sets in $\Delta(Q)$.

Lemma 4. $\operatorname{im}\delta \cap \widetilde{Q} \subseteq \operatorname{span}\{\widetilde{f}_{1\dot\cup F} - \widetilde{\gamma}_{1\dot\cup F}\colon 1 \notin F,\ 1\,\dot\cup\, F \in \Delta(Q)\}$.

Proof. The proof is omitted from this extended abstract. □

Lemma 5. *There is a subspace* $\widetilde{Q'}$ *of* $\widetilde{Q}$ *such that*

$$\dim\delta\widetilde{Q'} = |\{F \in \Delta(Q)\colon 1 \notin F,\ 1\,\dot\cup\, F \in \Delta(Q)\}|.$$

Proof. Let $\widetilde{Q'} = \operatorname{span}\{\widetilde{f}_F - \widetilde{\gamma}_F\colon F \in \Delta(Q),\ 1 \notin F,\ 1\,\dot\cup\, F \in \Delta(Q)\}$. The proof that $\delta\widetilde{Q'}$ has the appropriate dimension is omitted from this extended abstract. □

Theorem 6. *For any pair of simplicial complexes* $L \subseteq K$,

$$\beta_j(K,L) \le \beta_j(\Delta(K),\Delta(L))$$

for all j.

Proof. The sets in equations (1) and (2) that count the homology of $\Delta(K)$, $\Delta(L)$, and $\Delta(Q)$ overlap in a nice way. In particular, if we let

$$\begin{aligned} C_{KQ} &= \{F \in \Delta(Q)\colon 1 \notin F,\ 1\,\dot\cup\, F \notin \Delta(K)\},\\ C'_{LQ} &= \{G \in \Delta(Q)\colon 1 \in G,\ G-1 \in \Delta(L)\},\\ C_{LQ} &= \{F \in \Delta(L)\colon 1 \notin F,\ 1\,\dot\cup\, F \in \Delta(Q)\}, \end{aligned}$$

and

$$C_{KL} = \{F \in \Delta(L)\colon 1 \notin F,\ 1\,\dot\cup\, F \notin \Delta(K)\},$$

then it is not hard to see, from equations (1) and (2), that

$$\begin{aligned} \beta_j(\Delta(K)) &= |(C_{KQ})_j| + |(C_{KL})_j|,\\ \beta_j(\Delta(L)) &= |(C_{LQ})_j| + |(C_{KL})_j|, \end{aligned}$$

and

$$\beta_j(\Delta(K),\Delta(L)) = |(C_{KQ})_j| + |(C'_{LQ})_j|.$$

An easy bijection shows that

$$|(C_{LQ})_j| = |(C'_{LQ})_{j+1}|.$$

We can find "corresponding" subspaces in $\Lambda[K]$; define

$$\begin{aligned} A_{KQ} &= (\ker\delta \cap \widetilde{Q})/(\operatorname{im}\delta \cap \widetilde{Q}), \\ A'_{LQ} &= (\operatorname{im}\delta \cap \widetilde{Q})/(\delta\widetilde{Q}), \\ A_{LQ} &= (\delta^{-1}\widetilde{Q})/(\ker\delta + \widetilde{Q}), \end{aligned}$$

and

$$A_{KL} = (\ker\delta + \widetilde{Q})/(\operatorname{im}\delta + \widetilde{Q}).$$

Then we can show

$$\begin{aligned} \beta_j(K) &= \dim(A_{KQ})_j + \dim(A_{KL})_j, \\ \beta_j(L) &= \dim(A_{LQ})_j + \dim(A_{KL})_j, \\ \beta_j(K,L) &= \dim(A_{KQ})_j + \dim(A'_{LQ})_j, \end{aligned}$$

and

$$\dim(A_{LQ})_j \cong \dim(A'_{LQ})_{j+1}.$$

(The formulas for $\beta_j(L)$ and $\beta_j(K,L)$ follow readily from Lemma 3, and the formula for $\beta_j(K)$ is a routine exercise in linear algebra. The isomorphism between A_{LQ} and A'_{LQ} is [Du, Lemma 3.6]. Details are omitted from this extended abstract.)

Then, because algebraic shifting preserves homology,

$$\dim(A_{LQ})_j + \dim(A_{KL})_j = \beta_j(L) = \beta_j(\Delta(L)) = |(C_{LQ})_j| + |(C_{KL})_j| \tag{3}$$

and

$$\dim(A_{KQ})_j + \dim(A_{KL})_j = \beta_j(K) = \beta_j(\Delta(K)) = |(C_{KQ})_j| + |(C_{KL})_j|. \tag{4}$$

Now, by Lemma 4,

$$\dim(\operatorname{im}\delta \cap \widetilde{Q})_{j+1} \le |\{G \in \Delta(Q)_{j+1} \colon 1 \in G\}|,$$

and by Lemma 5,

$$\begin{aligned} \dim(\delta\widetilde{Q})_{j+1} &\ge \dim(\delta\widetilde{Q'})_{j+1} \\ &= |\{G \in \Delta(Q)_{j+1} \colon 1 \in G,\ G - \{1\} \in \Delta(Q)\}|. \end{aligned}$$

Since $\Delta(L)$ is the complement of $\Delta(Q)$ with respect to $\Delta(K)$, then,

$$\begin{aligned}\dim(A'_{LQ})_{j+1} &= \dim((\operatorname{im}\delta \cap \widetilde{Q})/(\delta\widetilde{Q}))_{j+1} = \dim(\operatorname{im}\delta \cap \widetilde{Q})_{j+1} - \dim(\delta\widetilde{Q})_{j+1} \\ &\leq |\{G \in \Delta(Q)_{j+1}\colon 1 \in G,\ G - \{1\} \in \Delta(L)\}| \\ &= |(C'_{LQ})_{j+1}|,\end{aligned}$$

and so

$$\dim(A_{LQ})_j \leq |(C_{LQ})_j|.$$

Equation (3) then implies

$$\dim(A_{KL})_j \geq |(C_{KL})_j|,$$

so by equation (4),

$$\dim(A_{KQ})_j \leq |(C_{KQ})_j|,$$

and so finally

$$\beta_j(K,L) = \dim(A_{KQ})_j + \dim(A'_{LQ})_j \leq |(C_{KQ})_j| + |(C'_{LQ})_j| = \beta_j(\Delta(K),\Delta(L)).$$

□

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