Solitary Bases for Irreducible Representations of Semisimple Lie Algebras*

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Abstract

The main results of this paper were found while addressing the question: what do the "nice" bases for the irreducible representations of semisimple Lie algebras look like? Using the Gelfand-Zetlin bases for the irreducible representations of $gl(n, \mathbb{C})$ as our model, we take a combinatorial approach to this question by associating a certain kind of directed graph to each weight basis for an irreducible representation of a semisimple Lie algebra. These directed graphs are of combinatorial interest in their own right, and we show that these are connected, rank symmetric, rank unimodal, and strongly Sperner posets. We will view these posets as discrete invariants on the set of all weight bases for a given representation. In this way we split the set of weight bases into smaller subsets. We will take particular interest in the smallest subsets, that is, those containing essentially only one weight basis (and its scalar multiples). We call such a basis a *solitary* basis. We show that the Gelfand-Zetlin bases are solitary, and then we describe solitary bases for the fundamental representations of $sp(2n, \mathbb{C})$ and $so(2n+1, \mathbb{C})$, and for the adjoint representations of the simple Lie algebras. These modest but somewhat striking preliminary results also suggest that there could be a deep relationship between solitary bases and the combinatorial structure and relative "efficiency" of their associated posets. One other compelling curiosity: we will see that the coefficients for the actions of the generators for the Lie algebras on these solitary bases are rational numbers.

Résumé

La question centrale de cet article est d'étudier quelles formes ont les bases intéressantes des représentations d'algèbre de Lie semi-simples. Les résultats présentés ici sont motivés par cette question et nous nous inspirons aussi de la construction des bases de Gelfand-Zetlin pour les représentations irréductibles de $gl(n, \mathbb{C})$. Notre approche est combinatoire: nous associons un certain graphe dirigé à chacune des bases pondérées d'une représentation irréductible d'une algèbre semi-simple. Ces graphes dirigés ont un intérêt combinatoire comme tel; nous montrons que ce sont des ordres partiels connexes, gradués symmétriques, unimodals et fortement Sperner. Nous voyons que ces ordres partiels peuvent être considérés comme des invariants sur l'ensemble des bases pondérées d'une représentation donnée. De cette façon nous partitionnons l'ensemble des bases pondérées en classes. Nous portons ici un intérêt particulier aux classes contenant une seule base pondérée et ses multiples scalaires. Nous appelons une telle base *solitaire*. Nous montrons que les bases de Gelfand-Zetlin sont solitaires. Nous décrivons aussi les bases solitaires des représentations fondamentales de $sp(2n, \mathbb{C})$ et $so(2n+1, \mathbb{C})$, et des représentations adjointes des algèbres de Lie semi-simples. Ces résultats modestes sont des préliminaires frappants et ils suggèrent la possibilité d'une relation plus profonde entre d'une part les bases solitaires, et d'autre part la structure combinatoire et "l'efficacité" relative de leurs ordres partiels associés. Nous voyons aussi que l'action sur les bases solitaires des générateurs de l'algèbre de Lie est à coefficients rationnel; ceci est une autre curiosité intéressante.

1 Introduction

The Lie algebra $gl(n, \mathbb{C})$ can be thought of as an algebra of $n \times n$ matrices, and one can view $gl(n-1, \mathbb{C})$ as the subalgebra inside $gl(n, \mathbb{C})$ consisting of those $n \times n$ matrices whose last column and last row are all zeros. Let V be an irreducible representation of $gl(n, \mathbb{C})$. In 1950, Gelfand

*Some of the results in this paper are contained in the author's doctoral thesis written under the supervision of Robert A. Proctor.

and Zetlin explicitly described a canonical basis for V that "respects the chain of subalgebras" $gl(1,\mathbb{C}) \subset gl(2,\mathbb{C}) \subset \cdots \subset gl(n,\mathbb{C})$ ([GZ]; see [NT] for another derivation). The uniqueness of this basis follows from the fact that any $gl(n-1,\mathbb{C})$-irreducible component of V (regarded now as a $gl(n-1,\mathbb{C})$-module via the induced action) appears at most once in the decomposition of V.

This construction by Gelfand and Zetlin is the prototype for what we call an *explicit construction* of a representation of a semisimple Lie algebra. Such an explicit construction consists of a set of objects which freely generate a vector space of appropriate dimension, together with a procedure or recipe for acting on these objects with generators from the Lie algebra. In addition, we require that this procedure have no recursive calculations (that is, it should immediately specify the coefficients for the actions of the generators).

Our constructions of the fundamental representations of $sp(2n,\mathbb{C})$ in [Don1] appear to form the first "non-routine" infinite family of explicitly constructed representations of simple Lie algebras on weight bases found since the Gelfand-Zetlin constructions for $gl(n,\mathbb{C})$. Our symplectic constructions are realized on two infinite families of distributive lattices, which we have called the "KN" and "De Concini" symplectic lattices. (These are so named because we originally borrowed the labels of Kashiwara and Nakashima [KN] and of De Concini [DeC] to construct these lattices.) We have also constructed the fundamental representations of $so(2n+1,\mathbb{C})$ on two infinite families of distributive lattices that are odd orthogonal analogs of the symplectic lattices in a certain sense.

In Section 3 we will describe how to associate a certain directed graph (called a "supporting diagram") to each weight basis for a representation of a semisimple Lie algebra by viewing the representing matrices for the generators of the Lie algebra as "incidence matrices." This allows us to visualize the differences between weight bases and to bring combinatorial methods to bear on the problem of constructing representations. For example, in order to construct the fundamental representations of $sp(2n,\mathbb{C})$ and $so(2n+1,\mathbb{C})$ in [Don1], we reversed the procedure of Section 3: we first found the directed graphs (distributive lattices in these cases) and then worked backwards to produce the bases and the actions.

Recently we realized that the Gelfand-Zetlin basis is unique in another sense: the only other bases with the same supporting diagram are its scalar relatives. (In the language of Section 3, we say this basis is *solitary*.) This led us to consider the question: are the bases associated to the KN and De Concini symplectic and odd orthogonal lattices unique in this same sense? The answer is yes; for the precise statement of this result see Section 4. There we will also say how our pursuit of this question has improved our understanding of *why* in each case there are two supports that seem to work equally well. In the symplectic case, we will also say how to locate crystal graphs inside each support in order to see that corresponding KN and De Concini lattices have the same number of edges (so they are "equally efficient"). We have also begun to build a small stock of examples of diagrams for other algebras, and we discuss these in Section 4 as well. Most notably among these, we have found all possible "efficient" weight bases for the adjoint representations of the simple Lie algebras. The bases we produce for the fundamental representations of $sp(2n,\mathbb{C})$ and $so(2n+1,\mathbb{C})$, and for the adjoint representations, have one other salient feature: the coefficients for the actions are positive, rational numbers.

Those who are comfortable with the language of posets and lattices, and who are familiar with the representation theory of semisimple Lie algebras as in [Hum], may find it easier to skip ahead to Section 3 and refer back to Section 2 as necessary.

2 Definitions and Notation

In this section, we define the main combinatorial structures we will be working with, and briefly review some pertinent facts about representations of semisimple Lie algebras. Let P be a *poset* (partially ordered set) and let x and y be elements of P. We write $x \to y$ if x is covered by y in P (i.e. $x \leq z \leq y$ in P implies that $x = z$ or $z = y$). The *Hasse diagram* (or *order diagram*) of a poset P is the directed graph whose nodes are the elements of P and whose directed edges are given by the covering relations in P. We will not usually bother to distinguish a poset from its Hasse diagram. If $k \geq 1$, the *k-element chain* $[\mathbf{k}]$ is the totally ordered set $\{1 < 2 < \cdots < k\}$. The *dual* of a poset P is the set $P^* := \{x^*\}_{x \in P}$ together with the partial ordering $y^* \leq x^*$ if and only if $x \leq y$ in P.

A *ranked poset of length* l is a partially ordered set P together with a partition $P = \cup_{i=0}^{l} P_i$ into $l+1$ ranks P_i, $0 \leq i \leq l$, such that elements of P_i cover only elements in P_{i-1}. Define the *rank function* $\rho : P \longrightarrow \{0, \ldots, l\}$ by $\rho(x) := i$ if $x \in P_i$. A ranked poset P is *rank symmetric* if $|P_i| = |P_{l-i}|$ for $0 \leq i \leq l$. A ranked poset is *rank unimodal* if there is an m such that $|P_0| \leq |P_1| \leq \cdots \leq |P_m| \geq |P_{m+1}| \geq \cdots \geq |P_l|$. It is *strongly Sperner* if for every $k \geq 1$, the largest union of k antichains is no larger than the largest union of k ranks. A ranked poset is *Peck* if it is rank symmetric, rank unimodal, and strongly Sperner.

A *lattice* L is a poset such that any two elements x and y of L have a least upper bound (called the *join* of x and y, and denoted $x \vee y$) and a greatest lower bound (called the *meet* of x and y, and denoted $x \wedge y$). Notice that a lattice is necessarily connected and has a unique maximal element and a unique minimal element. A lattice L is *modular* if it is ranked and its rank function satisfies $\rho(x) + \rho(y) = \rho(x \vee y) + \rho(x \wedge y)$ for all x and y in L. One can see that a lattice L is modular if and only if for elements x and y of L we have

$$x \wedge y \to x \quad \text{and} \quad x \wedge y \to y \iff x \to x \vee y \quad \text{and} \quad y \to x \vee y.$$

This follows from Proposition 3.3.2 of [Sta], and says in effect that modular lattices have no "open vees." A lattice L is *distributive* if for all x, y, and z in L we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. It is not hard to see that distributive lattices are modular.

A subset $\mathcal{T} \subseteq P$ is an *order ideal* of P if $y \in \mathcal{T}$ and $x \leq y \Rightarrow x \in \mathcal{T}$. The poset $J(P)$ of all order ideals of P ordered by inclusion is always a distributive lattice. If L is a distributive lattice, an element $x \in L$ is *join irreducible* if it covers exactly one other element. If P is the poset of join irreducibles of L (with the induced order from L), then $L = J(P)$. So $J(P) = J(Q)$ if and only if $P = Q$. A distributive lattice $L = J(P)$ is ranked of length $|P|$, with $L_i = \{\mathcal{T} \in J(P) : |\mathcal{T}| = i\}$, $0 \leq i \leq |P|$.

Some distributive lattices for Section 4. Let N be a positive integer and let λ be a shape with no more than $N-1$ rows. (A "shape" is a collection of boxes arranged into left-justified rows, with each row having at least as many boxes as the row below it.) Define $L(\lambda, N)$ to be the distributive lattice of semistandard Young tableaux of shape λ and with entries from $\{1, 2, \ldots, N\}$, ordered by *reverse* componentwise comparison. If λ is a column of length k, we set $L(k, N-k) := L(\lambda, N)$.

It can be seen that $L(k, N-k) = J([\mathbf{k}] \times [\mathbf{N-k}])$, the distributive lattice of order ideals on this product of chains. We can view $L(k, N-k)$ as a distributive lattice of partitions (ordered by inclusion) as follows. For positive integers m and n, we define an (m,n) *partition* to be an m-tuple τ such that $n \geq \tau_1 \geq \cdots \geq \tau_m \geq 0$. That is, the partition τ fits inside an $m \times n$ box. The (n,m) *conjugate partition* τ' is obtained by "flipping" the partition τ across its main diagonal. In other words, $\tau'_i := \max\{j : \tau_j \geq i\}$ for $1 \leq i \leq n$. Let $1 \leq k \leq N-1$, and let T be a tableau in $L(k, N-k)$, thought of as a k-tuple $(T_1, \ldots, T_k)$ such that $1 \leq T_1 < \cdots < T_k \leq N$. We associate a $(k, N-k)$ partition τ to this tableau T by the rule $\tau_i = N - k + i - T_i$.

Following [RS], a $(k, N-k)$ partition τ is *Andrews* if $\tau_i - \tau'_i \leq N - 2k$ for $1 \leq i \leq d$ where d^2 is the size of the Durfee square of τ, viz. $d = \max\{i : i \leq \tau_i\} = \max\{j : j \leq \tau'_j\}$. Define *Andrews*$(k, N-k)$ to be the distributive lattice of Andrews partitions in $L(k, N-k)$. When $N = 2n$ and $1 \leq k \leq n$, let $L_C^{KN}(k, 2n-k) :=$ *Andrews*$(k, 2n-k)$. Let $\tilde{\tau}$ be the "middle" of τ, formed by removing the first $n-k$ and last $n-k$ columns of the Ferrers diagram for τ. More precisely, $\tilde{\tau}$ is the (k, k) partition whose conjugate $\tilde{\tau}'$ is given by $\tilde{\tau}' = (\tau'_{n-k+1}, \ldots, \tau'_n)$. Let the symplectic lattice $L_C^{DeC}(k, 2n-k)$ be the distributive lattice of partitions τ in $L(k, 2n-k)$ such that $\tilde{\tau}'$ is a (k, k) Andrews partition. In [Don2], we define the two "odd orthogonal" analogs to these symplectic lattices, which we denote $L_B^{KN}(k, 2n+1-k)$ and $L_B^{DeC}(k, 2n+1-k)$.

A tableau T is covered by a tableau U in $L(\lambda, N)$ if U is obtained from T by changing an $i+1$ entry in T to an i, where $1 \leq i \leq N-1$. Attach the "color" i to the edge $T \rightarrow U$ in the Hasse diagram for $L(\lambda, N)$. Let $N = 2n$ and let $1 \leq k \leq n$. The lattices $L_C^{KN}(k, 2n-k)$ and $L_C^{DeC}(k, 2n-k)$ are distributive sublattices of $L(k, 2n-k)$, and so "inherit" its edge colors. Now recolor the edges of the symplectic lattices by changing an edge of color i to an edge of color $2n-i$ whenever $n+1 \leq i \leq 2n-1$.

Representations of semisimple Lie algebras. To fix notation, we give a quick review of some of the main definitions and results concerning representations of semisimple Lie algebras. Following chapter 3 of [Hum] let Φ be a root system of rank n in $\mathbb{R}^n$. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a choice of simple roots. Let $\check{\Phi}$ be the dual root system, with simple roots $\check{\Delta} = \{\check{\alpha}_1, \ldots, \check{\alpha}_n\}$. Let $\{\omega_1, \ldots, \omega_n\}$ be the associated *fundamental weights*, defined by $(\omega_i, \check{\alpha}_j) = \delta_{ij}$, where $(\cdot, \cdot)$ denotes the inner product on $\mathbb{R}^n$. Let Λ denote the *lattice of weights*, that is, the $\mathbb{Z}$-linear combinations of the fundamental weights. A weight $\lambda \in \Lambda$ is said to be *dominant* if $\lambda = \sum m_i\omega_i$, with $m_i \geq 0$. Let $\omega_0 := 0$ be the zero weight.

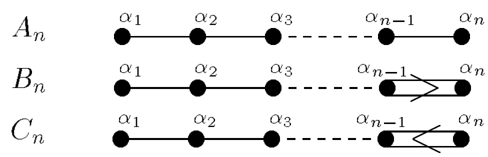
Let $\mathcal{L}$ be the complex semisimple Lie algebra associated to this root system. Recall that $\mathcal{L}$ has $3n$ generators $\{x_i, y_i, h_i\}_{i=1}^n$ known as the *Chevalley generators*, and these satisfy the *Serre relations* ([Hum], pp. 96, 99). The subalgebra $\mathcal{H}$ spanned by $\{h_1, \ldots, h_n\}$ is called a *Cartan subalgebra* of $\mathcal{L}$. A representation V of $\mathcal{L}$ is a (complex) vector space V together with a map of Lie algebras $\phi : \mathcal{L} \rightarrow gl(V)$. This map makes the vector space V into an *$\mathcal{L}$-module*. We use the lower case x_i, y_i, and h_i when we are thinking of the generators as elements of $\mathcal{L}$, and the upper case X_i, Y_i, and H_i when we are thinking of the images of the generators in $gl(V)$.

In this abstract, we will be concerned only with finite-dimensional representations. Let V be a finite-dimensional representation of $\mathcal{L}$, with $\phi : \mathcal{L} \rightarrow gl(V)$. Following chapter 6 of [Hum], let $\mu \in \Lambda$, and define the weight space V_μ of V by

$$V_\mu := \{v \in V \mid H_i.v = (\mu, \check{\alpha}_i)v \ \ \textit{for all}\ 1 \leq i \leq n\},$$

where $H_i = \phi(h_i)$. Then it can be seen that V is a (vector space) direct sum of its weight spaces, so that $\mathcal{H}$ acts diagonally on V. An element $v \in V_\mu$ is said to have weight μ, and we let $wt(v) := \mu$. A basis for V which respects this decomposition is called a *weight basis*. We say that two weight bases are *multiscalar equivalent* if (after an appropriate reordering) they differ by a diagonal change of basis matrix. If V is irreducible, then there is a vector $v^+ \in V$ (unique up to nonzero scalar multiplication) such that $X_i.v^+ = 0$ for all i. Moreover, v^+ has weight λ, where λ is dominant, and we call λ the *highest weight* for the representation V. If μ is a weight for this irreducible representation (i.e. $V_\mu \neq 0$), then $\lambda - \mu = \sum k_i\alpha_i$, where each k_i is a non-negative integer. The *weight diagram* for V is the set $\Pi(\lambda) := \{\mu \in \Lambda | V_\mu \neq 0\}$, together with the partial order $\mu \leq \nu$ in $\Pi(\lambda)$ if and only if $\nu - \mu = \sum k_i\alpha_i$, where each k_i is a non-negative integer. One of the central results of [Hum] (Corollary 21.2, p. 113) is that the irreducible representations of $\mathcal{L}$ are indexed by dominant weights in the following sense: to every dominant weight there is an irreducible representation V with highest weight λ. Moreover, if V and W are irreducible and have highest weight λ, then V and W are isomorphic as $\mathcal{L}$-modules.

When $\mathcal{L}$ is simple of rank n, it will be convenient to identify $\mathcal{L}$ with its root system $\mathcal{X}_n$, where $\mathcal{X} \in \{A, B, C, D, E, F, G\}$. We will let $\mathcal{L}(\lambda)$ denote the irreducible representation of $\mathcal{L}$ with highest weight λ. (So, for example, for us $C_n(\omega_k)$ denotes the irreducible representation of the Lie algebra C_n with highest weight ω_k.) We will also refer to the classical Lie algebras by their usual names by identifying A_n with $sl(n+1, \mathbb{C})$, B_n with $so(2n+1, \mathbb{C})$, C_n with $sp(2n, \mathbb{C})$, and D_n with $so(2n, \mathbb{C})$. The following are pictures of the Dynkin diagrams for A_n, B_n, and C_n, respectively. Here, $\alpha_1, \ldots, \alpha_n$ are simple roots.



3 Representation Diagrams and Supporting Diagrams

We are now ready to describe in precise terms the combinatorial setting for the main results of Section 4. Let Φ be a root system of rank n, and let $\mathcal{L}$ be the associated complex semisimple Lie algebra. Let $\{v_{\mathbf{t}}\}_{\mathbf{t} \in I}$ be any weight basis for a representation V of $\mathcal{L}$ of dimension d. (Here, I is an index set of order $|I| = d = \dim(V)$.) With respect to a suitable ordering on these basis vectors, we can think of the operators X_i, Y_i, and H_i in $gl(V)$ as $d \times d$ matrices. Now think of each basis vector as a vertex. The pairs of matrices $\{X_i, Y_i\}_{i=1}^{n}$ specify incidence relations between vertices as follows: place a directed edge of "color" i from the basis vector $v_{\mathbf{s}}$ to the basis vector $v_{\mathbf{t}}$ if either the $(\mathbf{t}, \mathbf{s})$-entry of the matrix X_i is non-zero, or the $(\mathbf{s}, \mathbf{t})$-entry of the matrix Y_i is non-zero, or both are non-zero. Attach two coefficients to each directed edge of color i: an "x-coefficient" corresponding to the appropriate matrix entry for the generator X_i, and a "y-coefficient" corresponding to the appropriate matrix entry for Y_i. Lemma 20.1 of [Hum] states that if a vector v has weight μ in a representation V, then $X_i.v \in V_{\mu+\alpha_i}$ and $Y_i.v \in V_{\mu-\alpha_i}$. Thus any two vertices in our directed graph can have at most one edge between them, and in addition, the directed graph will have no loops. We call this directed graph with colored edges and with coefficients attached to each edge a *representation diagram* for the representation V of $\mathcal{L}$. This diagram (which we normally denote by P) encodes all the information for the actions of the generators X_i and Y_i on V with respect to the basis $\{v_{\mathbf{t}}\}_{\mathbf{t} \in I}$, and we say that P *realizes the representation* V. The *supporting diagram* (or just the *support*) of a representation diagram P (or of an associated weight basis $\{v_{\mathbf{t}}\}_{\mathbf{t} \in I}$) is the edge-colored directed graph $S[P]$ obtained from P by ignoring the coefficients on the edges.

We make the following observations. First, notice that the number of supports for a given representation is finite. In addition, one can see that if two bases are multiscalar equivalent (see Section 2), and have representation diagrams P and Q respectively, then their supporting diagrams $S[P]$ and $S[Q]$ are the same. Moreover, the product of the "x" and "y" coefficients for an edge in P equals the product of coefficients on the corresponding edge in Q. It is not hard to show that any support for a representation V of $\mathcal{L}$ is a ranked poset. Next, note that the connected components of the diagram correspond to subspaces of V that are stable under the action of $\mathcal{L}$ (but these need not be irreducible). So when V is irreducible, it follows that any supporting diagram will be connected. That these posets are Peck follows by restricting to the action of a "principal $sl(2, \mathbb{C})$" inside $\mathcal{L}$ (see [Pr2]) and applying Proctor's "Peck Poset Theorem" [Pr1]. So we get:

Proposition 3.1 *If V is an irreducible representation of $\mathcal{L}$, then any support for V is the Hasse diagram for a connected, rank symmetric, rank unimodal, and strongly Sperner poset.* □

One antecedent for this notion of a representation diagram can be found in [Pr1], [Pr2], and [Pr5], where Proctor was mainly interested in $sl(2,\mathbb{C})$ actions. As an example, he proved that the lattices $L(\lambda, n)$ are Peck [Pr5] by using the fact that these lattices can be viewed as representation diagrams for the irreducible representations of $gl(n,\mathbb{C})$ due to the construction of Gelfand and Zetlin [GZ].

Let $\mathcal{W}(V)$ be the collection of all possible weight bases for a representation V of $\mathcal{L}$. Let $\mathcal{M}(V)$ be the set of all equivalence classes of multiscalar equivalent weight bases. Let $\mathcal{D}(V)$ and $\mathcal{S}(V)$ respectively denote the collections of representation diagrams and supporting diagrams for V. We have found the following picture to be a convenient way to keep track of these different notions:

$$\begin{tikzcd} & \mathcal{D} \arrow[dr] & \\ \mathcal{W} \arrow[ur] \arrow[dr] & & \mathcal{S} \\ & \mathcal{M} \arrow[ur] & \end{tikzcd}$$

For example, the arrow $\mathcal{D} \to \mathcal{S}$ indicates passing from a representation diagram to the associated supporting diagram. Evidently, the diagram commutes.

Definitions 3.2 A supporting diagram S (or any associated weight basis or representation diagram) for V is *edge-minimizing* if the supporting diagram for any other weight basis for V has at least as many edges as S. Define the *efficiency* of a supporting diagram (or any associated weight basis or representation diagram) to be the number of edges in the support. Call a supporting diagram S (or any associated weight basis or representation diagram) *solitary* if the weight basis associated to S is unique up to multiscalar equivalence (i.e. there is only one equivalence class in $\mathcal{M}(V)$ with support S).

Any representation has an edge-minimizing weight basis, but there is no guarantee that it will have a solitary basis. Minimizing the number of edges in a supporting diagram is similar in spirit to making the representing matrices $\{X_i, Y_i\}_{i=1}^n$ as collectively sparse as possible. However, these two issues are distinct, since it is possible for an edge in a representation diagram to have one zero coefficient and one non-zero coefficient.

Following our observations above about multiscalar equivalent weight bases, we see that if two representation diagrams have the same solitary support, then the products of coefficients on corresponding edges are the same. So we say that the "products of edge-coefficients are fixed" for solitary supports. In all the examples of solitary supports that we know of, these fixed products are positive, rational numbers.

One can obtain solitary bases for an irreducible representation of a (non-simple) semisimple Lie algebra $\mathcal{L}$ by "tensoring" together solitary bases for certain irreducible representations of the simple Lie algebras that comprise $\mathcal{L}$. More precisely, let V and W be irreducible representations of two semisimple Lie algebras $\mathcal{K}$ and $\mathcal{L}$, respectively. Then V and W are irreducible representations of $\mathcal{K} \oplus \mathcal{L}$ (where we let $\mathcal{K}$ act trivially on W and $\mathcal{L}$ act trivially on V), and so is $V \otimes W$. Moreover, all of the irreducible representations of $\mathcal{K} \oplus \mathcal{L}$ can be realized in this way. Suppose $\{v_{\mathbf{s}}\}_{\mathbf{s}\in I}$ and $\{w_{\mathbf{t}}\}_{\mathbf{t}\in J}$ are solitary bases for V and W respectively. Recently we have shown that the weight basis $\{v_{\mathbf{s}} \otimes w_{\mathbf{t}}\}_{(\mathbf{s},\mathbf{t})\in I\times J}$ is a solitary basis for the irreducible representation $V \otimes W$ of $\mathcal{K} \oplus \mathcal{L}$.

We conclude this section with a description of how representation diagrams behave when we restrict to the action of a subalgebra. Let $\mathcal{L}$ be semisimple of rank n. Let $J \subset \{1, 2, \ldots, n\}$, and consider the (semisimple) subalgebra $\mathcal{K}$ generated by $\{x_i, y_i, h_i\}_{i\in J}$. Notice that the Dynkin diagram for $\mathcal{K}$ is obtained from the Dynkin diagram for $\mathcal{L}$ by removing those nodes corresponding to the simple roots α_i whenever $i \notin J$. Let $V := \mathcal{L}(\lambda)$ be an irreducible representation of $\mathcal{L}$.

Now let $\{v_{\mathbf{t}}\}_{\mathbf{t}\in I}$ be a weight basis for V, and let P be the corresponding representation diagram. Form another diagram Q by removing from P all edges whose colors are not in the set J. Then Q is a representation diagram for $\mathcal{K}$ corresponding to the induced action of $\mathcal{K}$ on V. We say that the basis $\{v_{\mathbf{t}}\}_{\mathbf{t}\in I}$ (and the associated representation diagram P) *respects the restriction to* $\mathcal{K}$ if the connected components of Q realize *irreducible* representations of $\mathcal{K}$. More generally, if $J_1 \subset J_2 \subset \cdots J_m \subset \{1,\ldots,n\}$ is any sequence of proper subsets, consider the associated "chain" of subalgebras $\mathcal{K}_1 \subset \cdots \subset \mathcal{K}_m \subset \mathcal{L}$. Given a representation diagram P for $\mathcal{L}$, form diagrams $Q_m, Q_{m-1},\ldots,Q_1$ by successively removing edges from P (as described above). We say that P (or any associated weight basis) *respects the chain of subalgebras* $\mathcal{K}_1 \subset \cdots \subset \mathcal{K}_m \subset \mathcal{L}$ if the connected components of Q_i realize *irreducible* representations of $\mathcal{K}_i$, where $1 \leq i \leq m$.

4 Main Examples and Results

In the previous section we described how to associate a representation diagram and a support to any weight basis. One difficulty in describing the set of supports for a family of representations is that very few infinite families of representations have been constructed explicitly. We have been able to construct several small infinite families of irreducible representations, and in the following examples we consider the supporting diagrams for these constructions. These results give some preliminary evidence that edge-minimizing supports, modular lattice supports, solitary supports, and supports with positive, rational edge coefficients are somehow related.

Example 0 *Representations of* $sl(2,\mathbb{C})$*.* If we are going to make any sense of our notion of efficiency, then we should be able to say something about efficient bases for representations of $sl(2,\mathbb{C})$. It is not hard to see that the only possible support for $A_1(k\omega_1)$ is a chain of length k, and that this support is solitary (using the explicit basis of section 7 of [Hum], for example). In fact, in any representation diagram for $A_1(k\omega_1)$, the product of the coefficients on any edge will be a positive integer. The following proposition says in effect that the connected components of an edge-minimizing representation diagram for a representation V of $sl(2,\mathbb{C})$ correspond to irreducible components in the decomposition of V. (A direct sum of chains is just their disjoint union.)

Proposition 4.1 *Let P be a representation diagram for some representation of $sl(2,\mathbb{C})$. Then P is edge-minimizing if and only if the supporting diagram for P is a direct sum of chains.* $\square$

Example 1 *Gelfand-Zetlin bases.* Recalling our discussion of Gelfand-Zetlin bases from the introduction, let us now restrict our attention to the case $A_n = sl(n+1,\mathbb{C})$. One can view A_{n-1} inside A_n as the subalgebra generated by $\{x_i, y_i, h_i\}_{i=1}^{n-1}$ (that is, the subalgebra whose generators correspond to the $n-1$ leftmost nodes of the Dynkin diagram for A_n). Let λ be a dominant weight, and use the induced action to regard $A_n(\lambda)$ as an A_{n-1}-module; each irreducible A_{n-1}-module occurring in the decomposition of $A_n(\lambda)$ does so only once. In light of this, the Gelfand-Zetlin basis for $A_n(\lambda)$ is the *unique* weight basis (up to multiscalar equivalence) which respects the chain of subalgebras $A_1 \subset \cdots \subset A_{n-1} \subset A_n$. In [GZ] (or [NT]), this basis is indexed by the so-called "Gelfand patterns," which are easily converted to semistandard Young tableaux using [Pr4]. If $\lambda = a_1\omega_1 + a_2\omega_2 + \cdots + a_n\omega_n$, then let $shape(\lambda)$ be the shape with a_n columns of length n, a_{n-1} columns of length $n-1$, etc. It can be seen that the support for this Gelfand-Zetlin basis is the edge-colored distributive lattice $L(shape(\lambda), n+1)$ defined in Section 2. For brevity, we will refer to this basis as the "GZKN" basis, and we will let $L_A^{KN}(shape(\lambda), n+1) := L(shape(\lambda), n+1)$. (In addition, we note that the edge coefficients supplied by [NT] are rational numbers, and the product of the coefficients on any edge is a positive, rational number.)

Now let A'_{n-1} be the subalgebra inside A_n generated by $\{x_i, y_i, h_i\}_{i=2}^{n}$, so it is the subalgebra corresponding to the *rightmost* $n-1$ nodes of the Dynkin diagram for A_n. There is a basis for $A_n(\lambda)$

which respects the chain of subalgebras $A_n \supset A'_{n-1} \supset \cdots \supset A'_2 \supset A'_1$, and which can be obtained as follows. Let $V := A_n(\lambda)$. Let A_n^{sym} be the image of A_n under the automorphism (actually, an involution) of A_n induced by the obvious symmetry in the Dynkin diagram for A_n. So A_n^{sym} is an isomorphic copy of $sl(n+1,\mathbb{C})$ in A_n having generators $x_i^{sym} = x_{n+1-i}$, $y_i^{sym} = y_{n+1-i}$, and $h_i^{sym} = h_{n+1-i}$, where $1 \leq i \leq n$. Then by the induced action, the representation V becomes an irreducible A_n^{sym}-module of highest weight $\lambda^{sym} = a_n\omega_1^{sym} + a_{n-1}\omega_2^{sym} + \cdots + a_1\omega_n^{sym}$. Take the GZKN basis for V (regarded here as an A_n^{sym}-module). Its support is the edge-colored distributive lattice $L(shape(\lambda^{sym}), n{+}1)$. (A simple bijection can be used to show that $L(shape(\lambda^{sym}), n{+}1) \cong L(shape(\lambda), n{+}1)^*$ as edge-colored posets.) Finally, to view the support for this basis when we again regard V as an A_n-module, simply recolor the edges of $L(shape(\lambda^{sym}), n{+}1)$ by the rule $i \mapsto n{+}1{-}i$, where $1 \leq i \leq n$. We call this the "GZDeC" basis for $A_n(\lambda)$, and let $L_A^{DeC}(shape(\lambda), n{+}1)$ denote the resulting supporting diagram. (Normally, the lattices $L_A^{KN}(shape(\lambda), n{+}1)$ and $L_A^{DeC}(shape(\lambda), n{+}1)$ are distinct, although they coincide when λ is a fundamental weight, for example.) Summarizing, we have the following theorem:

Theorem (Gelfand-Zetlin) *Let λ be a dominant weight for A_n, and let $shape(\lambda)$ be the corresponding shape. Then the Gelfand-Zetlin bases GZKN and GZDeC for the irreducible representation $A_n(\lambda)$ are solitary, their supports are the distributive lattices $L_A^{KN}(shape(\lambda), n+1)$ and $L_A^{DeC}(shape(\lambda), n+1)$ respectively, and the fixed products of edge coefficients for these solitary supports are positive, rational numbers.* □

Example 2 *Symplectic and odd orthogonal lattices.* (See Section 2 for definitions.) The main result of [Don1] was:

Theorem *With the positive, rational edge coefficients specified in* [Don1]*, the symplectic lattices $L_C^{KN}(k, 2n-k)$ and $L_C^{DeC}(k, 2n-k)$ are representation diagrams for the* k*th fundamental representation of $sp(2n,\mathbb{C})$.* □

We have also produced positive, rational edge coefficients for $L_B^{KN}(k, 2n+1-k)$ and $L_B^{DeC}(k, 2n+1-k)$, and have shown that these lattices are representation diagrams for the kth fundamental representation of $so(2n+1,\mathbb{C})$. As an immediate consequence we can conclude that the symplectic lattices $L_C^{KN}(k, 2n-k)$ and $L_C^{DeC}(k, 2n-k)$ and the odd orthogonal lattices $L_B^{KN}(k, 2n+1-k)$ and $L_B^{DeC}(k, 2n+1-k)$ are Peck. (In particular, this confirms Reiner and Stanton's conjecture that the lattices $Andrews(k, 2n-k) = L_C^{KN}(k, 2n-k)$ defined in [RS] have the strong Sperner property.) The symplectic lattices have other similarities. They are both distributive sublattices of $L(k, 2n-k)$, and they have the same rank generating function. However, using posets of join irreducibles (see [Don2]), one can see that these lattices are isomorphic as posets if and only if $k = 1$ or $k = n$.

The following two results are more recent. The proof that these symplectic lattices are solitary actually gives insight into the distinguishing features of these supports (we say more about this at the end of this example). In the proof of part A of this theorem, we consider the induced action of the subalgebra A_{n-1} generated by $\{x_i, y_i, h_i\}_{i=1}^{n-1}$ (that is, the subalgebra inside C_n whose generators correspond to the $n-1$ leftmost nodes of the Dynkin diagram for C_n).

Theorem 4.2

A. *Regarded as supporting diagrams for the* k*th fundamental representation of $sp(2n,\mathbb{C})$, the symplectic lattices $L_C^{KN}(k, 2n-k)$ and $L_C^{DeC}(k, 2n-k)$ are solitary.*

B. *As supporting diagrams, the symplectic lattices $L_C^{KN}(k, 2n-k)$ and $L_C^{DeC}(k, 2n-k)$ are equally efficient (that is, they have the same number of edges).*

Outline of proof of part A. The proof begins by demonstrating that when edges of color n are removed from $L_C^{KN}(k, 2n-k)$ (respectively, $L_C^{DeC}(k, 2n-k)$), the connected components of the resulting representation diagram for A_{n-1} correspond to certain $L_A^{KN}(\mu, n)$ (respectively, $L_A^{DeC}(\mu, n)$). In fact, there are exactly $k+1$ such components in each case, and each shape μ has at most two columns.

Let $V := C_n(\omega_k)$. With a knowledge of the connected components described in the previous paragraph, one can show that any representation diagram for V with support $L_C^{KN}(k, 2n-k)$ (resp. $L_C^{DeC}(k, 2n-k)$) determines a weight basis $\{v_1^{(0)}, \ldots, v_{j_0}^{(0)}, \ldots, v_1^{(k)}, \ldots, v_{j_k}^{(k)}\}$ for V such that:

(a) $V_i := \text{span}\{v_1^{(i)}, \ldots, v_{j_i}^{(i)}\}$ is stable under the induced action of A_{n-1}, and is irreducible as an A_{n-1}-module.

(b) The basis $\{v_1^{(i)}, \ldots, v_{j_i}^{(i)}\}$ for V_i is the GZKN (resp. GZDeC) basis for the A_{n-1}-module V_i. (Let $v_1^{(i)}$ be the maximal vector.)

(c) Regarding V as a C_n-module, then $v_1^{(i)}$ has weight $wt(v_1^{(i)}) = \omega_i + \omega_{n-k+i} - \omega_n$, where $0 \leq i \leq k$.

One can see that the list of weights in (c) has no redundancies, and each of the corresponding weight spaces is one-dimensional. Putting (b) and (c) together, we see that there is at most one basis for V (up to multiscalar equivalence) that has $L_C^{KN}(k, 2n-k)$ (resp. $L_C^{DeC}(k, 2n-k)$) as its supporting diagram.

Outline of proof of part B. Our first step is to locate a copy of the *crystal graph* $\mathcal{G}(C_n, \omega_k)$ associated to the dominant weight ω_k for C_n inside each of $L_C^{KN}(k, 2n-k)$ and $L_C^{DeC}(k, 2n-k)$. That is, we want to view $\mathcal{G}(C_n, \omega_k)$ as an edge-colored subgraph of these lattices. In the KN case, this is immediate since the labels of [KN] were developed for the purpose of explicitly describing crystal graphs. Let $\phi : L_C^{KN}(k, 2n-k) \longrightarrow L_C^{DeC}(k, 2n-k)$ be the bijection (of sets) described by Sheats ([She], Appendix). The bijection ϕ has the following property: if an edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in $L_C^{KN}(k, 2n-k)$ is contained in the the subgraph of $L_C^{KN}(k, 2n-k)$ corresponding to $\mathcal{G}(C_n, \omega_k)$, then $\phi(\mathbf{s}) \xrightarrow{i} \phi(\mathbf{t})$ in $L_C^{DeC}(k, 2n-k)$. So this bijection locates a copy of $\mathcal{G}(C_n, \omega_k)$ inside $L_C^{DeC}(k, 2n-k)$.

In general, the bijection ϕ is not a poset isomorphism; however, it does give a correspondence between two large subsets of the edges in these lattices. Moreover, this bijection also suggests how the remaining edges can be put in a one-one correspondence. $\square$

In fact, our experience with these symplectic representation diagrams leads us to speculate that they are edge-minimizing. We have an argument similar to the proof of part A above that shows that the odd orthogonal lattices $L_B^{KN}(k, 2n+1-k)$ and $L_B^{DeC}(k, 2n+1-k)$ are solitary as supporting diagrams for the fundamental representations of $so(2n+1, \mathbb{C})$. We have found a "crystal graph preserving" bijection from $L_B^{KN}(k, 2n+1-k)$ to $L_B^{DeC}(k, 2n+1-k)$, and we believe that an argument similar to the proof of part B above will show that these odd orthogonal lattices are equally efficient.

We should also add that for $1 \leq k \leq n$, the KN symplectic lattice $L_C^{KN}(k, 2n-k)$ respects the chain of subalgebras $A_1 \subset \cdots \subset A_{n-1} \subset C_n$, where A_m is the subalgebra of C_n corresponding to the m leftmost nodes in the Dynkin diagram for C_n (but this is false for the De Concini symplectic lattice when $n \geq 3$ and $k = 2$, for example). On the other hand, the De Concini symplectic lattice $L_C^{DeC}(k, 2n-k)$ respects the chain of subalgebras $C_n \supset C_{n-1} \supset \cdots \supset C_2 \supset C_1$, where C_m is the subalgebra of C_n corresponding to the m rightmost nodes in the Dynkin diagram for C_n, with $C_1 = A_1$ (but this is false for the KN symplectic lattice when $n \geq 3$ and $k = 2$, for example). The preceding two sentences remain true if we replace the symplectic lattices with their odd orthogonal counterparts, and replace the symbol "C" with the symbol "B."

Example 3 *Multiplicity free representations.* In this example we answer the question: when does an irreducible representation $\mathcal{L}(\lambda)$ have a unique support? We say that a representation is *multiplicity free* if all its weight spaces have dimension one. For example, minuscule representations are multiplicity free. For any weight basis $\{v_{\mathbf{t}}\}$ for a multiplicity free representation, the placement of colored edges between the basis vectors is severely limited by the following fact: if $v \in V$ has weight μ, then $X_i.v \in V_{\mu+\alpha_i}$ and $Y_i.v \in V_{\mu-\alpha_i}$. One can use this fact to see that a multiplicity free representation has a unique supporting diagram (and this support is solitary). For irreducible representations, the converse is also true:

Theorem 4.3 *An irreducible representation $\mathcal{L}(\lambda)$ has unique support if and only if $\mathcal{L}(\lambda)$ is multiplicity-free.* □

We have a "uniform construction" of all irreducible multiplicity free representations that goes as follows. Since all weight spaces have dimension one in any such representation, we can use the weights themselves as our vertices. Place a directed edge of color i from the weight μ to the weight ν if $\mu + \alpha_i = \nu$. The "i-components" of this edge-colored graph are all chains, so by Example 0, the product of the coefficients on any edge of color i must be a positive integer. (The i-components are just the connected components that are left over when the edges of colors other than i are all removed.) We have a procedure that assigns positive, integral coefficients to the edges of this edge-colored graph, and we can confirm that this choice of edge coefficients indeed produces a representation diagram. We only use general reasoning at each step of this uniform construction.

For simple Lie algebras, the minuscule representations are $A_n(\omega_k)$, $B_n(\omega_n)$, $C_n(\omega_1)$, $D_n(\omega_1)$, $D_n(\omega_{n-1})$, $D_n(\omega_n)$, $E_6(\omega_1)$, $E_6(\omega_6)$, and $E_7(\omega_7)$. The other irreducible multiplicity free representations that we know of are $A_n(k\omega_1)$, $A_n(k\omega_n)$, $B_n(\omega_1)$, $C_2(\omega_2)$, $C_3(\omega_3)$, and $G_2(\omega_1)$.

Proposition 4.4 *If $\mathcal{L}(\lambda)$ is one of the multiplicity free representations of the previous paragraph, then its supporting diagram is a distributive lattice.* □

See [Pr2] for an explicit description of the posets of join irreducibles of these distributive lattices for the minuscule cases.

Example 4 *Adjoint representations.* Let $\mathcal{L}$ be simple of rank n. Then the adjoint representation of $\mathcal{L}$ is irreducible, and its highest weight is the highest long root. If more than one root length occurs in the irreducible root system associated to $\mathcal{L}$ (i.e. $\mathcal{L} \in \{B_n, C_n, F_4, G_2\}$), then the highest short root is also a dominant weight. We call the associated irreducible representation the *short adjoint representation* of $\mathcal{L}$. These representations are "almost" multiplicity free: only the zero weight space (which corresponds to the Cartan subalgebra $\mathcal{H}$ in the case of the adjoint representation) has multiplicity. Still, there are many possible representation diagrams.

We have explicitly produced n distinguished representation diagrams for the adjoint representation of $\mathcal{L}$, one for each simple root $\{\alpha_i\}_{i=1}^n$. For now, let us refer to these as the "efficient adjoint constructions." Let m be the number of short simple roots for $\mathcal{L}$. We also have m distinguished representation diagrams for the short adjoint representation of $\mathcal{L}$, which we will refer to as the "efficient short adjoint constructions." The supports for these representation diagrams are pairwise distinct, and each support is a *modular lattice* (see Section 2). The edge coefficients are all positive, rational numbers. If the Dynkin diagram for $\mathcal{L}$ does not branch (i.e. $\mathcal{L} \in \{A_n, B_n, C_n, F_4, G_2\}$), then exactly two of these efficient adjoint constructions have *distributive lattice support.* These distributive supports "correspond" to the simple roots at each end of the Dynkin diagram for one of these simple algebras. The adjoint representations of D_n, E_6, E_7, and E_8 cannot be realized on distributive lattices. We proved the following theorem by proceeding case by case, although there are uniform aspects to our proof.

Theorem 4.5 *Let $\mathcal{L}$ be a simple Lie algebra of rank n, with m short simple roots. The supports for the n efficient adjoint constructions (respectively, the m efficient short adjoint constructions)* *mentioned above are precisely the edge-minimizing supports for the adjoint (resp. short adjoint) representation. Moreover, these are precisely the modular lattice supports for the adjoint (resp. short adjoint) representation, and these supports are all solitary.* $\square$

5 Questions and Remarks

The results of the previous section show that the irreducible representations of $sl(n, \mathbb{C})$, the fundamental representations of $sp(2n, \mathbb{C})$ and $so(2n+1, \mathbb{C})$, the adjoint representations of the simple Lie algebras, and many* irreducible multiplicity free representations can be realized on modular lattice supports. Moreover, each of these supports is solitary, and the fixed products of edge coefficients are positive, rational numbers. (And except for the representations $A_n(\lambda)$, in all these cases we have provided positive, rational edge coefficients.) In certain cases (adjoint representations of simple Lie algebras, irreducible multiplicity free representations), we know that these supports are edge-minimizing.

In both the KN and De Concini symplectic cases we have been able to q-ize the edge coefficients in order to obtain explicit realizations of the "fundamental" representations of the associated quantized enveloping algebra $\mathcal{U}_q(C_n)$. (This has already been done for the Gelfand-Zetlin bases for the irreducible representations of $gl(n, \mathbb{C})$; for example, see [NT].) We expect to be able to q-ize our other constructions as well.

The examples of Section 4 suggest the following questions about the structure of supporting diagrams. Is an edge-minimizing support always a modular lattice, and vice-versa? Will edge-minimizing supports or modular lattice supports always be solitary? (The converse is false if we look at the "KN" support for $C_2(3\omega_1)$.) If a representation diagram is edge-minimizing, will its connected components realize irreducible representations? Do edge-minimizing representation diagrams also minimize the total number of nonzero entries in the matrices $\{X_i, Y_i\}_{i=1}^n$? (The converse is false for $A_2(\omega_1+\omega_2)$.) Will important bases such as Lusztig's canonical basis be edge-minimizing? solitary? have modular lattice support? Will the fixed products of edge coefficients for solitary supports always be positive, rational numbers? Will there be a corresponding representation diagram with positive, rational edge coefficients?

In the future we hope to be able to construct the irreducible representation $B_n(k\omega_1)$ on the distributive lattice $L_B^{RS}(k\omega_1, n) := Good(k, 2n)$ defined by Reiner and Stanton in [RS]. We know how the edges of this lattice should be colored. We also know that if $L_B^{RS}(k\omega_1, n)$ is a supporting diagram for $B_n(k\omega_1)$, then it will respect the chain of subalgebras $A_1 \subset \cdots \subset A_{n-1} \subset B_n$, where A_m is the subalgebra of B_n corresponding to the m leftmost nodes in the Dynkin diagram for B_n. Moreover, when all edges with color n are removed from $L_B^{RS}(k\omega_1, n)$, the resulting connected components can be shown to correspond to certain $L_A^{KN}(\mu, n)$. Now, each irreducible A_{n-1}-module appearing in the decomposition of $B_n(k\omega_1)$ (regarded as an A_{n-1}-module) does so only once. So if $L_B^{RS}(k\omega_1, n)$ is a supporting diagram, it is solitary. Presumably the GZKN coefficients can be attached to the edges of color i when $1 \leq i < n$, so we only need to determine the coefficients on the edges of color n. We have also found distributive lattices "$L_B^{DeC}(k\omega_1, n)$" that will hopefully serve as the De Concini analogs in this case.

We have speculated that the symplectic lattices are edge-minimizing. This is automatically true when $k = 1$ since $C_n(\omega_1)$ is minuscule. Case $k = 2$ is covered by Theorem 4.5. Will the crystal graph always appear inside any support for $C_n(\omega_k)$? If so, then perhaps we could argue that each of $L_C^{KN}(k, 2n-k)$ and $L_C^{DeC}(k, 2n-k)$ adds the least number of edges to this subgraph needed to

*To make this claim for *all* such irreducible representations, we only need to check that our list in Section 4 of irreducible multiplicity free representations for the simple Lie algebras is a complete list.

make it a support for $C_n(\omega_k)$.

More generally, we could ask: will a supporting diagram always contain the associated crystal graph as a subgraph? In the case of representations of $sl(2,\mathbb{C})$, the implications of this question are surprising. An affirmative answer would imply that any connected supporting diagram for $sl(2,\mathbb{C})$ (including $L(k,N-k)$, $L(\lambda,N)$, $L_C^{KN}(k,2n-k)$, $L_C^{DeC}(k,2n-k)$, etc) has a symmetric chain decomposition.

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