Formal Power Series and Algebraic Combinatorics 1998
Extended Abstract For Lecture

SYMPLECTIC MATROIDS THE EASY WAY

Timothy Y. Chow
University of Michigan Department of Mathematics
525 E. University Ave. #2072, Ann Arbor, MI 48109
tchow@umich.edu

Summary. After a brief exposition of Gelfand and Serganova's theory of Coxeter matroids, we present our main result: an axiomatization of symplectic matroids in terms of independent sets. We hope that this contribution to the basic theory of Coxeter matroids will make this somewhat esoteric subject more accessible. Our result is also a small step towards the long-term goal of a "combinatorial Erlanger program" for discrete symplectic and orthogonal geometry.

Résumé. Après une briève présentation de la théorie des matroïdes de Coxeter par Gelfand et Serganova, nous présentons notre résultat principal: une axiomatization des matroïdes symplectiques en termes d'ensembles indépendants. On espère que cette contribution à la théorie des matroïdes de Coxeter rendra le sujet plus accessible. Il s'agit d'une étape vers un "programme combinatoire d'Erlanger" dans la géométrie symplectique et orthogonale discrète.

Introduction

Symplectic matroids are not so well known, so they require more than the average amount of motivation. So although the main result of this paper is small, we shall give a "big picture" introduction.

Symplectic matroids occur at the confluence of two streams in modern combinatorics: Coxeter groups and matroid theory. Recall that a matroid may be defined as a pair $(E, \mathscr{I})$ where E is a finite set and $\mathscr{I}$ is a nonempty family of subsets of E that satisfies the following conditions:

(1) if $J \in \mathscr{I}$ and $I \subseteq J$ then $I \in \mathscr{I}$;

(2) if $I \in \mathscr{I}$ and $J \in \mathscr{I}$ and $|I| < |J|$, then there exists $x \in J \setminus I$ such that $I \cup \{x\} \in \mathscr{I}$.

For example, if we let E be a set of vectors in a vector space and $\mathscr{I}$ the family of linearly independent subsets of E, then $(E, \mathscr{I})$ is clearly a matroid. For this reason, in any matroid, the members of $\mathscr{I}$ are called *independent sets* and the independent sets of maximum cardinality are called *bases.* For another example, let G be a finite undirected graph with edge set E, and let $\mathscr{I}$ be the family of (edge sets of) acyclic subgraphs of G. It is not hard to show that $(E, \mathscr{I})$ is a matroid.

Matroid theory is a major branch of modern combinatorics. The above two examples already suggest its importance, and for further motivation one can simply browse through the numerous beautiful theorems and conjectures in [6] or any other text on matroids.

Coxeter groups are a class of groups that include finite subgroups of $GL_n(\mathbb{R})$ that are generated by reflections, and Weyl groups arising from Lie theory. They typically arise in combinatorics when some combinatorial concept is shown to be definable in terms of the symmetric group. The symmetric group is a Coxeter group, so by replacing the term "symmetric group" in such a definition by an arbitrary Coxeter group, one can often obtain "Coxeter analogues" of the original concept. In particular, the Weyl groups of classical semisimple Lie groups give rise to "symplectic" and "orthogonal" analogues. Doing this often sheds new light on the original concept and suggests ways of applying methods from representation theory or geometry to combinatorial problems.

This suggests the following questions: can matroids be defined in terms of the symmetric group? If so, what do the Coxeter analogues of matroids look like? J. P. S. Kung [3, 4] seems to have been the first to ask and answer these questions. Later, I. M. Gelfand and V. V. Serganova [2] suggested a different definition of such analogues (which they called "WP-matroids"). In both cases the work has a strong geometric flavor, and one exciting possibility is that these "Coxeter matroids" may form the foundation for discrete symplectic and orthogonal geometry, in the same way that oriented matroids form the basis for MacPherson's theory of combinatorial differentiable manifolds [5].

However, before this (in Kung's words) "combinatorial Erlanger program" can be carried out, many fundamental questions must first be answered. For instance, what is the precise connection between Kung's matroids and WP-matroids? The answer to this is not obvious. Kung was motivated by invariant theory, and WP-matroids were originally defined in terms of the rather technical concept of Bruhat order on parabolic quotients of Coxeter groups. These complicated definitions make it hard to see what is happening.

A basic fact about matroids is that they admit a wide variety of equivalent elementary axiomatizations: independent sets, circuit elimination, basis exchange, etc. But so far no analogous elementary axiomatizations for WP-matroids are known. Borovik, Gelfand and White [1] have made some progress in obtaining such elementary axiomatizations in the symplectic case, but their paper does not, for example, give an independent set axiomatization of symplectic matroids. Indeed, for some time it was suspected that finding such an axiomatization might be an intractable problem.

The main result of this paper is an independent set axiomatization for (Gelfand-Serganova) symplectic matroids. In addition to answering a very natural question, this result provides one of the simplest ways to date of explaining what a symplectic matroid is to someone with no background in matroids or Coxeter groups.

It is natural to ask if this result can be used to obtain other elementary axiomatizations of symplectic WP-matroids, or if the result can be extended to other classes of WP-matroids. There is indeed hope of this, although preliminary attempts have run up against technical difficulties.

We conclude this introduction with one final remark. The reader may still feel that symplectic matroids are very special structures. In fact, a wide variety of objects that appear in the literature—including ordinary matroids, metroids, Lagrangian matroids, and Δ-matroids—are all essentially special cases of symplectic matroids. Most theorems about symplectic matroids therefore automatically yield corollaries about these classical combinatorial structures.

The Main Result

As mentioned before, Gelfand and Serganova define WP-matroids in terms of the Bruhat order on parabolic quotients of Coxeter groups. But as suggested by the title, we are trying to do things as simply as possible here, so we will take advantage of the results in [1] and define symplectic matroids in a way that requires no explicit mention of these concepts. Readers familiar with Coxeter groups who want the full story should see [7].

Let $E_{\pm n}$ be the set $\{\pm 1, \pm 2, \pm 3, \ldots, \pm n\}$. For brevity we will sometimes write the minus sign on top, e.g., we will write $\bar{1}$ for -1. If w is a permutation of $E_{\pm n}$ and $B = \{b_1, b_2, \ldots, b_k\}$ is a subset of $E_{\pm n}$, then we define

$$wB \stackrel{\text{def}}{=} \{wb_1, wb_2, \ldots, wb_k\}.$$

We define a partial ordering $\leq$ on k-element subsets of $E_{\pm n}$ by letting

$$\{a_1, a_2, \ldots, a_k\} \leq \{b_1, b_2, \ldots, b_k\}$$

if $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 < \cdots < b_k$ and $a_i \leq b_i$ for all i. (In other words, to compare two k-element subsets, sort their elements into increasing order and compare them componentwise.) We say that a permutation w of $E_{\pm n}$ is *admissible* if $w(-x) = -w(x)$ for all $x \in E_{\pm n}$. If $S \subseteq E_{\pm n}$, define

$$\bar{S} \stackrel{\text{def}}{=} \{-s \mid s \in S\}.$$

We say that S is *admissible* if $S \cap \bar{S} = \varnothing$. We can now define a symplectic matroid.

Definition. A *symplectic matroid* is a pair $(E_{\pm n}, \mathscr{B})$ where $\mathscr{B}$ is a nonempty family of equinumerous admissible subsets of $E_{\pm n}$ satisfying the following *maximality property:* for every admissible permutation w of $E_{\pm n}$ there exists a unique $B \in \mathscr{B}$ such that $wA \leq wB$ for all $A \in \mathscr{B}$.

The family $\mathscr{B}$ is called the family of *bases* of the symplectic matroid. If $(E_{\pm n}, \mathscr{B})$ is a symplectic matroid, we define its *family* $\mathscr{I}$ *of independent sets* by

$$\mathscr{I} \stackrel{\text{def}}{=} \{I \subseteq E_{\pm n} \mid I \subseteq B \text{ for some } B \in \mathscr{B}\}.$$

An example of a symplectic matroid is $(E_{\pm 3}, \mathscr{B})$ where

$$\mathscr{B} = \{1\bar{3}, 2\bar{3}, \bar{1}2, \bar{1}3\}.$$

Here $1\bar{3}$ is to be understood as shorthand for the set $\{1, \bar{3}\}$. The family of independent sets in this example is

$$\mathscr{I} = \{\varnothing, 1, \bar{1}, 2, 3, \bar{3}\} \cup \mathscr{B}.$$

The definition of a symplectic matroid does not look much like the definition of a matroid, but in fact there exists a characterization of matroids in terms of a maximality property. Let E_n be the set $\{1, 2, \ldots, n\}$ and define a partial ordering $\leq$ on k-element subsets of E_n in exactly the same way we did for $E_{\pm n}$. Then we have the following standard result.

Theorem. *Let $\mathscr{B}$ be a nonempty family of equinumerous subsets of E_n. Then $\mathscr{B}$ is the family of bases of some matroid $(E_n, \mathscr{I})$ if and only if $\mathscr{B}$ satisfies the following maximality property: for every permutation w of E_n there exists a unique $B \in \mathscr{B}$ such that $wA \leq wB$ for all $A \in \mathscr{B}$.*

So the definition of a symplectic matroid is reasonable. Now for our main result.

Theorem. *A subset-closed family $\mathscr{I}$ of admissible subsets of $E_{\pm n}$ is the family of independent sets of a symplectic matroid if and only if it has the following property:*

> *If I and J are members of $\mathscr{I}$ such that $|I| < |J|$ and such that for all $y \in J \setminus I$, the set $\{y\} \cup I$ is not in $\mathscr{I}$, then $I \cup J$ is inadmissible and there exists $x \notin I$ such that both $\{x\} \cup I$ and $\{\bar{x}\} \cup I \setminus \bar{J}$ are in $\mathscr{I}$.*

We remark that part of this theorem—the part about $I \cup J$ being inadmissible—was previously known and is essentially Theorem 14 of [1].

The proof of the theorem, which we shall not give here in this abstract, relies on Borovik, Gelfand, and White's greedy algorithm for symplectic matroids [1, Theorem 16]. While not trivial, the proof is not excessively difficult. Most of the difficulty lay in finding the correct statement of the result.

Acknowledgments

This work was supported in part by a National Science Foundation Postdoctoral Fellowship. Part of the work was done while I was a general member of the Mathematical Sciences Research Institute. I thank Neil White for introducing me to this problem and for helpful discussions and encouragement.

References

1. A. V. Borovik, I. M. Gelfand, and N. L. White, Symplectic matroids, *J. Alg. Combin.,* accepted for publication.

2. I. M. Gelfand and V. V. Serganova, Combinatorial geometries and torus strata on homogeneous compact manifolds, *Russ. Math. Surv.* **42** (1987), 133–168.

3. J. P. S. Kung, An Erlanger program for combinatorial geometries, Ph.D. thesis (1978), Massachusetts Institute of Technology, Cambridge, MA.

4. J. P. S. Kung, Bimatroids and invariants, *Advances in Math.* **30** (1978), 238–249.

5. R. MacPherson, Combinatorial differential manifolds, in *Topological Methods in Modern Mathematics: A Symposium in Honor of John Milnor's Sixtieth Birthday,* Publish or Perish, Houston, 1993.

6. J. G. Oxley, *Matroid Theory,* Oxford University Press, New York, 1992.

7. N. L. White, The Coxeter matroids of Gelfand et al., *Contemp. Math.* **197** (1996), 401–409.