The Terwilliger Algebras of Bipartite P- and Q-polynomial Association Schemes

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Abstract. Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a symmetric association scheme with $D \geq 3$, and suppose Y is bipartite P- and Q-polynomial. Let T denote the Terwilliger algebra with respect to any vertex x. We prove that any irreducible T-module W is both thin and dual-thin in the sense of Terwilliger. We produce two bases for W and describe the action of T on these bases. We prove that the isomorphism class of W as a T-module is determined by two parameters, the endpoint and diameter of W. We find the multiplicities with which the irreducible T-modules occur in the standard module for T.

Résumé. Soit $Y = (X, \{R_i\}_{0 \leq i \leq D})$ un association-scheme symétrique avec $D \geq 3$. Supposons que Y est bi-partie et a la propriété des polynômes P et Q. Soit T l'algébre de Terwilliger correspondant á un apex x arbitraire. Nous montrons qu'un T-module W irréductible est *thin* et *dual-thin* au sens de Terwilliger. Nous donnons deux bases de W et décrivons l'action de T sur ces bases. Nous montrons que, la class d'isomorphisme de W est déterminée par deux paramétres, le diametre et le terminus de W. Nous trouvons les multiplicités des T-modules irréductible dans le module ordinaire de T.

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1 Introduction. The Terwilliger algebra of a commutative association scheme was introduced in [19]. This algebra is a finite dimensional, semisimple $\mathbb{C}$-algebra, and is noncommutative in general. The Terwilliger algebra has been used to study P- and Q-polynomial schemes [19], group schemes [1], [3], strongly regular graphs [22], Doob schemes [18], and schemes over the Galois rings of characteristic four [15]. Other work involving this algebra can be found in [10], [9], [11], [12], [13], [14], [20], [21], and [23].

The Terwilliger algebra is particularly well-suited for studying P- and Q-polynomial schemes; nevertheless, it was shown in [19] that the intersection numbers of these schemes do not completely determine the structure of the algebra. In this talk, we consider the Terwilliger algebra of a bipartite P- and Q-polynomial scheme. Examples of such schemes are the bipartite dual polar graphs $D_D(q)$, the D-cubes $H(D, 2)$, the folded cubes (antipodal quotient of $H(2D, 2)$), the cycles of even length, and the graph of cocliques in the Hoffman-Singleton graph. There are also several families of feasible intersection arrays for which existence has not yet been determined. We show that under the added bipartite assumption, the intersection numbers of the scheme completely determine the structure of the algebra.

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To describe our results, let $Y = (X, \{R_i\}_{0\le i\le D})$ denote a symmetric association scheme with $D \ge 3$ (Definitions appear in section 2 below). Suppose Y is bipartite P- and Q-polynomial. Fix any $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra of Y with respect to x. T acts faithfully on the vector space $\mathbb{C}^X$ by matrix multiplication; we refer to $\mathbb{C}^X$ as the *standard module*. Since T is semi-simple, $\mathbb{C}^X$ decomposes into a direct sum of irreducible T-modules.

Let W denote an irreducible T-module contained in $\mathbb{C}^X$. We show that W is thin and dual thin in the sense of Terwilliger. We produce two bases for W with respect to which the action of T is particularly simple. To describe this action, we use two sets of scalars, the *intersection numbers* of W and the *dual intersection numbers* of W. We compute these scalars in terms of the eigenvalues of Y, the dual eigenvalues of Y, and two additional parameters, called the *endpoint* and *diameter* of W (Theorems 3.1, 3.2). We show that the endpoint and diameter of W determine its isomorphism class as a T-module (Theorem 3.3).

Combining our above results, we find a recurrence which gives the multiplicities with which the irreducible T-modules occur in $\mathbb{C}^X$. Using the recurrence, we obtain formulas for the multiplicities of the irreducible T-modules (Theorem 3.4).

We now give some of the details.

2 Association Schemes.

A D-class **symmetric association scheme** is a pair $Y = (X, \{R_i\}_{0\le i\le D})$, where X is a non-empty finite set, and where

(i) $\{R_i\}_{0\le i\le D}$ is a partition of $X \times X$;

(ii) $R_0 = \{xx \mid x \in X\}$;

(iii) $R_i^t = R_i$ for $0 \le i \le D$, where $R_i^t = \{yx \mid xy \in R_i\}$;

(iv) For all h, i, j $(0 \le h, i, j \le D)$, and for all $x, y \in X$ such that $xy \in R_h$, the scalar $p^h_{ij} = |\{z \in X | xz \in R_i, \text{ and } zy \in R_j\}|$ is independent of x, y. $\square$

The integers p^h_{ij} are known as the **intersection numbers** for Y.

Let $Y = (X, \{R_i\}_{0\le i\le D})$ denote a symmetric association scheme, and let $\mathrm{Mat}_X(\mathbb{C})$ denote the algebra of matrices over $\mathbb{C}$ with rows and columns indexed by X. For $0 \le i \le D$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } xy \in R_i, \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in X). \tag{1}$$

Then $A_0, ..., A_D$ form a basis for a semisimple subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$, known as the **Bose-Mesner algebra**. Y is said to be **P-polynomial** (with respect to the ordering $A_0, ..., A_D$) whenever each matrix A_i is a polynomial of degree exactly i in A, where $A = A_1$. When Y is P-polynomial, the graph structure induced on X by the relation R_1 is said to be **distance-regular**. In this case, Y is called **bipartite** if the corresponding graph is bipartite.

Let $E_0, ..., E_D$ denote the primitive idempotents of M. Y is said to be **Q-polynomial** (with respect to the ordering $E_0, ..., E_D$) whenever each matrix E_i is polynomial of degree exactly i in E_1, in the algebra $(M, \circ)$, where $\circ$ denotes entry-wise multiplication.

We now recall the dual Bose-Mesner algebra. Fix any vertex $x \in X$. For each integer i $(0 \leq i \leq D)$, let $E_i^* = E_i^*(x)$ (resp. $A_i^* = A_i^*(x)$) denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with y, y entry $(A_i)_{xy}$ (resp. $|X|(E_i)_{xy}$) $(y \in X)$. The matrices $A_0^*, ..., A_D^*$ form a basis for a semisimple subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$, known as the **dual Bose-Mesner algebra** for Y with respect to x. $E_0^*, ..., E_D^*$ are the primitive idempotents for M^*.

By the **Terwilliger algebra** for Y with respect to x, we mean the subalgebra $T = T(x)$ of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We remark that T is finite dimensional and semisimple. For the remainder of this section, let Y denote a bipartite P- and Q-polynomial scheme. Our main result is the determination of the irreducible T-modules in this bipartite case. To describe an irreducible T-module W, we use the following parameters. We define the **endpoint** to be $r := min\{i | E_i^* W \neq 0\}$, the **dual endpoint** to be $t := min\{i | E_i W \neq 0\}$, and the **diameter** to be $d := |\{i | E_i^* W \neq 0\}| - 1$.

We next consider the parameterization of Y. For $0 \leq i \leq D$, let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with the idempotent E_i (resp. E_i^*). It is well-known that the intersection numbers for Y are determined by θ_0,...,θ_D and θ_0^*,...,θ_D^*. Moreover, as long as Y is not one of a small number of exceptional schemes, there exist complex scalars q, s^* such that

$$\begin{aligned} \theta_i &= h(q^{-i} - q^{i-D}), & (2) \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)(1 - s^* q^{i+1}) q^{-i}, & (3) \end{aligned}$$

for $0 \leq i \leq D$, where

$$h = \frac{q^D(1 - s^* q^3)}{(q-1)(1 - s^* q^{D+2})}, \qquad \theta_0^* = \frac{h^*(q^D - 1)(1 - s^* q^2)}{q(q^{D-1} + 1)}, \qquad h^* = \frac{(q^D + q^2)(q^D + q)}{q(q^2 - 1)(1 - s^* q^{2D})}. \quad (4)$$

This theorem is due to Leonard [16]. For notational convenience, we say Y is of **type I** whenever (2)-(4) hold.

3 Results.

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a bipartite P- and Q-polynomial association scheme with $D \geq 3$. Fix any $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra for Y with respect to x. Let W denote an irreducible T-module with endpoint r, dual endpoint t, and diameter d. We want to describe the action of the Terwilliger algebra on W. It is known that

$$E_r^* v, \; E_{r+1}^* v, \; \ldots, E_{r+d}^* v \qquad (5)$$

is a basis for W, where v is any nonzero element in $E_t W$. With respect to this basis, the matrix representing multiplication by A^* is $diag(\theta_r^*, \theta_{r+1}^*, ..., \theta_{r+d}^*)$. The matrix representing multiplication by A is of the form

$$\begin{pmatrix} a_0(W) & b_0(W) & & & & \mathbf{0} \\ c_1(W) & a_1(W) & b_1(W) & & & \\ & c_2(W) & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & a_{d-1}(W) & b_{d-1}(W) \\ \mathbf{0} & & & & c_d(W) & a_d(W) \end{pmatrix}, \qquad (6)$$

for some scalars $a_i(W)$, $b_i(W)$, $c_i(W)$ $(0 \le i \le d)$. These scalars are given by the following.

3.1 Theorem. *(Caughman [5]) Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote a bipartite P- and Q-polynomial association scheme of type I, with diameter $D \ge 3$. Fix any $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra for Y with respect to x. Let W denote an irreducible T-module with endpoint r, dual endpoint t, and diameter d. Then the scalars $a_i(W)$, $b_i(W)$, $c_i(W)$ from line (6) are given by*

$$\begin{aligned} c_i(W) &= \frac{h(q^i-1)(1-s^*q^{2r+d+i+1})}{q^{d+t}(1-s^*q^{2r+2i+1})} \qquad (0 \le i \le d-1), \\ c_d(W) &= h(q^{-t}-q^{t-D}), \end{aligned} \tag{7}$$

$$b_0(W) = h(q^{-t}-q^{t-D}), \tag{8}$$

$$b_i(W) = \frac{h(q^d-q^i)(1-s^*q^{2r+i+1})}{q^{d+t}(1-s^*q^{2r+2i+1})} \qquad (1 \le i \le d),$$

$$a_i(W) = 0 \qquad (0 \le i \le d), \tag{9}$$

where q, s^*, h are as in lines (2)-(4). □

Referring to Theorem 3.1, there is a second basis for W that is of interest, dual to the basis in (5). It has the form $E_t u, E_{t+1}u, \ldots, E_{t+d}u$, where u is any nonzero element in $E_r^* W$. With respect to this basis, the matrix representing multiplication by A has the form $diag(\theta_t, \theta_{t+1}, ..., \theta_{t+d})$. The matrix representing multiplication by A^* is tridiagonal as in (6); we denote the entries of this matrix by $a_i^*(W)$, $b_i^*(W)$, $c_i^*(W)$ $(0 \le i \le d)$. These scalars are given by the following.

3.2 Theorem. *(Caughman [5]) Let Y, T, and W be as in Theorem 3.1. Then the scalars $a_i^*(W)$, $b_i^*(W)$, $c_i^*(W)$ are given by*

$$c_i^*(W) = \frac{h^*(q^{2i}-1)(1-s^*q^{2d+2r-2i+2})}{q^{2i+r}(q^{d-2i+1}+1)(q^{d-2i}+1)} \qquad (0 \le i \le d-1), \tag{10}$$

$$c_d^*(W) = \frac{h^*(q^d-1)(1-s^*q^{2r+2})}{q^{r+1}(q^{d-1}+1)}, \tag{11}$$

$$b_0^*(W) = \frac{h^*(q^d-1)(1-s^*q^{2r+2})}{q^{r+1}(q^{d-1}+1)}, \tag{12}$$

$$b_i^*(W) = \frac{h^*(q^{2d-2i}-1)(1-s^*q^{2r+2i+2})}{q^{2i+r+1}(q^{d-2i-1}+1)(q^{d-2i}+1)} \qquad (1 \le i \le d), \tag{13}$$

$$a_i^*(W) = \theta_r^* - b_i^*(W) - c_i^*(W) \qquad (0 \le i \le d), \tag{14}$$

where q, s^*, h are as in lines (2)-(4). □

3.3 Theorem. *(Caughman [5]) With the notation of Theorem 3.1, the following hold.*

(i) $2t + d = D$; *in particular,* $D - d$ *is even.*

(ii) $\frac{1}{2}(D-d) \ \leq \ r \ \leq \ D-d.$

(iii) The isomorphism class of W as a T-module is determined by r and d. □

3.4 Theorem. *(Caughman [5]) Let Y and T be as in Theorem 3.1. For any integers r, d let $mult(r,d)$ denote the multiplicity with which the irreducible T-module with endpoint r and diameter d appears in the standard module $\mathbb{C}^X$. Then these multiplicities are determined by the intersection numbers of Y. For example,*

(i) $mult(0,D) = 1.$

(ii) $$mult(1,D-2) = \frac{(q^D-q)(1-s^*q^2)}{(q-1)(1-s^*q^{D+2})}.$$

(iii) $$mult(2,D-2) = \frac{(q^D-1)(q^D-q)(1-s^*q^4)(1+s^*q^{D+1})}{q(q^2-1)(1-s^*q^{D+2})(1-s^*q^{2D})}.$$

(iv) $$mult(2,D-4) = \frac{(q^D-1)(q^D-q^3)(1-s^*q^2)(1-s^*q^4)(1-s^*q^{2D+1})}{q(q-1)(q^2-1)(1-s^*q^{D+2})(1-s^*q^{D+3})(1-s^*q^{2D})}.$$

(v) $$mult(3,D-4) = \frac{(q^D-1)(q^D-q)(q^D-q^3)(1-s^*q^2)(1-s^*q^6)(1+s^*q^{D+1})}{q^3(q-1)(q^2-1)(1-s^*q^{D+2})(1-s^*q^{D+4})(1-s^*q^{2D-2})}.$$

where q, s^*, h are as in lines (2)-(4). □

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