Exact enumeration of 1342-avoiding permutations
A close link with labeled trees and planar maps

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Abstract

Solving the first nonmonotonic, longer-than-three instance of a classic enumeration problem, we obtain the generating function $H(x)$ of all 1342-avoiding permutations of length n as well as an *exact* formula for their number $S_n(1342)$. While achieving this, we bijectively prove that the number of indecomposable 1342-avoiding permutations of length n equals that of labeled plane trees of a certain type on n vertices recently enumerated by Cori, Jacquard and Schaeffer, which is in turn known to be equal to the number of rooted bicubic maps enumerated by Tutte in 1963. Moreover, $H(x)$ turns out to be algebraic, proving the first nonmonotonic, longer-than-three instance of a conjecture of Noonan and Zeilberger. We also prove that $\sqrt[n]{S_n(1342)}$ converges to 8, so in particular, $\lim_{n\to\infty} \frac{S_n(1342)}{S_n(1234)} = 0$.

Résumé

En prouvant la première spécialisation d'un problème de dénombrement classique dans les cas nonmonotoniques, on obtient la fonction génératrice $H(x)$ des permutations de longueur n évitant le motif 1342, et aussi une formule exacte pour leur nombre $S_n(1342)$. En la trouvant on prouve, d'une façon bijective, que le nombre des permutations indécomposables de longueur n évitant le motif 1342 est égal à celui des arbres plans d'un certain genre, qui étaient récemment dénombrés par Cori, Jacquard et Schaeffer. De plus, ce nombre est aussi égal au nombre de cartes planaires pointées bicubiques, dénombrées par Tutte en 1963. En outre, $H(x)$ est algébrique, ce qui prouve une spécialisation d'une conjecture de Zeilberger et Noonan. Nous prouvons aussi que $\sqrt[n]{S_n(1342)}$ converge vers 8, donc en particulier $\lim_{n\to\infty} \frac{S_n(1342)}{S_n(1234)} = 0$.

1 Introduction

1.1 Our main results

In this paper we are going to prove an exact formula for the number $S_n(1342)$ of 1342-avoiding permutations of length n showing that

$$S_n(1342) = \frac{(7n^2-3n-2)}{2}\cdot(-1)^{n-1} + 3\sum_{i=2}^{n} 2^{i+1}\cdot\frac{(2i-4)!}{i!(i-2)!}\cdot\binom{n-i+2}{2}\cdot(-1)^{n-i},$$

by first proving that the ordinary generating function $H(x)$ for these numbers $S_n(1342)$ has the following simple form:

$$H(x) = \frac{32x}{-8x^2+20x+1-(1-8x)^{3/2}}.$$

This is the first result we know of which provides an exact formula for the number $S_n(q)$ of permutations of length n avoiding a given pattern q that is longer than three and is not 1234. The formula for $q = 1234$ is given in [7]. Until recently it has not even been known that $S_n(1342) < c^n$ for some constant c. In [1] this upper bound with $c = 9$ was proved. This paper's result pushes down this c to 8, and proves that it is optimal. (Definitions and more background information can be found in the next subsection).

In our proof, we are going to present a new link between the enumeration of permutations avoiding the pattern 1342 and the that of $\beta(0,1)$-trees, a class of labeled trees recently introduced in [4]. We will show that the number $I_n(1342)$ of *indecomposable* 1342-avoiding permutations of length n is equal to the number of $\beta(0,1)$-trees on n nodes. The set $D_n^{\beta(0,1)}$ of $\beta(0,1)$-trees on n nodes is known [4] to be equinumerous to the set of rooted bicubic maps on $2(n-1)$ vertices, and an exact formula for the number t_n of these is provided in [18]. Therefore, $I_n(1342) = |D_n^{\beta(0,1)}| = t_n = 3\cdot 2^{n-2}\cdot\frac{(2n-2)!}{(n+1)!(n-1)!}$. So combinatorially, the number of all n-permutations avoiding 1342 will be shown to be equal to that of plane forests on n vertices in which each component is a $\beta(0,1)$-tree. To our best knowledge, this is the first time when permutations avoiding a given pattern are shown to have such a close connection with some planar maps, though recently 2-stack-sortable permutations have been shown to be equinumerous to nonseparable planar maps [5] [6].

Examining the generating function $H(x)$ we will prove and disprove several conjectures for the pattern 1342. In fact, $H(x)$ turns out to be algebraic, proving a conjecture of Noonan and Zeilberger [9] for the first time for a nonmonotonic pattern which is longer than three. We will see that $\sqrt[n]{S_n(1342)} \to 8$, which disproves a conjecture of Stanley and implies the surprising fact that $\lim_{n\to\infty}\frac{S_n(1342)}{S_n(1234)} = 0$.

1.2 Definitions and Background

In what follows permutations of length n will be called n-permutations. Let $q = (q_1, q_2, \ldots, q_k)$ be a k-permutation and let $p = (p_1, p_2, \ldots, p_n)$ be an n-permutation. We say that p contains a q-subsequence if there exists $1 \leq i_{q_1} < i_{q_2} < \ldots < i_{q_k} \leq n$ such that $p_{i_1} < p_{i_2} < \ldots < p_{i_k}$. We say that p avoids q if p contains no q-subsequence. For example, p avoids 132 if it cannot be written as $\cdots, a, \cdots, b, \cdots c, \cdots$ so that $a < c < b$.

It is a natural and easy-looking question to ask how many n-permutations avoid a given pattern q. Throughout this paper, this number will be denoted by $S_n(q)$.

However, this problem turns out to be very hard. Exact answers have only been known for the easy case of patterns of length 3 [11]. In that case $S_n(q) = C_n$, the n-th Catalan number, for any such pattern q. If q is longer than three, then the most exact result is due to Regev [10] and says that for all n, $S_n(1234\cdots k)$ asymptotically equals $c\frac{(k-1)^{2n}}{n^{(k^2-2k)/2}}$, where c is a constant given by a multiple integral. The major problem of this area is to prove the conjecture of Wilf and Stanley [17] from 1990, stating that for each pattern q there is an absolute constant λ so that $S_n(q) < \lambda^n$ holds. This has been proven for the case of length 4 and some longer patterns in [1]. The general case is, however, still open.

Stanley [16] conjectured that $\sqrt[n]{S_n(q)}$ converges to $(k-1)^2$ where k is the length of q. The results of this paper will indicate the *disproof* of this conjecture for the pattern 1342, in fact, our formula for $S_n(1342)$ clearly implies that $\sqrt[n]{S_n(1342)} \to 8$. This shows in particular that $\frac{S_n(1342)}{S_n(1234)} \to 0$ when $n \to \infty$ as $\sqrt[n]{S_n(1234)} \to 9$ by the above results of Regev [10] and Gessel [7]. We would like to point out the surprising nature of this discovery: while $S_n(q) = \binom{2n}{n}/(n+1)$ for *any* pattern q of length three, for the case of length four there are sequences $S_n(q)$ that are not only different from each other, but their quotient also converges to 0.

We note that while there are 24 permutation patterns of length 4, for many of them the sequences $S_n(q)$ are identical. In fact, there are only three different classes of patterns from this point of view, [19], [12]; they can be represented by 1342, 1234 and 1324, respectively.

Recently, attention has been paid to the problem of counting the number $S_{n,r}(q)$ of permutations of length n containing a *given number* r (as opposed to 0) of subsequences of a certain type q.

In [9] Noonan and Zeilberger conjectured that for *any* given subsequence q and for *any* given r, $S_{n,r}(q)$ is a p-recursive function of n. The present author has proved this conjecture for any r when $q = 132$. Beyond the case of length 3, however, there have been no nonmonotic instances solved prior to this paper, not even for the case of $r = 0$. Even in the case of length 3, exact formulae have only been given for $r = 1$ ([8] for $q = 123$ and [3] for 132). The case of monotonic patterns and $r = 0$ has been handled in [20]. Figure 1 shows the present state of research on permutations avoiding given patterns of length 4, including the contributions of this paper.

	$S_n(q) < c^n$	formula	p-recursive
1234	known [10]	known [7]	known [20]
1342	known [1]	known this paper	known this paper
1324	known [1]	open	open

Figure 1

The definition and basic properties of p-recursive functions can be found in [13].

We are going to prove our results by showing connections between these permutations and rooted bicubic maps. These are planar maps with 2-colorable vertices which in addition all have degree three and a distinguished "root" edge and face. We will make good use of a class of labeled rooted trees called $\beta(0,1)$-trees introduced in [4]. Their definition is at the beginning of the next section.

2 The correspondence between trees and permutations

Cori, Jacquard and Schaeffer [4] give the following definition in their study of planar maps.

Definition 1 *A rooted plane tree with nonnegative integer labels $l(v)$ on each of its vertices v is called a $\beta(0,1)$-tree if it satisfies the following conditions:*

- *if v is a leaf, then $l(v) = 0$,*
- *if v is the root and $v_1, v_2, \cdots, v_k$ are its children, then $l(v) = \sum_{i=1}^{k} l(v_k)$,*
- *if v is an internal node and $v_1, v_2, \cdots, v_k$ are its children, then $l(v) \leq 1 + \sum_{i=1}^{k} l(v_k)$.*

Example 1 Two $\beta(0,1)$-trees are shown in Figures 2a and 2b.

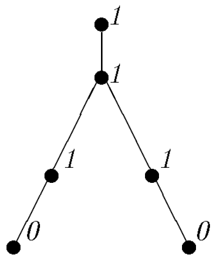

Figure 2a

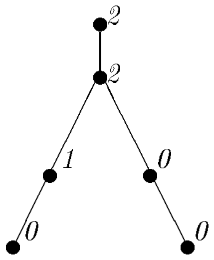

Figure 2b

Denote $D_n^{\beta(0,1)}$ the set of all $\beta(0,1)$-trees on n vertices. A *branch* of a rooted tree is a tree whose top is one of the root's children. Some rooted trees may have only one branch, which doesn't necessarily mean they consist of a single path.

We start by treating two special types of $\beta(0,1)$-trees on n vertices. First we set up a bijection f from the set of all 1342-avoiding n-permutations starting with the entry 1 and the set of $\beta(0,1)$-trees on n vertices consisting of one single path. Note that the former is the set of 231-avoiding permutations of the set $\{2,3,4\cdots,n\}$ which have been prepended with the entry 1.

So let $p=(p_1p_2\cdots p_n)$ be an 1342-avoiding n-permutation so that $p_1=1$. Take an unlabeled tree on n nodes consisting of a single path and give the label $l(i)$ to its ith node $(1\leq i\leq n-1)$ by the following rule:

$l(i)=\#\{j\leq i \text{ so that } p_j>p_s \text{ for at least one } s>i\,\}$.

Finally, let $l(n)=l(n-1)$. In words, $l(i)$ is the number of entries on the left of p_i (inclusive) which are larger than at least one entry on the right of p_i. We note that this way we could define f on the set of *all* n-permutations starting with the entry 1, but in that case, f would not be a bijection. (For example, the images of 1342 and 1432 would be identical).

Example 2 If $p=14325$, then the labels of the nodes of $f(p)$ are, from the leaf to the root, 0,1,2,0,0.

Lemma 1 *f is a bijection from the set of of all $\beta(0,1)$-trees on n vertices consisting of one single path to the set of 1342-avoiding n-permutations starting with the entry 1.*

Proof: It is easy to see that f indeed maps into the set of $\beta(0,1)$-trees : $l(i+1)\leq l(i)+1$ for all i because there can be at most one entry counted by $l(i+1)$ and not counted by $l(i)$, namely the entry p_{i+1}. All labels are certainly nonnegative and $l(1)=0$.

To prove that f is a bijection, we show that it has an inverse, that is, for any $\beta(0,1)$-tree T consisting of a single path, we can find a unique permutation p such that $f(p) = T$. We claim that given T, we can recover the entry n of the preimage p. First note that p was 1342-avoiding and started by 1, so any entry on the left of n must have been smaller than any entry on the right of n. In particular, the node preceding n must have label 0. Moreover, n is the leftmost entry p_i of p so that $p_j > 0$ for all $j \geq i$ if there is such an entry at all, and $n = p_n$ if there is none. That is, n corresponds to the node which starts the uninterrupted sequence of strictly positive labels which ends in the last node, if there is such a sequence, and corresponds to the last node otherwise. To see this, note that n is the largest of all entries, so in particular it is always larger than at least one entry in any nonempty set of entries.

Once we located where n was in p, we can simply delete the node corresponding to it from T and decrement all labels after it by 1. (If this means deleting the last node, we just change $l(n-1)$ $l(n-2)$ to satisfy the root-condition). We can indeed do this because the node preceding n had label 0, and the node after n had a positive label, by our algorithm to locate n. Then we can proceed recursively, by finding the position of the entries $n-1, n-2, \cdots, 1$ in p. This clearly defines the inverse of f, so f is a bijection. $\diamond$

Lemma 2 *The number of $\beta(0,1)$-trees on n vertices with all labels equal to zero is C_{n-1}.*

Proof: These $\beta(0,1)$-trees are in fact unlabeled plane trees. It is well known [14] that the number of such trees is C_{n-1}. $\diamond$

Example 3 The permutation 341256 corresponds to the $\beta(0,1)$-tree with all labels equal to 0 shown in Figure 3.

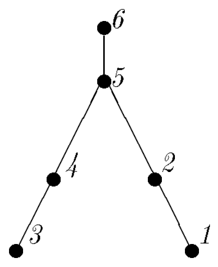

Figure 3

An easy way to read off the corresponding permutation once we have its entries written to the corresponding nodes is the well-known *preorder* reading: for every node, first write down the entries associated with its children from left to right, then the entry associated with the node itself, and do this recursively for all the children of the node.

Note that such a $\beta(0,1)$-tree has only one branch if and only if the next-to-last element of the indecomposable 132-avoiding n-permutation corresponding to it is $n-1$.

An entry of a permutation which is smaller than all the entries preceding it is called a *left-to-right minimum.*

Definition 2 *Two $n-$permutations x and y are said to be in the same* class *if the left-to-right minima of x are the same as those of y and they are in the same positions.*

Example 4 34125 and 35124 are in the same class since their left-to-right minima are 3 and 1, and they are located at the same positions. 3142 and 3412 are not in the same class.

Proposition 1 *Each nonempty class C of n-permutations contains exactly one 132-avoiding permutation.*

Definition 3 *The normalization $N(p)$ of an n-permutation p is the only 132-avoiding permutation in the class C containing p.*

Definition 4 *A permutation is called* indecomposable *if it cannot be cut into two parts so that any entry before the cut is larger than any entry after the cut. Otherwise, we say that the permutation is* decomposable

Proposition 2 *A permutation p is indecomposable if and only $N(p)$ is indecomposable.*

Corollary 1 *If p is an indecomposable n-permutation, then $N(p)$ always ends with the entry n.*

Proof: Note that the only way for a 132-avoiding n-permutation to be indecomposable is for it to end with n. If p is a 132-avoiding n permutation and n is not the last entry, then we may cut it just after the entry n. Then the statement follows from Proposition 2. $\diamond$

Now we are in a position to prove our theorem about the number of indecomposable 1342-avoiding permutations.

Theorem 1 *The number of indecomposable n-permutations which avoid the pattern 1342 is*

$$I_n(1342) = t_n = 3 \cdot 2^{n-2} \cdot \frac{(2n-2)!}{(n+1)!(n-1)!}$$

Proof: We are going to set up a bijection F between these permutations and $D_n^{\beta(0,1)}$. This will be an extension of the bijection f of lemma 1. As the size of $D_n^{\beta(0,1)}$ is known to be equal to t_n [4], this will prove our claim.

Let p be an indecomposable 1342-avoiding n-permutation. Take $N(p) = r$. By corollary 1 its last element is n. Define $F(r)$ to be the $\beta(0,1)$-tree S associated to r by the bijection of lemma 2. Now write the entries of p to the nodes of S so that for all i, the p_i is written to the node where r_i was written in S. In particular, the left-to-right minima remain unchanged. Figure 4 shows how we associate the entries of the permutation 361542 to the nodes of S, which is the image of $r = N(p) = 341256$.

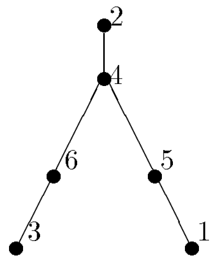

Figure 4

Now we define the label of each node for this new $\beta(0,1)$-tree T and obtain $F(p)$ this way. (As an unlabeled tree, T will be isomorphic to S, but its labels will be different). Let i be the ith node of T in the preorder reading. So p_i is the ith entry of p, and it is therefore associated to node i, while $l(i)$ is the label of this node. We say that p_i *beats* p_j if there is an element p_h so that p_h, p_i, p_j are written in this order (so $h < i < j$) and they form a 132-pattern. Moreover, we say that p_i *reaches* p_k if there is a subsequence $p_i, p_{i+a_1}, \cdots p_{i+a_t}, p_k$ of entries so that $i < i + a_1 < i + a_2 < \cdots < i + a_t < k$ and that any entry in this subsequence beats the next one. For example, in the permutation 361542, the entry 6 beats 5 and 4, 5 beats 4 and 2, and 4 beats 2, while 6 reaches 2 (of course, each entry reaches all those elements it beats, too). Then let

$l(i) = \#\{j$ descendants of i (including i itself) so that there is at least one $k > i$ for which p_j reaches $p_k\}$,

and let $F(p)$ be the $\beta(0,1)$-tree defined by these labels. (Recall that a descendant of i is an element of the tree whose top element is i). First, it is easy to see that F indeed maps into the set of $\beta(0,1)$-trees: if v is an internal node and $v_1, v_2, \cdots, v_k$ are its children, then $l(v) \leq 1 + \sum_{i=1}^{k} l(v_k)$ because there can be at most one entry counted by $l(v)$ and not counted by any of its children's label, namely v itself. All labels are certainly nonnegative and all leaves, that is, the left-to-right minima, have label 0.

If $p = 361542$, then $F(p)$ is the $\beta(0,1)$-tree shown in Figure 5b.

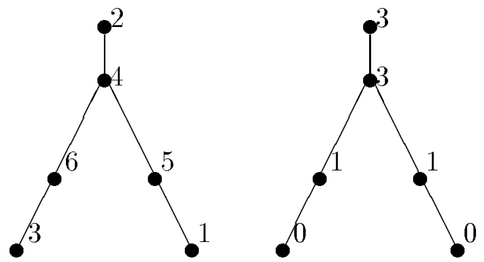

Figure 5a Figure 5b

To prove that F is a bijection, it suffices to show that it has an inverse, that is, for any $\beta(0,1)$-tree $T \in D_n^{\beta(0,1)}$, we can find a unique permutation p so that $F(p) = T$. (This proves that F is one-to-one and onto). We again claim that given T, we can recover the node j which has the entry n of the preimage p associated to it, and so we can recover the position of n in the preimage.

Proposition 3 *Suppose $p_n \neq n$, that is, n is not associated to the root vertex. Then each ancestor of n, including n itself, has a positive label. If $p_n = n$, then $l(n) = 0$ and thus there is no vertex with the above property.*

Proof: If $p_n = n$, then nothing beats it, thus $l(n) = 0$. Suppose p_n is not the root vertex.

To prove our claim it is enough to show that for any node i which is an an ancestor of j, there is an entry p_k so that p_k is an ancestor of p_i and $n = p_j$ reaches k. Indeed, this would imply that the entry $p_j = n$ is counted by the label $l(i)$ of i. Now let $a_m = p_1 > a_2 > \cdots a_1 = 1$ be the left-to-right minima of p so that n is located between a_r and a_{r+1}. Then n certainly beats all elements located between a_r and a_{r+1} as a_r can play the role of 1 in the 132-pattern. Clearly, n must beat at least one entry y_1 on the right of a_{r+1} as well, otherwise p would be decomposable by cutting it right before a_{r+1}. If y_1 is on the right of i, then we are done. If not, then y_1 must beat at least one entry y_2 which is on the other side of a_{r_1+1}, where y is located between a_{r_1} and a_{r_1+1} for the same reason, and so on. This way we get a subsequence $y_1, y_2, \cdots$ so that n reaches each of the y_t, and this subsequence eventually gets to the right of i, since in each step we bypass at least one left-to-right minimum. Thus the proposition is proved. $\diamond$

Proposition 4 *Suppose $p_n \neq n$. Then n is the* leftmost *entry of p which has the property that each of its ancestors has a positive label.*

Proof: Suppose p_k and n both have this property and that p_k is on the left of n. (If there are several candidates for the role of p_k, choose the rightmost one). If p_k beats an element y on the right of n by participating in the 132-pattern $x\ p_k\ y$, then $x\ p_k\ n\ y$ is a 1342-pattern, which is a contradiction. So p_k does not beat such an element y. In other words, all elements after n are smaller than all elements before p_k. Still, p_k must reach elements on the right of n, thus it beats some element v between p_k and n. This element v in turn beats some element w on the right of n by participating in some 132-pattern $t\ v\ w$. However, this would imply that $t\ v\ n\ w$ is a 1342-pattern, a contradiction, which proves our claim. $\diamond$

Therefore, we can recover the entry n of p from T. Then we can proceed as in the proof of Lemma 1, that is, just delete n, subtract 1 from the labels of its ancestors and iterate this procedure to get p. If any time during this procedure we find that the current root is associated to the maximal entry that hasn't been associated to other vertices yet, and the tree has more than one branch at this moment, then deleting the root vertex will split the tree into smaller trees. Then we continue the same procedure on each of them. The set of the entries associated to each of these smaller trees is uniquely determined because T as an unlabeled tree determines the left-to-right minima of p. Therefore, we can always recover p in this way from T. This proves that F is a bijection.

Thus we have set up a bijection between the set of indecomposable 1342-avoiding n-permutations and $D_n^{\beta(0,1)}$. We know from [4] that $|D_n^{\beta(0,1)}| = t_n = 3 \cdot 2^{n-2} \cdot \frac{(2n-2)!}{(n+1)!(n-1)!}$ and therefore the Theorem is proved. $\diamond$

Note that in particular, F maps 132-avoiding permutations into $\beta(0,1)$-trees with all labels equal to 0 and permutations starting with the entry 1 into $\beta(0,1)$-trees consisting of a single path.

Corollary 2 $S_n(1342)$ *equals the number of plane forests on n vertices in which each component is a $\beta(0,1)$-tree.*

3 Enumerative results

Tutte [18] has obtained the numbers t_n by first computing a translate of their generating function

$$F(x) = \sum_{n=1}^{\infty} 3 \cdot 2^{n-1} \cdot \frac{(2n-2)!}{(n+1)!(n-1)!} x^n = \frac{8x^2 + 12x - 1 + (1-8x)^{3/2}}{32x}. \tag{1}$$

By theorem 1, the coefficients of this generating function are the numbers $I_n(1342)$. Therefore, the generating function of *all* 1342-avoiding permutations is given by the following theorem.

Theorem 2 *Let $s_n = S_n(1342)$ and let $H(x) = \sum_{n=0}^{\infty} s_n x^n$. Then*

$$H(x) = \sum_{i \geq 0} F^i(x) = \frac{1}{1 - F(x)} = \frac{32x}{-8x^2 + 20x + 1 - (1-8x)^{3/2}}. \tag{2}$$

Theorem 3 *For all $n \geq 0$ we have*

$$S_n(1342) = \frac{(7n^2-3n-2)}{2} \cdot (-1)^{n-1} + 3\sum_{i=2}^{n} 2^{i+1} \cdot \frac{(2i-4)!}{i!(i-2)!}\binom{n-i+2}{2}(-1)^{n-i}. \tag{3}$$

Proof: Multiply both the numerator and the denominator of $H(x)$ by $(-8x^2+20x+1)+(1-8x)^{3/2}$. After simplifying we get

$$H(x) = \frac{(1-8x)^{3/2} - 8x^2 + 20x + 1}{2(x+1)^3}. \tag{4}$$

As $(1-8x)^{3/2} = 1 - 12x + \sum_{n\geq 2} 3 \cdot 2^{n+2} x^n \frac{(2n-4)!}{n!(n-2)!}$, formula (4) implies our claim. $\diamond$

So the first few values of $S_n(1342)$ are 1,2,6,23,103,512,2740,15485,91245,555662. One sees easily that the expression on the right hand side of (3) is dominated by the last summand; in fact, the alternation in sign assures that this last summand is larger than the whole right hand side if $n \geq 8$. As $\frac{(2n-4)!}{n!(n-2)!} < \frac{8^{n-2}}{n^{2.5}}$ by Stirling's formula, we have proved the following exponential upper bound for $S_n(1342)$.

Corollary 3 *For all n, we have $S_n(1342) < 8^n$.*

It is routine to check that for $n \geq 3$ the numbers $I_n = t_n$ satisfy the recurrence $t_n = \frac{8n-12}{n+1} \cdot t_{n-1}$. Therefore, $\sqrt[n]{t_n} \rightarrow 8$. Using this formula we can disprove a conjecture of Stanley claiming that for all permutation patterns q of length k, the sequence $\sqrt[n]{S_n(q)}$ converges to $(k-1)^2$.

Theorem 4 $\sqrt[n]{s_n} = \sqrt[n]{S_n(1342)} \rightarrow 8$ *when $n \rightarrow \infty$.*

In particular, $\lim_{n\rightarrow\infty} \frac{S_n(1342)}{S_n(1234)} = 0$, for $\lim_{n\rightarrow\infty} \sqrt[n]{S_n(1234)} = 9$ [7] [10]. This implies that $S_n(1342) < S_n(1234)$ if n is large enough. However, using the formulae of Theorem 3 and [7], we can easily show that this is true for *all* $n \geq 6$. (This has been shown by a long argument in [1]).

Formula (2) enables us to prove that the sequence $S_n(1342)$ is p-recursive in n, solving an instance of the conjecture of Zeilberger and Noonan mentioned in the Introduction. Indeed, $H(x)$ is rational in x and $\sqrt{1-8x}$, so in particular, it is certainly algebraic and so it is D-finite. Therefore $S_n(1342)$ is p-recursive as claimed.

References

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