Higher Specht Polynomials for Complex Reflection Group $G(r,p,n)$

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Abstract

A basis of the quotient ring P/J_+ is given, where P is the ring of polynomials and J_+ is the ideal generated by the fundamental invariants under the action of the complex reflection group $G(r,p,n)$.

Résumé

On donne une base de l'anneau quotient P/J_+, où P est un anneau de polynômes et J_+ est l'idéal engendré par les invariants fondamentaux sous l'action du groupe des réflexions complexes $G(r,p,n)$.

1 Introduction

This note concerns with a certain graded module over the imprimitive complex reflection group $G(r,p,n)$ [ST]. The group $G(r,p,n)(r,p,n \geq 1,\ p|r)$ consists of the monomial matrices of size n with non-zero entries $\zeta^j (0 \leq j < r)$ such that the d-th power of the product of all non-zero entries is equal to 1, where we denote by ζ a primitive r-th root of 1, and $d = r/p$. In some special cases, $G(r,p,n)$ is isomorphic to the Weyl group:

$$\begin{aligned} G(1,1,n) &= W(A_{n-1})\ , \\ G(2,1,n) &= W(B_n) = W(C_n)\ , \\ G(2,2,n) &= W(D_n)\ , \\ G(6,6,2) &= W(G_2)\ . \end{aligned}$$

Also it is naturally identified as a normal subgroup of the wreath product $G(r,n) = (\mathbf{Z}/r\mathbf{Z})^n \wr \Sigma_n = \{(\zeta^{i_1}, \ldots, \zeta^{i_n}; \sigma) \mid i_k \in \mathbf{N},\ \sigma \in \Sigma_n\}$, whose product is given by

$$(\zeta^{i_1}, \ldots, \zeta^{i_n};\ \sigma)(\zeta^{j_1}, \ldots, \zeta^{j_n};\ \tau) = (\zeta^{i_1 + j_{\sigma^{-1}(1)}}, \ldots, \zeta^{i_n + j_{\sigma^{-1}(n)}};\ \sigma\tau)\ .$$

Let $P = \mathbf{C}[x_1, \cdots, x_n]$ on which the group $G(r,n)$ acts as follows :

$$((\zeta^{i_1}, \cdots, \zeta^{i_n};\sigma)f)(x_1, \ldots, x_n) = f(\zeta^{i_{\sigma(1)}}x_{\sigma(1)}, \cdots, \zeta^{i_{\sigma(n)}}x_{\sigma(n)}),$$

for $(\zeta^{i_1}, \cdots, \zeta^{i_n};\sigma) \in G(r,n)$. It is known that the fundamental invariants under this action are given by the elementary symmetric functions $e_j(x_1^r, \cdots, x_n^r)$, $1 \leq j \leq n$. Let J'_+ be the ideal of P generated by these fundamental invariants and $R' = P/J'_+$ be the quotient ring. It is also known that $G(r,n)$-module R' is isomorphic to the group ring $\mathbf{C}G(r,n)$, namely the left regular representation. A description of all the irreducible components of R' has known in [ATY], in terms of what we call "*higher SPecht polynomials*". (see also [TY] for the case $r = 1$.) The irreducible representation of $G(r,n)$ are parametrized by the r-tuple of Young diagrams $(\lambda^0, \ldots, \lambda^{r-1})$ with $|\lambda^o| + \cdots + |\lambda^{r-1}| = n$. In [ATY] (and [TY]) a combinatorics of Young diagrams are utilized to determine a basis for each irreducible component of R'.

Now we consider the restriction of the above action of $G(r,n)$ on P to the subgroup $G(r,p,n)$. The fundamental invariants are $e_j(x_1^r, \ldots, x_n^r)$ $(1 \leq j \leq n-1)$ and $e_j(x_1^d, \ldots, x_n^d)$. Denote by J_+ the ideal generated by these polynomials and let $R = P/J_+$. The representation of $G(r,p,n)$ on R is again isomorphic to the left regular representation. Our problem is to describe the irreducible components of R as well as their bases. The key for our description is the Clifford theory [S] for a finite group G and its normal subgroup H.

2 Higher Specht polynomials for $G(r,n)$

Here we recall the results of [ATY] on irreducible decomposition of the graded $G(r,n)$-module

$$R' = P/J'_+ \ ,$$

where $P = \mathbf{C}[x_1, \cdots x_n]$ and $J'_+ = ((e_1(x_1^r, \ldots, x_n^r), \ldots, e_n(x_1^r, \ldots, x_n^r))$. As is well known the irreducible representations of $G(r,n)$ are parametrized by the set $\mathcal{P}_{r,n}$ of r-tuples of Young diagrams $\lambda = (\lambda^0, \ldots \lambda^{r-1})$ with $|\lambda^0| + \cdots + |\lambda^{r-1}| = n$. By filling each cell with a positive integer in such a way that every j $(1 \leq j \leq n)$ occurs once, we obtain an *r-tableau* $T = (T^0, \ldots, T^{r-1})$ of shape $\lambda = (\lambda^0, \ldots, \lambda^{r-1})$. When the number k occurs in the component T^ν, we may write $k \in T^\nu$. The set of r-tableaux of shape λ is denoted by $\mathrm{Tab}(\lambda)$. An r-tableau $T = (T^0, \ldots, T^{r-1})$ is said to be *standard* if the numbers are increasing on each column and each row of T^ν $(0 \leq \nu < r)$. The set of standard r-tableaux of shape λ is denoted by $\mathrm{STab}(\lambda)$.

Let $S = (S^0, \ldots, S^{r-1}) \in \mathrm{STab}(\lambda)$. We associate a word $w(S)$ in the following way. First we read each column of the component S^0 from the bottom to the top starting from the left. We continue this procedure for the tableau S^1 and so on. For the word $w(S)$ we define index $i(w(S))$ inductively as follows. The number 1 in the word $w(S)$ has the index $i(1) = 0$. If the number k has index $i(k) = p$ and the number $k+1$ is sitting on the left (resp. right) of k, then $k+1$ has index $p+1$ (resp. p). Finally, assigning the index to the corresponding cell, we get a shape $\lambda = (\lambda^0, \ldots, \lambda^{r-1})$, each cell filled with a nonnegative integer, which is denoted by $i(S) = (i(S)^0, \ldots, i(S)^{r-1})$. Let $T = (T^0, \ldots, T^{r-1})$ be an r-tableau of shape λ. For each component T^ν $(0 \le \nu < r)$, the Young symmetrizer e_{T^ν} of T^ν is defined by

$$e_{T^\nu} = \frac{1}{\alpha_{T^\nu}} \sum_{\sigma \in R(T^\nu) \tau \in C(T^\nu)} \mathrm{sgn}(\tau)\tau\sigma \ ,$$

where α_{T^ν} is the product of the hook lengths for the shape λ^ν. For $S \in \mathrm{STab}(\lambda)$ and $T \in \mathrm{Tab}(\lambda)$, define the higher Specht polynomial $\Delta_{S,T}(x)$ by

$$\Delta_{S,T}(x) = \prod_{\nu=0}^{r-1} \left\{ e_{T^\nu}(x_{T^\nu}^{ri(S)^\nu}) \prod_{k \in T^\nu} x_k^\nu \right\} \ ,$$

where $x_{T^\nu}^{ri(S)^\nu} = \prod_{k \in T^\nu} x_k^{ri(k)}$.

The following is a fundamental result of [ATY] on the higher Specht polynomials for $G(r, n)$.

Theorem 1

1. *The space* $V_S(\lambda) = \sum_{T \in \mathrm{Stab}(\lambda)} \mathbf{C}\Delta_{S,T}(x)$ *affords an irreducible representation of the complex reflection group* $G(r, n)$*.*
2. *For* $S_1 \in \mathrm{STab}(\lambda)$ *and* $S_2 \in \mathrm{STab}(\mu)$*, the representations* $V_{S_1}(\lambda)$ *and* $V_{S_2}(\mu)$ *are isomorphic if and only if* S_1 *and* S_2 *has the same shape,* i.e., $\lambda = \mu$*.*
3. *The set* $\{\Delta_{S,T}(x) \mid T \in \mathrm{STab}(\lambda)\}$ *gives a basis for* $V_S(\lambda)$*.*
4. *The irreducible decomposition*

$$R' = \bigoplus_{\lambda \in \mathcal{P}_{r,n}} \bigoplus_{S \in \mathrm{STab}(\lambda)} V_S(\lambda)$$

holds as representations of $G(r, n)$

3 Review of the Clifford theory

We briefly review the Clifford theory following [Stem]. Let H be a normal subgroup of a finite group G such that the quotient group G/H is cyclic. We have in mind the case $G = G(r, n)$ and $H = G(r, p, n)$. Let C denote the group of 1-dimensional representations, or characters, $(\delta, \mathbf{C}_\delta)$ of G such that $H \subset \mathrm{Ker}\ \delta$. In other words, C is the group of the characters of G/H, which is isomorphic to G/H. Two irreducible representations (ϕ, V) and (ψ, W) of G are said to be *associates* if there exists $\delta \in C$ such that $\psi = \delta \otimes \phi$. For a fixed irreducible representation (ϕ, V) of G, let

$$C_\phi = \{\delta \in C \mid \phi = \delta \otimes \phi\}$$

be the stabilizer of ϕ and let $(\delta, \mathbf{C}_\delta)$ be a generator of C_ϕ. There exists a G-module isomorphism $A : V \longrightarrow \mathbf{C}_\delta \otimes V$. If $|C_\phi| = e$, then A^e commutes with G and, by Schur's lemma, A^e is a nonzero scalar. By normalizing the constant, we assume that $A^e = \mathbf{1}_V$

and call such A the *associator* of (ϕ, V). Choose an associator A for (ϕ, V) and let $V = \oplus_{j=0}^{e-1} E^{(\ell)}$ denote the eigenspace decomposition of A, where $E^{(\ell)}$ is the eigenspace with the eigenvalue $e^{\frac{-2\pi i \ell}{e}}$. Since $H \subset \mathrm{Ker}\delta$, each $E^{(\ell)}$ is an H-module. Moreover the $E^{(\ell)}$'s are inequivalent irreducible H-modules of the same dimension $\dim V/e$. The Frobenius reciprocity tells us that $\mathrm{Ind}_H^G E^{(\ell)}$ is the multiplicity free direct sum of all the associates of (ϕ, V).

From these results, we can conclude that the irreducible representations of H are parametrized by the pairs $(\mathcal{O}, \varepsilon)$ consisting of a C-orbit $\mathcal{O}$ through an irreducible representation of G and a character $\varepsilon \in C$ that stabilizes $\mathcal{O}$.

4 Higher Specht polynomials for $G(r, p, n)$

We now apply the Clifford theory to the case $G = G(r, n)$ and $H = G(r, p, n)$. Define the linear character δ of $G(r, n)$ by $\delta(\zeta^{i_1}, \ldots, \zeta^{i_n}; \sigma) = \zeta^{i_1 + \cdots i_n}$ so that our cyclic group is $C =< \delta^d > \cong \mathbf{Z}/p\mathbf{Z}$. Define the *shift operator* sh on $\mathcal{P}_{r,n}$ by

$$\mathrm{sh}(\lambda^0, \ldots, \lambda^{r-1}) = (\lambda^{r-1}, \lambda^0, \ldots, \lambda^{r-2}) \, .$$

By the realization of the irreducible representations of $G(r, n)$ described in Section 1, we see that $\mathbf{C}_\delta \otimes V_S(\lambda) \cong V_S(\mathrm{sh}(\lambda))$ for any $S \in \mathrm{STab}(\lambda), \lambda \in \mathcal{P}_{r,n}$. Hence the C-orbits are characterized by $\mathcal{P}_{r,n}/\sim$, where we denote $\lambda \sim \mu$ if $\mu = \mathrm{sh}^{dj}\lambda$ for some $j = 0, 1, \ldots, p-1$. For convenience we will denote $\mathrm{Sh} = \mathrm{sh}^d$. For $\lambda \in \mathcal{P}_{r,n}$, let $b(\lambda)$ be the minimal j such that $\mathrm{Sh}^j\lambda = \lambda$, i.e., $b(\lambda) = |\{\mu \in \mathcal{P}_{r,n} \mid \lambda \sim \mu\}|$ and put $e(\lambda) = p/b(\lambda)$. The stabilizer C_λ of the C-orbit through λ is a subgroup of C generated by $\delta^{b(\lambda)d}$, so that $|C_\lambda| = e(\lambda)$ and $|C/C_\lambda| = b(\lambda)$. The corresponding associator is denoted by A_λ. In other words the associator A_λ is realized on $V_S(\lambda)$ by

$$A_\lambda \left(\Delta_{S,T}(x)\right) = \Delta_{S,\mathrm{Sh}^{b(\lambda)}(T)}(x) \qquad (T \in \mathrm{Tab}(\lambda)) \, .$$

For $h = 1, 2, \ldots, r$, let

$$\mathrm{STab}(\lambda)_h = \{T = (T^0, \ldots, T^{r-1}) \in \mathrm{STab}(\lambda) \mid 1 \in T^\nu, \ 0 \le \nu < h\}.$$

Note that, if $T \in \mathrm{STab}(\lambda)_{db(\lambda)}$, then the standard r-tableaux

$$T, (\mathrm{Sh})^{b(\lambda)}T, (\mathrm{Sh})^{2b(\lambda)}T, \ldots, (\mathrm{Sh})^{(e(\lambda)-1)b(\lambda)}T$$

are all distinct. Let $\lambda = (\lambda^0, \ldots, \lambda^{r-1})$ be an element of $\mathcal{P}_{r,n}$. Fix $S \in \mathrm{STab}(\lambda)$ and $\ell = 0, 1, \ldots, e(\lambda) - 1$. For each $T \in \mathrm{STab}(\lambda)$, we define a polynomial

$$\Delta_{S,T}^{(\ell)}(x) := \sum_{k=0}^{e(\lambda)-1} \zeta^{\ell k d b(\lambda)} \Delta_{S,\mathrm{Sh}^{kb(\lambda)}T}(x) \, ,$$

as an element of $R' = P/J'_+$. Since $\Delta_{S,T_1}^{(\ell)}(x)$ coinsides with $\Delta_{S,T_2}^{(\ell)}(x)$ if $T_1 \sim T_2$ in $\mathrm{STab}(\lambda)$, we only have to consider the polynomial associated with $T \in \mathrm{STab}(\lambda)_{db(\lambda)}$.

Let $\mathcal{D}_S^{(T)}$ $(S, T \in \mathrm{STab}(\lambda))$ denote the set $\{\Delta_{S,\mathrm{Sh}^{kb(\lambda)}T}(x) \mid k = 0, \ldots, e(\lambda)-1\}$. Then we have a disjoint union

$$\{\Delta_{S,T} \mid T \in \mathrm{STab}(\lambda)\} = \coprod_{T \in \mathrm{STab}(\lambda)_{db(\lambda)}} \mathcal{D}_S(T) ,$$

for each $S \in \mathrm{STab}(\lambda)$. Since $\{\Delta_{S,T}(x) \mid T \in \mathrm{STab}(\lambda)\}$ are linearly independent over $\mathbf{C}$, the polynomials $\{\Delta_{S,T}^{(\ell)} \mid T \in \mathrm{STab}(\lambda)_{db(\lambda)}\}$ are also linearly independent for each ℓ and S.

Lemma 2 *Let S and T be standard r-tableaux of shape λ and $\ell = 0, 1, \ldots e(\lambda)-1$. Then the polynomial $\Delta_{S,T}^{(\ell)}(x)$ does not vanish in $R = P/J_+$ if and only if $S \in \mathrm{STab}(\lambda)_d$.*

We now have a family of polynomials

$$\{\Delta_{S,T}^{(\ell)}(x) \mid S \in \mathrm{STab}(\lambda)_d,\ T \in \mathrm{STab}(\lambda)_{db(\lambda)},\ \ell = 0, 1, \ldots, e(\lambda)-1\} ,$$

as elements of $R = P/J_+$. We call these polynomials the *higher Specht polynomials* for the complex reflection group $G(r, p, n)$.

Theorem 3

1. *The space $V_S^{(\ell)}(\lambda) = \sum_{T \in \mathrm{STab}(\lambda)} \mathbf{C}\Delta_{S,T}^{(\ell)}(x)$ affords an irreducible representation of the complex reflection group $G(r, p, n)$.*

2. *The set $\{\Delta_{S,T}^{(\ell)}(x) \mid T \in \mathrm{STab}(\lambda)_{db(\lambda)}\}$ gives a basis for $V_S^{(\ell)}(\lambda)$.*

3. *The left regular representation $R = P/J_+$ of $G(r, p, n)$ has an irreducible decomposition*

$$R = \bigoplus_{\lambda \in \mathcal{P}_{r,n}} \bigoplus_{S \in \mathrm{STab}(\lambda)_d} \bigoplus_{\ell = o}^{e(\lambda)-1} V_S^{(\ell)}(\lambda).$$

Finally we give a "hook length formula"for the Poincaré series following [S]. Let $\lambda = (\lambda^0, \ldots, \lambda^{r-1}) \in \mathcal{P}_{r,n}$ and set

$$A_j(\lambda) = \sum_{\nu=0}^{r-1} \nu |((\mathrm{Sh})^j \lambda)^\nu| ,$$

where $((\mathrm{Sh})^j\lambda)^\nu$ denotes the ν-th component of the shape $(\mathrm{Sh})^j\lambda$. Then the Poincaré series is given by

$$M_\lambda(q) = \frac{q^{r(\sum_{\nu=0}^{r-1} n(\lambda^\nu))}(1-q^{dn}) \prod_{k=1}^{n-1}(1-q^{rk})}{\prod_{\nu=0}^{r-1} \prod_{x \in \lambda^\nu}(1-q^{rh(x)})} \sum_{j=0}^{b(\lambda)-1} q^{A_j(\lambda)} ,$$

where $h(x)$ is the hook length of the cell x.

5 Examples

In this section, we give some examples of higher Specht polynomials. First we consider $G(2,1,4) = W(B_4)$. Let $\lambda = \left(\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} , \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \right)$, $T_1 = \left(\begin{array}{c} 1 \\ 2 \end{array} , \begin{array}{c} 3 \\ 4 \end{array} \right)$, $T_2 = \mathrm{sh}(T_1) = \left(\begin{array}{c} 3 \\ 4 \end{array} , \begin{array}{c} 1 \\ 2 \end{array} \right)$ and $S = \left(\begin{array}{c} 1 \\ 4 \end{array} , \begin{array}{c} 2 \\ 3 \end{array} \right)$, so that $i(S) = \left(\begin{array}{c} 0 \\ 2 \end{array} , \begin{array}{c} 0 \\ 1 \end{array} \right)$. The higher Specht polynomials associated with (S, T_1) and (S, T_2) are, respectively,

$$\begin{aligned} \Delta_{S,T_1}(x) &= \{\frac{1}{2}(id - s_1)x_2^4\}\{\frac{1}{2}(id - s_3)x_4^2\}x_3x_4 \\ &= \frac{1}{4}(x_2^4 - x_1^4)(x_4^2 - x_3^2)x_3x_4 , \end{aligned}$$

$$\begin{aligned} \Delta_{S,T_2}(x) &= \{\frac{1}{2}(id - s_3)x_4^4\}\{\frac{1}{2}(id - s_1)x_2^2\}x_1x_2 \\ &= \frac{1}{4}(x_2^2 - x_1^2)(x_4^4 - x_3^4)x_1x_2 . \end{aligned}$$

Here $s_1 = (12)$ and $s_3 = (34)$ are transpositions and 1 stands for the identity. Next consider the case $G(2,2,4) = W(D_4)$, where $d = 1$. For the above λ, we see that $b(\lambda) = 1$ and $e(\lambda) = 2$. Therefore the 6-dimensional representation $V_S(\lambda)$ of $G(2,1,4)$ decomposes into 2 irreducible components $V_S^{(0)}(\lambda)$ and $V_S^{(1)}(\lambda)$ under $G(2,2,4)$, each of which is 3-dimensional. Accordingly the higher Specht polynomial associated with (S, T_1) decomposes to

$$\Delta_{S,T_1}^{(0)}(x) = \Delta_{S,T_1}(x) + \Delta_{S,T_2}(x) ,$$

and

$$\Delta_{S,T_1}^{(1)}(x) = \Delta_{S,T_1}(x) - \Delta_{S,T_2}(x).$$

If we take $S_1 = \left(\begin{array}{c} 2 \\ 3 \end{array} , \begin{array}{c} 1 \\ 4 \end{array} \right)$ so that $i(S_1) = \left(\begin{array}{c} 1 \\ 2 \end{array} , \begin{array}{c} 0 \\ 2 \end{array} \right)$, then

$$\begin{aligned} \Delta_{S_1,T_1}(x) &= \{\frac{1}{2}(id - s_1)x_1^2x_2^4\}\{\frac{1}{2}(id - s_3)x_4^4\}x_3x_4 \\ &= \frac{1}{4}(x_1^2x_2^4 - x_1^4x_2^2)(x_4^4 - x_3^4)x_3x_4 \\ &= \frac{1}{4}(x_1x_3^3 - x_1^3x_2)(x_4^4 - x_3^4)x_1x_2x_3x_4 , \end{aligned}$$

which does not survive in R.

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