Belyi Functions for Archimedean Solids

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> Without doubt the authentic type of these figures exists in the mind of God the Creator and shares His eternity.
>
> J. Kepler [13] (cited in [6])

Abstract

The notion of a Belyi function is a main technical tool which relates the combinatorics of maps (i.e., graphs embedded into surfaces) with Galois theory, algebraic number theory, and the theory of Riemann surfaces. In this paper we compute Belyi functions for a class of semi-regular maps which correspond to the so-called Archimedean solids.

Résumé

La notion de fonction de Belyi est un outil technique qui relie la combinatoire des cartes (c'est-à-dire, des graphes plongés sur des surfaces) avec la théorie de Galois, la théorie des nombres algébriques et la théorie des surfaces de Riemann. Dans cet article nous calculons les fonctions de Belyi pour une classe des cartes semi-regulières, correspondant à ce qu'on appelle les solides d'Archimède.

1 Introduction

The title of the present paper attempts to link together traditional and contemporary mathematics. The name of Archimedes represents tradition; that of Belyi, one of the most recent advances in Galois theory, known (even in the English literature) under the French name of the theory of "Dessins d'Enfants" (which means "Children's drawings"), see for example [22]. The theory of "Dessins d'Enfants" was called so by its founder Alexandre Grothendieck [10]. The core notion of the theory is that of a *Belyi function*. In the planar case (which is the only case we consider here) a Belyi function is a rational function defined on the Riemann complex sphere which has at most three critical values, namely, 0, 1 and ∞. A pre-image of the segment $[1, \infty]$ under such a function is a planar (hyper)map. The coefficients of such a function are algebraic numbers. This leads to

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an action of the absolute Galois group on planar maps. See details in Section 2, and for a more extended presentation see [22].

An explicit computation of a Belyi function corresponding to a given map is reduced to a solution of a system of algebraic equations. It may turn out to be extremely difficult. To give an idea of the level of difficulty, we mention that our attempts to compute Belyi functions for some maps with only six edges took us several months, and the result was achieved only after using some advanced Gröbner bases software [8] and numerous consultations given by its author (this story will be related elsewhere [19]). The situation changes dramatically when the map in question possesses rich symmetries. Thus, for Platonic solids the Belyi functions (under other name) were computed by Felix Klein as early as in 1875 (see, for example, [14], Sections 10–14 of Chapter II, and also [5]). Our aim is to continue this study and to compute Belyi functions for a class of semi-regular maps. There are many possible candidates for such a class; for this paper we have chosen the class that is often called Archimedean solids; see details in Section 3. Our biggest "accomplishment" (if we don't take into account the infinite series of prisms and antiprisms) is a map of 180 edges.

Of course, we were attracted by the subject not only because we wished to "beat the records", not even because of its geometric beauty. We hope to extract from our computations some interesting facts concerning explicit construction of Galois groups and other related topics; see in this respect [15], [16], which examine only one of our figures, namely, the truncated icosahedron. The connection of regular and semi-regular maps to Galois theory is after all not so surprising, if one recalls the important role played in this theory by the icosahedron [14].

2 Belyi functions

For more details concerning Belyi functions see [24], [5] and [23].

Let $\overline{\mathbb{C}}$ denote the Riemann complex sphere, and let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function. For almost all $w \in \overline{\mathbb{C}}$ the equation $f(z) = w$ has the same number n of solutions; n is called the degree of f ($\deg f = n$). Those values of $w \in \overline{\mathbb{C}}$ for which the equation has less than n solutions (that is, some of the solutions are multiple) are called *critical values*. The multiple roots $z \in \overline{\mathbb{C}}$ of the equation $f(z) = w$ are called *critical points*.

Definition 2.1 (Belyi function) A rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called a *Belyi function* if all its critical values are in $\{0, 1, \infty\}$.

Definition 2.2 (clean Belyi function) A Belyi function f is called *clean* if all the points z such that $f(z) = 1$ are critical points of degree 2.

If f is a clean Belyi function, the set $f^{-1}([1, \infty])$ is a *planar map*; for an arbitrary Belyi function it is a *planar hypermap* (for the notion of a hypermap see [4]). This means that it is a connected graph drawn on the Riemann sphere of the variable z, and the complement of this graph is a disjoint union of sets homeomorphic to the open disk; these sets are called *faces* of the map. The following additional information is useful for understanding the relation between the Belyi function and the map:

- The vertices of the map are the points of $f^{-1}(\infty)$, the vertex degree being equal to the multiplicity of the corresponding root of the equation $f(z) = \infty$.
- Inside each face there is a unique root of the equation $f(z) = 0$, its multiplicity being equal to the face degree. We call this point the *center* of the face.
- The "open" edges of the map are disjoint intervals of the set $f^{-1}([1, \infty[)$. Inside each edge there is a root of the equation $f(z) = 1$. We call this point the *middle point* of the edge. In the case of a map this root always has multiplicity 2; in the case of a hypermap its multiplicity may be arbitrary.
- Besides the points mentioned above, there are no other critical points of f.
- If we join the center of each face to the vertices and middle points adjacent to this face, we obtain a *canonical triangulation* of the (hyper)map. We call a triangle *positive* (resp. *negative*), if its vertices taken in the counter clockwise direction are labelled as $0, 1, \infty$ (resp. $0, \infty, 1$). The image under the Belyi function of all positive triangles is the upper half-plane, and for the negative ones, the lower half-plane.

Theorem 2.3 (Riemann's existence theorem) *For any hypermap the corresponding Belyi function exists and is unique up to a linear fractional transformation of the variable z.*

Remark 2.4 The above theorem is true not only for "polyhedral" maps but also for arbitrary maps: loops and multiple edges are allowed, as well as vertices of degree 1 and 2. The only condition is that the corresponding graph must be connected.

The same map may be drawn on the plane in infinitely many ways. But one of its geometric forms is distinguished: it is the form given as the preimage of the segment $[1, \infty]$ via the Belyi function corresponding to the map. We will call this particular drawing of the map its *dessin d'enfant.*

Example 2.5 The following is the Belyi function for the dodecahedron (see Figure 1):

$$f_{\text{dodeca}}(z) = 1728 \, \frac{\left(z^{10} - 11\, z^5 - 1\right)^5 z^5}{\left(z^{20} + 228\, z^{15} + 494\, z^{10} - 228\, z^5 + 1\right)^3}$$

The roots of the polynomial $C(z) = z^{20} + 228\, z^{15} + 494\, z^{10} - 228\, z^5 + 1$ in the denominator are the coordinates of the 20 vertices of the dodecahedron; degree 3 in the denominator $C(z)^3$ reflects the fact that all the vertices are of degree 3. The roots of the polynomial $A(z) = (z^{10} - 11\, z^5 - 1)z$ in the numerator are the centers of the faces, all of them of degree 5. We "see" here only 11 faces; the 12th one is situated at ∞. Its degree is also 5, because the total degree of the numerator is 55, and that of the denominator is 60.

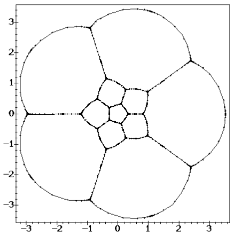

Figure 1: Dessin d'enfant of a dodecahedron

Finally, if we compute $f_{\rm dodeca}(z)-1$, we find

$$f_{\rm dodeca}(z)-1=-\frac{(z^{10}+1)^2\,(z^{20}-522\,z^{15}-10006\,z^{10}+522\,z^5+1)^2}{(z^{20}+228\,z^{15}+494\,z^{10}-228\,z^5+1)^3}$$

Thus, all the 30 roots of the equation $f_{\rm dodeca}(z)=1$ are of multiplicity 2, and they are situated at the roots of the polynomial $B(z)=(z^{10}+1)(z^{20}-522\,z^{15}-10006\,z^{10}+522\,z^5+1)$. These are the edge middle points.

For completness one must also verify that there are no other critical points of $f_{\rm dodeca}$, that is, that $f'_{\rm dodeca}(z)=0$ implies $f_{\rm dodeca}(z)\in\{0,1,\infty\}$.

Belyi functions for all Platonic solids were computed in [5]. The reader may find some hints on the methods of their computation in Section 4.

Tetrahedron (Dessin d'enfant: Figure 2)

$$f_{\rm tetra}(z)=-64\frac{(z^3+1)^3}{(z^3-8)^3\,z^3},\qquad f_{\rm tetra}(z)-1=-\frac{(z^6+20\,z^3-8)^2}{(z^3-8)^3\,z^3}.$$

Cube (Dessin d'enfant: Figure 3)

$$f_{\rm cube}(z)=-108\frac{(z^4+1)^4\,z^4}{(z^8-14\,z^4+1)^3},\qquad f_{\rm cube}(z)-1=-\frac{(z^{12}+33\,z^8-33\,z^4-1)^2}{(z^8-14\,z^4+1)^3}.$$

Remark 2.6 It is obvious that if f is a Belyi function for a map, then $1/f$ is a Belyi function for the dual map. Therefore, for the octahedron and for the icosahedron

$$f_{\rm octa}(z)=\frac{1}{f_{\rm cube}(z)},\qquad f_{\rm icosa}(z)=\frac{1}{f_{\rm dodeca}(z)}.$$

However, if the center of the "outer" face of the initial map was situated at ∞, which is only natural, then for the dual map it is one of its vertices which is situated at ∞, and that is not so convenient if we want to draw the dessin d'enfant. Therefore in practice this simple transformation needs a certain "remake", which is a linear fractional transformation of the variable z. This operation may not be entirely harmless, as the coefficients of the transformation may involve some irrational algebraic numbers even if the coefficients of the original function are rational.