A Bijection Between Intervals in The Fibonacci Posets

Darla Kremer
Murray State University
Murray, Kentucky 42071
dkremer@math.mursuky.edu

April 24, 1997

Abstract

For each word w in the Fibonacci lattices $\mathrm{Fib}(r)$ and $Z(r)$ we partition the interval $[\hat{0}, w]$ in $\mathrm{Fib}(r)$ into subposets called r-Boolean posets. In the case $r = 1$ those subposets are isomorphic to Boolean algebras. We also partition the interval $[\hat{0}, w]$ in $Z(r)$ into certain spanning trees of the r-Boolean posets. A bijection between those intervals is given in which each r-Boolean poset in $\mathrm{Fib}(r)$ corresponds to a spanning tree in $Z(r)$.

Résumé

Pour tout mot w appartenant aux treillis de Fibonacci $\mathrm{Fib}(r)$ et $Z(r)$ on partitionne l'intervalle $[\hat{0}, w]$ de $\mathrm{Fib}(r)$ en sous-posets, appellés r-posets de Boole. (Dans le cas $r = 1$ ces posets sont isomorphes à des algèbres de Boole). Pareillement, on partitionne l'intervalle $[\hat{0}, w]$ de $Z(r)$ en certains arbres maximaux de r-posets de Boole. On présente une bijection entre ces deux intervalles de sorte que tout r-poset de Boole dans $\mathrm{Fib}(r)$ corresponde à un arbre maximal de $Z(r)$.

1 Introduction

In a 1988 paper [8], R. Stanley introduced a class of partially ordered sets, called differential posets, defined independently by S. Fomin [1] who called them Y-graphs. The prototypical example of a differential poset is Young's lattice, Y, the lattice of integer partitions ordered by inclusion of Ferrers diagrams. Another important example is given by the Fibonacci r-differential poset, $Z(r)$, defined for each positive integer r.

In [6] and [8] Stanley defined the partially ordered set $\mathrm{Fib}(r)$ which has the same elements as $Z(r)$, but different covering relations. He showed that: (a) for both posets, the number of pairs of saturated chains from the smallest element, $\hat{0}$, to elements, w, of height n is $r^n n!$; (b) for every element w, $\mathrm{Fib}(r)$ and $Z(r)$ afford the same number of maximal chains in the interval $[\hat{0}, w]$; and in [9] that (c) they have the same number of elements in each such interval. Stanley's proofs of these results are algebraic. Statement (b) was also shown by Fomin [1]. In [2,4], Fomin gave a constructive proof of (a). In [5] K. O'Hara and this author gave a bijective proof of (b). A constructive proof of (c) is given in this paper.

The construction of this bijection partitions the intervals in $\mathrm{Fib}(r)$ into certain subposets, which we call r-Boolean posets, related to the lattice of subsets of the set $[n] = \{1, 2 \dots n\}$. At the same time, we partition the intervals in $Z(r)$ into trees, called r-Boolean trees which, for fixed n, have the same cardinality as the r-Boolean posets. The correspondence is then defined between r-Boolean posets and r-Boolean trees. An example of this correspondence is illustrated at the end of this paper.

2 Basic Definitions

In this section, we define the lattices $\mathrm{Fib}(r)$ and $Z(r)$, and give some related notation. We also define r-Boolean posets and trees. For definitions related to the theory of partially ordered sets and lattices see [7].

Definition 1: The Fibonacci Lattice, Fib(r)**.** ([8])

As a set, $\mathrm{Fib}(r) = A(r)^*$, the set of all finite words over the alphabet $A(r) = \{1, 2, \dots r, *\}$. The covering relations for $\mathrm{Fib}(r)$ are given by:
for $u, v \epsilon \mathrm{Fib}(r)$, $u \lessdot v$, if either

(i) v is obtained from u by changing any digit to a $*$, or

(ii) v is obtained from u by appending a digit to the end of u.

Equivalently, we could define the ordering on $\mathrm{Fib}(r)$ as: For words $u = u_1 u_2 \cdots u_l$ and $v = v_1 v_2 \cdots v_k$, $u \leq v$ if and only if $l \leq k$ and for each i, $1 \leq i \leq l$, $u_i \leq v_i$ where $\emptyset < a < *$ for all digits $a \in [r]$ and distinct digits are incomparable.

Definition 2: The Fibonacci Lattice, $Z(r)$**.** ([8])

The Fibonacci r-differential lattice, $Z(r)$, consists of all finite words over the alphabet $A(r) = \{1, 2, \dots, r, *\}$. (As sets, $Z(r)$=$\mathrm{Fib}(r)$.) An element u of $Z(r)$ is covered by an element v of $Z(r)$, $u \lessdot v$, if either:

(a) v is obtained from u by changing the first digit to a $*$, or

(b) v is obtained from u by inserting a digit to the left of the first digit (if no digit exists, then insert a digit in any position).

Definition 3: For $w \in A(r)^*$, let $i(w) =$ the number of nonzero digits in w, $j(w) =$ the number of $*'s$ in w, and $l(w) = i(w) + j(w) =$ the length of w. The rank function is the same for Fib(r) and $Z(r)$, and is given by $\rho(w) = i(w) + 2j(w)$.

We will sometimes use the notation $L(r)$ when referring to properties possessed by both lattices.

Definition 4: For integers $n \geq 1$ and $r \geq 1$, we define the r-*Boolean tree,* denoted $T_{(n,r)}$ to be a rooted tree whose vertices are labelled according to the following recursive rule:

- The root is given the label $(n+1)_r$, and
- Each vertex with a label k_i (for $1 < k \leq n+1$) has $r(k-1)$ children which are given the labels $1_1, 1_2, \ldots 1_r, 2_1, 2_2, \ldots 2_r, \ldots, (k-1)_1, (k-1)_2, \ldots, (k-1)_r$.

Observe that $T_{(n,1)}$ is a spanning tree for the Boolean algebra B_n, the lattice of subsets of $[n]$ ordered by inclusion, since each path from a fixed vertex to the root has labels which define a subset of n. In what follows, we define a structure for which $T_{(n,r)}$ will be a spanning tree.

Definition 5: For integers $n \geq 1$ and $r \geq 1$, define an r-*colored* k-*subset of* $[n]$ to be a k-subset of $[n]$ in which each element of the set is assigned a color from the set $\{1, 2, \ldots, r\}$. Let j_i denote the element j with the color i. There are

$$r^k \binom{n}{k}$$

r-colored k-subsets of $[n]$. The r-*Boolean poset,* $B_{(n,r)}$ is defined as the set of all r-colored k-subsets of $[n]$ ordered by reverse inclusion. Under this ordering, $\{1_1, 3_1, 4_2\} < \{1_1, 4_2\}$, but $\{1_1, 3_1, 4_2\}$ is not comparable to $\{1_1, 4_1\}$. When $r = 1$, $B_{(n,1)}$ is the usual Boolean algebra.

Note that

$$|B_{(n,r)}| = |T_{(n,r)}| = \sum_{k=0}^{n} r^k \binom{n}{k} = (r+1)^n.$$

3 The Correspondence

Proposition 1 *Let $w = w_1 w_2 \cdots w_{l(w)}$. The interval $[\hat{0}, w]_{Fib(1)}$ can be partitioned into disjoint sublattices of the form:*

One maximal $\hat{0} \to w$ chain, $C_F(w)$, and

for each $w_t = *$, $\rho(w_{t+1} w_{t+2} \cdots w_{l(w)})$ sublattices isomorphic to the Boolean lattice $B_{j(w_1 w_2 \cdots w_{t-1})}$.

Proof: There is a unique maximal $\hat{0} \to w$ chain in $[\hat{0}, w]_{\mathrm{Fib}(1)}$ which grows from left to right, always satisfying the condition that $x_i \leq w_i$. (e.g. if $w = 11 * * * 1 * 11 * * 1$, then $C_F(w) = \emptyset \lessdot 1 \lessdot 11 \lessdot 111 \lessdot 11* \lessdot 11*1 \lessdot 11** \lessdot 11**1 \lessdot 11*** \lessdot 11***1 \lessdot 11***11 \lessdot 11***1* \lessdot 11***1*1 \lessdot 11***1*11 \lessdot 11***1*111 \lessdot 11***1*11* \lessdot 11***1*11*1 \lessdot 11***1*11** \lessdot 11***1*11**1.$)

If $w = 1^k$, then $[\hat{0}, w]_{\mathrm{Fib}(1)}$ is the chain $\emptyset \lessdot 1 \lessdot 11 \lessdot \ldots \lessdot 1^k$.

Assume $w = u * v$. Then $u1v \lessdot w$. The elements in the interval $[1^{l(u)}1v, u1v]$ form a sublattice of Fib(1) isomorphic to $B_{j(u)}$, since there are

$$\binom{j(u)}{k}$$

ways of leaving v fixed and changing k *'s in u to 1's. Observe that the top of this copy of $B_{j(u)}$ begins at rank $\rho(w) - 1$.

Now, the element $u1v' \lessdot u1v$, where v' is obtained from v by deleting the last letter if it is a 1, or by changing the last letter of v from a $*$ to a 1. As before, the elements in $[1^{l(u)}1v', u1v']$ form a sublattice of Fib(1) isomorphic to $B_{j(u)}$ whose top element is on rank $\rho(w) - 2$. Continuing in this manner, we get $\rho(v)$ copies of $B_{j(u)}$ beginning on ranks $\rho(w) - 1, \rho(w) - 2, \ldots, \rho(w) - \rho(v)$.

Because each element in the Boolean algebra beginning on rank $\rho(w) - k$ has length $l(w) - k$, the intersection of these sublattices is empty. Furthermore, if we write $w = \overline{u} * \overline{v}$ for $\overline{v} \neq v$, $\overline{u} \neq u$ then $B_{j(u)} \cap B_{j(\overline{u})} = \emptyset$.

We have shown that the interval $[\hat{0}, w]$ contains the desired sublattices. On the other hand, if $x \in [\hat{0}, w]$, then $l(x) \leq l(w)$, and $x_i \leq w_i$ for $i = 1$ to $l(x)$. If $x_i = w_i$ for $1 \leq i \leq l(x) - 1$, then $x \in C_F(w)$. Otherwise, there exists a t such that $w_t = *$, and $x = x_1 x_2 \cdots x_{t-1} 1 w_{t+1} \cdots w_{l(x)-1} x_{l(x)}$, so

$$x \in [1^t w_{t+1} \cdots w_{l(x)-1} x_{l(x)}, w_1 w_2 \cdots w_{t-1} 1 w_{t+1} \cdots w_{l(x)-1} x_{l(x)}],$$

and x is contained in one of the Boolean sublattices. $\square$

Proposition 2 *Let $w = w_1 w_2 \cdots w_{l(w)}$. The interval $[\hat{0}, w]_{Z(1)}$ can be partitioned into disjoint sublattices of the form:*

One maximal $\hat{0} \to w$ chain, $C_Z(w)$, and

for each $w_t = *$, $\rho(w_{t+1} w_{t+2} \cdots w_{l(w)})$ subposets isomorphic to the tree $T_{j(w_1 w_2 \cdots w_{t-1})}$.

Proof: There is a unique maximal $\hat{0} \to w$ chain in $[\hat{0}, w]_{Z(1)}$ which first grows the $*$'s in w from left to right, then inserts the 1's from right to left. (e.g. if $w = 11***1*11**1$, then $C_Z(w) = \emptyset \lessdot 1 \lessdot * \lessdot *1 \lessdot ** \lessdot **1 \lessdot *** \lessdot ***1 \lessdot **** \lessdot ****1 \lessdot ***** \lessdot *****1 \lessdot ****** \lessdot ******1 \lessdot ****1**1 \lessdot ****11**1 \lessdot ***1*11**1 \lessdot 1***1*11** \lessdot 11***1*11**1.$)

If $w = 1^k$, then $[\hat{0}, w]_{Z(1)}$ is the chain $\emptyset \lessdot 1 \lessdot 11 \lessdot \ldots \lessdot 1^k$.

Assume $w = u * v$. Let $u' * v$ denote the element obtained from w by deleting the 1's from u. So $u' * v \leq w$. Then $u'1v \lessdot u' * v$. The elements in the interval $[1^{l(u')}1v, u'1v]$ form a subtree of $Z(1)$ isomorphic to $T_{j(u)}$. There are $j(u) = l(u')$ ways of leaving v fixed and changing a $*$ in u to a 1. Recursively, if the first 1 occurs in the k^{th} position of a word, then there are $k - 1$ ways of changing a $*$ to a 1, and leaving the tail fixed. Observe that the top of this copy of $T_{j(u)}$ begins at rank $\rho(w) - i(u) - 1$.

If v' is obtained from v by deleting the leftmost 1, if it exists, or by changing the last letter in v from a $*$ to a 1 if $v = *^{l(v)}$, then $u'1v' \lessdot u' * v' \lessdot u' * v$. The interval $[1^{l(u')}1v', u'1v']$ forms a subtree isomorphic to $T_{j(u)}$, and there are $\rho(v)$ disjoint copies of $T_{j(u)}$ in the interval $[\hat{0}, w]_{Z(1)}$. These trees begin on ranks $\rho(w) - i(u) - 1, \rho(w) - i(u) - 2, \ldots, \rho(w) - i(u) - \rho(v)$.

If $x \in [\hat{0}, w]_{Z(1)}$, then let t be the leftmost position such that $w = w'w_t v$ and let $x = x'x_t v$ where $w_t \neq x_t$.

If $w_t = *$ and $x_t = 1$, then $x = x'1v \leq w' * v$ if and only if $x'1v \leq *^{l(x')}1v \leq w' * v$. So, either $v = \emptyset$ and $x' = *^{l(x')}$, in which case $x \in C_Z(w)$, or $x \in [1^{l(x')}1v, *^{l(x')}1v]$.

If $w_t = 1$ and $x_t = *$, then $x = x' * v \leq w'1v$ if and only if $x' * v \leq *^{l(x')} * 1v \leq w'1v$. So, either $x' = *^{l(x')}$, in which case $x \in C_Z(w)$, or $x \in [1^{l(x')+1}1v, *^{l(x')+1}1v]$. □

The interval $[\hat{0}, w]$ with $w = 11 * * * 1 * 11 * * 1$ will have, in addition to the maximal chain $C_L(w)$, sublattices described by the table below:

w	11	$*$	$*$	$*$	1	$*$	11	$*$	$*$	1
Number		14	12	10		7		3	1	
Fib(1) Top ranks		B_0 $17, \ldots, 4$	B_1 $17, \ldots, 6$	B_2 $17, \ldots, 8$		B_3 $17, \ldots, 11$		B_4 $17, \ldots, 15$	B_5 17	
$Z(1)$ Top ranks		T_0 $15, \ldots, 2$	T_1 $15, \ldots, 4$	T_2 $15, \ldots, 6$		T_3 $14, \ldots, 8$		T_4 $12, \ldots, 10$	T_5 12	

Theorem 3 *There is a bijection between intervals $[\hat{0}, w]_{Fib(1)}$ and $[\hat{0}, w]_{Z(1)}$ which maps each Boolean subposet, B_n in Fib(1), to a spanning tree, T_n in $Z(1)$.*

Proof: The partitioning given by Propositions 1 and 2 shows that the intervals have the same number of elements. The bijection can be explicitly stated as follows:

The unique $\hat{0} \to w$ chain $C_F(w)$ is clearly in one-to-one correspondence with $C_Z(w)$. For $x \in C_F(w)$, $y \in C_Z(w)$, $x \leftrightarrow y$ if and only if $\rho(x) = \rho(y)$.

For $w = u * v$, if $v' \in C_F(v)$ is of rank $\rho(v') = m$, then there is a unique Boolean sublattice, $B_{j(u)}(1v')$ in $[\hat{0}, w]_{\mathrm{Fib}(1)}$ whose top element is $u1v'$.

Similarly, if $v'' \in C_Z(v)$ of rank $\rho(v'') = m$, and $u' = *^{j(u)}$, then there is a unique tree in $[\hat{0}, w]_{Z(1)}$ having top element $u'1v''$, $T_{j(u)}(1v'')$.

Thus, the correspondence $B_{j(u)}(1v') \leftrightarrow T_{j(u)}(1v'')$ is one to one and onto between subposets.

If $x \in B_{j(u)}(1v')$ then $x = \overline{u}1v'$ for some $\overline{u} \in [1^{l(u)}, u]_{\mathrm{Fib}(1)}$. Let $\overline{u}'$ be the word obtained from $\overline{u}$ by deleting the ones from the positions in which they occurred in u. Then $\overline{u}' \in [1^{l(u')}, *^{j(u)}]_{Z(1)}$ and $y = \overline{u}'1v'' \in T_{j(u)}(1v'')$. The correspondence $x \leftrightarrow y$ gives a bijection between elements within these subposets. □

In the case $w = 11***1*11**1$, the copy of B_3 beginning on rank 15 in Fib(1) will get mapped to the copy of T_3 beginning on rank 12 as illustrated below:

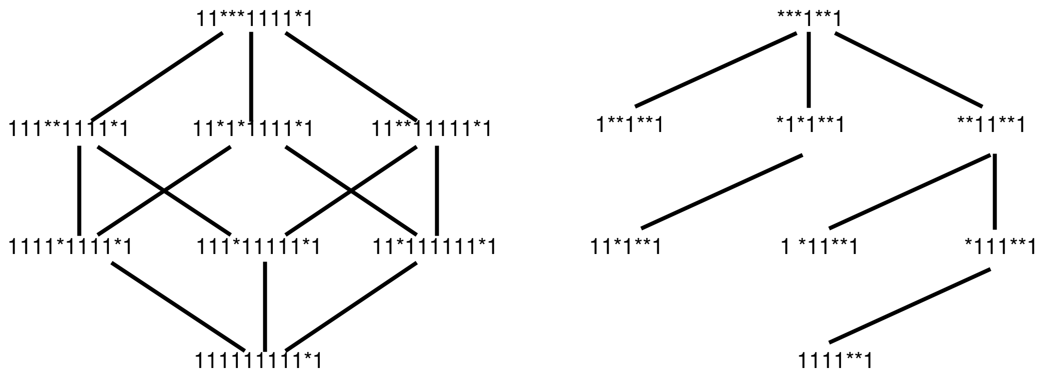

Figure 1

The case $r > 1$ is similar. Since there are r ways to change a $*$ to a digit, the interval is partitioned into slightly more complicated posets. We begin with the analogue to the unique maximal chains $C_F(w)$ and $C_Z(w)$ of Propositions 1 and 2. Figure 3a illustrates these lattices for the word $w = 2**1*2$.

Definition 6: We define $M_{\mathrm{Fib}(r)}(w)$ to be the unique sublattice of $[\hat{0}, w]_{\mathrm{Fib}(r)}$ which contains $\hat{0}$ and w and grows from left to right, always satisfying the condition that $x_i \leq w_i$. In this sublattice, if $w_t = *$, then the number of elements on rank $\rho(w_1 w_2 \cdots w_t) - 1$ is r while the number of elements on the other ranks, from 0 to $\rho(w)$, is 1.

Definition 7: Define $M_{Z(r)}(w)$ to be the unique sublattice of $[\hat{0}, w]_{Z(r)}$ which contains $\hat{0}$ and w and first grows the $j(w)$ $*'s$ in w from left to right, then inserts the digits of w from right to left. In this sublattice, the number of elements on odd ranks $1, 3, \ldots 2j(w)-1$ is r, while the number of elements on the ranks $0, 2, \ldots 2j(w), 2j(w)+1, \ldots \rho(w)$, is 1.

Note that
$$|M_{\mathrm{Fib}(r)}(w)| = |M_{Z(r)}(w)| = l(w) + rj(w).$$

Proposition 4 *Let $w = w_1 w_2 \cdots w_{l(w)}$. The interval $[\hat{0}, w]_{Fib(r)}$ can be partitioned into disjoint sublattices of the form:*

One copy of $M_{Fib(r)}(w)$, and

for each $w_t = *$, $r|M_{Fib(r)}(w_{t+1} w_{t+2} \cdots w_{l(w)})|$ subposets isomorphic to the r-Boolean lattice $B_{(j(w_1 w_2 \cdots w_{t-1}), r)}$.

Proof: This proof is similar to the proof of Proposition 1. We outline the essential differences. If $w = u*v$, then there are r elements of the form $uav \lessdot w$ where $a \in [r]$. For each of these r elements, and for each $v' \in |M_{\mathrm{Fib}(r)}(v)|$ we get a copy of $B_{(j(u),r)}$, having top element uav' (denoted $B_{(j(u),r)}(av')$). The elements of $B_{(j(u),r)}(av')$ represent the

$$r^k \begin{pmatrix} j(u) \\ k \end{pmatrix}$$

ways of leaving v' fixed and successively changing k $*$'s in u to digits.

If $x \in [\hat{0}, w]_{\mathrm{Fib}(r)}$, then either x agrees with w in at least the first $l(x)-1$ positions, and $x \in M_{\mathrm{Fib}(r)}(w)$, or x agrees with w in positions $t+1, \ldots l(x)-1$, and can be obtained from w by changing $1 \leq k \leq t$ $*$'s to digits (and $w_{l(x)}$ to $x_{l(x)}$ if they are different). So, $x \in B_{(j(w_1 \cdots w_{t-1}),r)}(x_t w_{t+1} \cdots w_{l(x)-1} x_{l(x)})$. □

The proof of Proposition 2 can be generalized to the case of $r \geq 1$ in a similar fashion to get the following partitioning in $Z(r)$.

Proposition 5 *Let $w = w_1 w_2 \cdots w_{l(w)}$. The interval $[\hat{0}, w]_{Z(r)}$ can be partitioned into disjoint sublattices of the form:*

One copy of $M_{Z(r)}(w)$, and

for each $w_t = *$, $r|M_{Z(r)}(w_{t+1} w_{t+2} \cdots w_k)|$ subposets isomorphic to the r-Boolean tree $T_{(j(w_1 w_2 \cdots w_{t-1}),r)}$.

Proof: If $w = u * v$, then for each $v' \in M_{Z(r)}(v)$, there are r rooted trees whose root label corresponds to the element $*^{j(u)} a v'$ for some $a \in [r]$ (denoted $T_{(j(u),r)}(av')$.) Each child labelled k_i corresponds to changing the k^{th} $*$ to the digit i in the first block of $*$'s appearing in the parent element. Thus, there are $r|M_{Z(r)}(v)|$ subposets isomorphic to the tree $T_{(j(u),r)}$.

If $x \in [\hat{0}, w]_{Z(r)}$, then we get the cases:

1. $x'av < w' * v$ if and only if $x'av \leq *^{l(x')} av \leq w' * v$, and $x \in M_{Z(r)}(w)$, or $x \in T_{(l(x'),r)}(av)$ having root $*^{l(x)} av$.

2. $x' * v < w'av$ if and only if $x' * v \leq *^{l(x')} * av \leq w'av$, and $x \in M_{Z(r)}(w)$, or $x \in T_{(l(x')+1,r)}(av)$ having root $*^{l(x)+1} av$.

3. $x'av < w'bv$ if and only if $x'av \leq *^{l(x')} * bv \leq w'bv$, and $x \in M_{Z(r)}(w)$, or $x \in T_{(l(x')+1,r)}(bv)$ having root $*^{l(x)+1} bv$.

□

The partitioning of the intervals $[\emptyset, 2**1*2]_{L(2)}$ into subposets is illustrated in Figure 2.

In the correspondence of Theorem 3 the map between maximal chains $C_F(w)$ and $C_Z(w)$ was rank preserving. It was this map (applied to the tail of w) which determined the correspondence between the set of subposets B_k and the set of trees T_k for fixed k. In the same way, we will define a map between $M_{\mathrm{Fib}(r)}(w)$ and $M_{Z(r)}(w)$ which will govern the correspondence between subposets of the same cardinality. As Figure 3a illustrates, this bijection cannot preserve rank.

Lemma 6 *For $w \in A(r)^*$, there is a bijection between the lattices $M_{Fib(r)}(w)$ and $M_{Z(r)}(w)$*

Proof: The bijection relies on knowing, for each $x' \lessdot x \in M_L(w)$, the position in which a digit is inserted or a digit is changed to a $*$ in x' in order to obtain x. First, beginning with w and working down to $\emptyset$, insert a 0 as a place holder each time a digit is deleted to get to the next lower rank in $M_{L(w)}$. The length of each new word in $M_L(w)$ will be equal to $l(w)$. In this new word is easy to see what change occurred between ranks. The bijection is defined as follows:

$\emptyset \leftrightarrow \emptyset$. Let $x = x_1x_2\cdots x_{l(w)}$, $y = y_1y_2\cdots y_{l(w)}$, $x' = x'_1x'_2\cdots x'_{l(w)}$, and $y' = y'_1y'_2\cdots y'_{l(w)}$ be such that $\emptyset \leq x' \lessdot x \leq w$ in $M_{\mathrm{Fib}(r)}(w)$ and $\emptyset \leq y' \lessdot y \leq w$ in $M_{Z(r)}(w)$. $x \leftrightarrow y$ if and only if for some t, $1 \leq t \leq l(w)$, either

1. $x'_t = 0 = y'_t$ and $x_t = a = y_t$, or
2. $x'_t, y'_t \in [r]$ and $x_t = * = y_t$.

To see that this is well-defined and one-to-one, notice that an element x in $M_{L(r)}(w)$ either covers a unique element or it covers r elements. If x is covered by r elements, then x is obtained by changing the digit which occurs in the same position of each of these r elements to a $*$. Since $|M_{\mathrm{Fib}(r)}(w)| = |M_{Z(r)}(w)|$ this defines a bijection. (See Figure 3a.) $\Box$

Theorem 7 *There is a bijection between intervals $[\hat{0}, w]_{Fib(r)}$ and $[\hat{0}, w]_{Z(r)}$ which maps each r-Boolean subposet, $B(n,r)$ in $Fib(r)$, to a spanning tree, $T(n,r)$ in $Z(r)$.*

Proof: Lemma 6 gives a bijection between the sublattices $M_{\mathrm{Fib}(r)}(w)$ and $M_{Z(r)}(w)$.

For $w = u * v$, $v' \in M_{\mathrm{Fib}(r)}(v)$ if and only if there is and element $v'' \in M_{Z(r)}(v)$ such that v' corresponds to v'' under the bijection of Lemma 6. For each $a \in [r]$, there is a unique r-Boolean subposet, $B_{(j(u),r)}(av')$ in $[\hat{0}, w]_{\mathrm{Fib}(r)}$ and a unique r-Boolean tree, $T_{(j(u),r)}(av'')$ in $[\hat{0}, w]_{Z(r)}$. Thus, the bijection of Lemma 6 gives us a correspondence between subposets.

Within subposets, the correspondence is as in Theorem 3, $x \in B_{(j(u),r)}(av')$ if and only if $x = \overline{u}av' \leftrightarrow \overline{u}'av'' = y$ if and only if $y \in T_{(j(u),r)}(av'')$ where $\overline{u}'$ is obtained from $\overline{u}$ by deleting the digits from the positions in which they occurred in u. An example of this bijection appears in Figures 2 and 3. $\Box$

Remark: In the case $w = *^k$, this bijection agrees with that of Theorem 3 for $r = 1$, and is just the identity map on $A(r)^*$.

The following diagram shows the partitioning of the intervals $[\emptyset, 2**1*2]_{\mathrm{Fib}(2)}$ and $[\emptyset, 2**1*2]_{Z(2)}$.

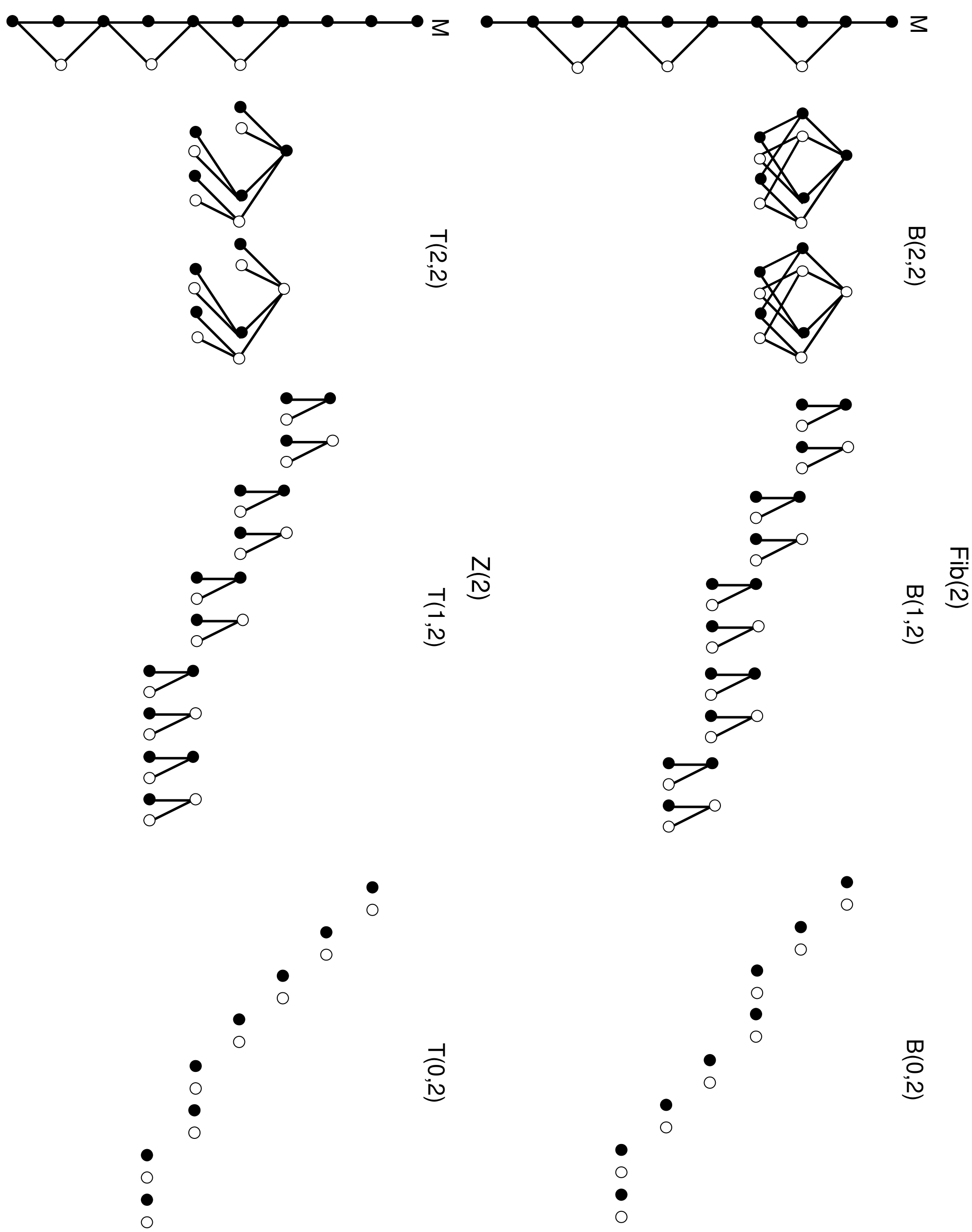

Figure 2

For $w = 2**1*2$ in $A(2)^*$, the lattices $M_{\mathrm{Fib}(r)}(w)$ and $M_{Z(r)}(w)$ are shown on the left, while the bijection between them is given on the right. Below, one subposet isomorphic to $B_{(2,2)}$ and the corresponding subtree isomorphic to $T_{(2,2)}$ is shown.

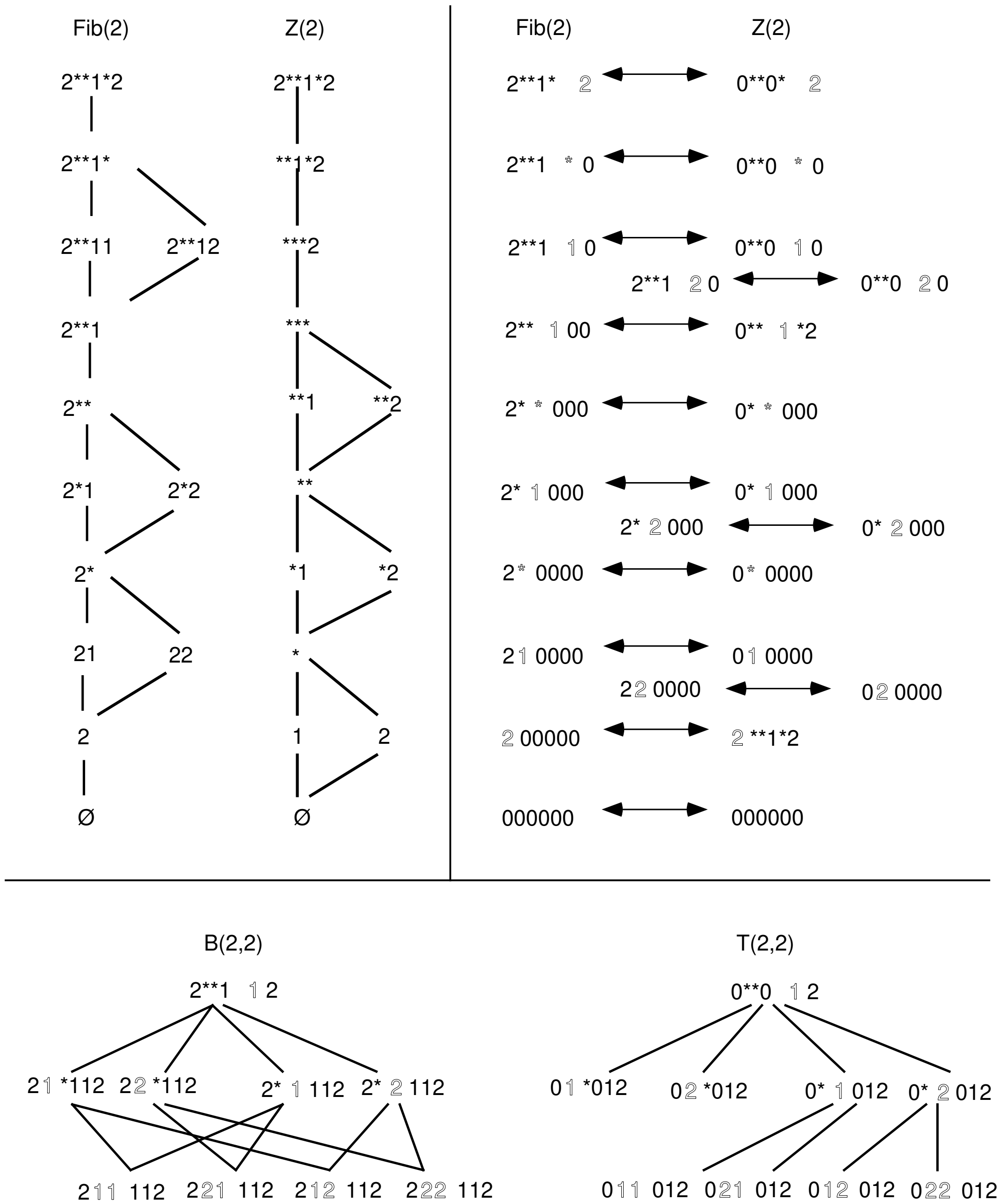

Figure 3a

The bijection between the Boolean subposets and trees in the interval $[\emptyset, 2{*}{*}1{*}2]_{L(2)}$:

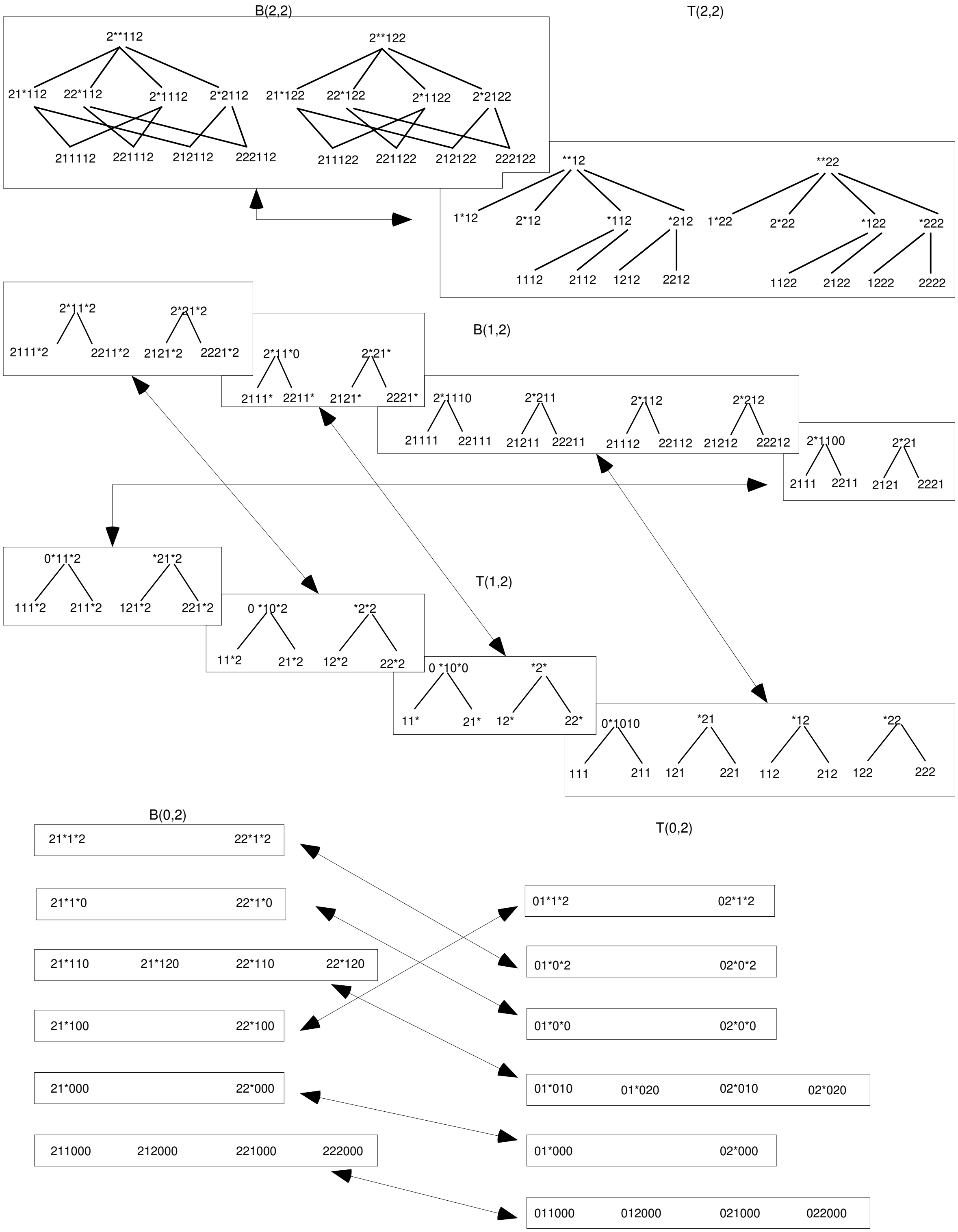

Figure 3b

4 REFERENCES

1. S. V. Fomin, Generalized Robinson-Schensted-Knuth Correspondence, *Zapiski Nauchn. Sem LOMI* **155** (1986), 156-175.

2. S. V. Fomin, Duality of Graded Graphs, Report No. 15 (1991/92), *Institut Mittag-Leffler* (1992).

3. S. V. Fomin, Duality of Graded Graphs, *J. of Algebraic Combinatorics* **3**, (1994), 357-404.

4. S. V. Fomin, Schensted Algorithms for Dual Graded Graphs, *J. of Algebraic Combinatorics* **4**, (1995), 5-45.

5. D. Kremer and K. O'Hara, "A Bijection Between Maximal Chains in Fibonacci Posets," *Journal of Combinatorial Theory* (Series **A**), to appear.

6. R. P. Stanley, The Fibonacci Lattice, *Fibonacci Quart.* **13** (1975), 215-232.

7. R. P. Stanley, "Enumerative Combinatorics, Vol. 1," Wadsworth & Brooks/Cole, Pacific Grove, CA 1986.

8. R. P. Stanley, Differential Posets, *J. Amer. Math. Soc.* **1** (1988), 919-961.

9. R. P. Stanley, Further Combinatorial Properties of Two Fibonacci Lattices, *Europ. J. Combinatorics* **11** (1990), 181-188.