Nilpotent orbit varieties and the atomic decomposition of the q-Kostka polynomials

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Abstract

We study the coordinate rings R_μ of scheme-theoretic intersections of nilpotent orbit closures with the diagonal matrices. de Concini and Procesi [5] proved a conjecture of Kraft [10] that these rings are isomorphic to the cohomology rings of the varieties constructed by Springer [21, 22]. The famous q-Kostka polynomial $K_{\lambda\mu}(q)$ is the Hilbert series for the multiplicity of the irreducible symmetric group representation indexed by λ in the ring R_μ. Lascoux and Schützenberger [13, 11] gave combinatorially a decomposition of $K_{\lambda\mu}(q)$ as a sum of "atomic" polynomials with non-negative integer coefficients, and Lascoux proposed a corresponding decomposition in the cohomology model.

Our work provides a geometric interpretation of the atomic decomposition. The Frobenius-splitting results of Mehta and van der Kallen [17] imply a direct-sum decomposition of the ideals of nilpotent orbit closures, arising from the inclusions of the corresponding sets. We carry out the restriction to the diagonal using a recent theorem of Broer [3]. This gives a direct-sum decomposition of the ideals yielding the R_μ, and a new proof of the atomic decomposition of the q-Kostka polynomials.

Résumé

Nous étudions l'anneau de coordonnées R_μ d'intersection (au sens des faisceaux) des clôtures d'orbites nilpotents avec les matrices diagonales. de Concini et Procesi [5] ont demontré une conjecture de Kraft [10] disant que ces anneaux sont isomorphes aux anneaus de cohomologie des variétés construites par Springer [21, 22]. Les célèbres q-polynômes de Kostka $K_{\lambda\mu}(q)$ sont les séries de Hilbert pour la multiplicité de la représentation irréductible du groupe symétrique indexée par λ dans l'anneau R_μ. Lascoux et Schützenberger [13, 11] ont donné une décomposition de $K_{\lambda\mu}(q)$ comme somme de polynômes "atomiques" à coefficients entiers non-negatifs et Lascoux avait proposé une décomposition correspondante dans le modèle cohomologique.

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Notre travail fournit une interprétation géométrique de cette décomposition atomique. Les résultats de Mehta et van der Kallen [17] sur la décomposition de Frobenius impliquent une décomposition en somme directe des idéaux de clôtures d'orbites nilpotents qui viennent de l'inclusion des ensembles correspondants. Nous explicitons la restriction à la diagonale en utilisant un théorème récent de Broer [3]. On obtient ainsi une décomposition en somme directe des idéaux constituant les R_μ, ainsi qu'une nouvelle démonstration de la décomposition atomique des q-polynômes de Kostka.

1 Introduction and Overview

The *q-Kostka polynomials* $K_{\lambda\mu}(q)$ have been central to numerous developments over the last two decades at the crossroads of combinatorics, algebra, and geometry. Defined through the expansion

$$s_\lambda(\mathbf{x}) = \sum_\mu K_{\lambda\mu}(q) P_\mu(\mathbf{x}; q) \tag{1.1}$$

expressing the Schur function $s_\lambda(\mathbf{x})$ as a linear combination of Hall-Littlewood polynomials $P_\mu(\mathbf{x}; q)$, they are polynomials in q which specialize upon setting $q = 1$ to the ordinary Kostka numbers $K_{\lambda\mu}$.

Foulkes [6], remarking that computations suggested that the polynomials $K_{\lambda\mu}(q)$ have non-negative integer coefficients, conjectured that they could be interpreted combinatorially. Lascoux and Schützenberger [19] proved the Foulkes conjecture by expressing $K_{\lambda\mu}(q)$ as a sum

$$K_{\lambda\mu}(q) = \sum_{T \in CS(\lambda,\mu)} q^{c(T)}, \tag{1.2}$$

in which T varies over the set $CS(\lambda, \mu)$ of column-strict tableaux of shape λ containing μ_1 ones, μ_2 twos, etc., and $c(T)$, the *charge* of T, is a certain intricate combinatorial statistic. Their result and subsequent work [12, 13, 11] reveal the remarkable combinatorial structures underlying the q-Kostka polynomials.

Several other algebraic and geometric interpretations of $K_{\lambda\mu}(q)$ have been given. Hotta and Springer [8] identified $K_{\lambda\mu}(q)$ as the Poincaré series for the multiplicities of the irreducible symmetric group representation V_λ in the cohomology groups of a certain algebraic variety X_μ. Lusztig [14, 15] described $K_{\lambda\mu}(q)$ as the local intersection homology Poincaré series of a nilpotent orbit variety, as an affine Kazhdan-Lusztig polynomial, and as a q-analog of weight multiplicities. Our concern here will be with one further interpretation, introduced by Kraft and de Concini-Procesi [10, 5], and recently given a simple and purely elementary treatment by Garsia and Procesi [7], which we now pause to describe.

In brief, Garsia and Procesi take as their starting point a family of rings described by [10, 5],

$$R_{(1^n)} = Q[x_1, \ldots, x_n]/(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})), \tag{1.3}$$

where $e_k(\mathbf{x})$ is the k-th elementary symmetric function, and

$$R_\mu = R_{(1^n)}/I_\mu \qquad \mu \text{ a partition of } n. \tag{1.4}$$

Here I_μ is an S_n-invariant ideal, generated by certain completely explicit *Tanisaki generators* [24]. Then setting

$$\tilde{K}_{\lambda\mu}(q) = q^{n(\mu)} K_{\lambda\mu}(1/q), \tag{1.5}$$

with $n(\mu) = \sum_i (i-1)\mu_i$, they prove that

$$\tilde{K}_{\lambda\mu}(q) = \sum_d \operatorname{mult}(\chi^\lambda, (R_\mu)_d) q^d, \tag{1.6}$$

where $(R_\mu)_d$ denotes the homogeneous component of degree d, and $\operatorname{mult}(\chi^\lambda, V)$ denotes the multiplicity of the irreducible character χ^λ in the character of an S_n module V. It happens that R_μ is isomorphic to the cohomology ring of the Hotta–Springer variety X_μ, but in the Garsia–Procesi treatment there is no need for recourse to this information.

The ideals I_μ satisfy $I_\mu \subseteq I_\nu$whenever $\mu \leq \nu$, with $\leq$ the dominance partial order on partitions. Thus the Garsia–Procesi interpretation immediately yields the coefficient-wise *monotonicity*

$$\tilde{K}_{\lambda\nu}(q) \leq \tilde{K}_{\lambda\mu}(q) \qquad \text{whenever } \mu \leq \nu \tag{1.7}$$

first described by Lascoux and Schützenberger [13]. In fact, much more holds: there is an *atomic decomposition*

$$\tilde{K}_{\lambda\mu}(q) = \sum_{\nu \geq \mu} R_{\lambda\nu}(q) \tag{1.8}$$

in which the *atom polynomials* $R_{\lambda\nu}(q)$ themselves have non-negative coefficients. Lascoux and Schützenberger [13, 11] originally obtained this atomic decomposition combinatorially, using certain co-charge preserving embeddings

$$\theta \colon CS(\lambda, \mu) \hookrightarrow ST(\lambda), \tag{1.9}$$

where $ST(\lambda) = CS(\lambda, 1^n)$ is the set of standard Young tableaux of shape λ, and the *co-charge* $\hat{c}(T)$ of a tableau T of weight μ is merely $n(\mu) - c(T)$.

In this paper we interpret the atomic decomposition (1.8) in the Garsia–Procesi setting, thus proving anew the non-negativity of the atom polynomials $R_{\lambda\nu}$. More precisely, we show (Theorem 6.2) that there is a direct-sum decomposition

$$R_{(1^n)} = \bigoplus_\nu A_\nu \tag{1.10}$$

of graded S_n-modules, such that for all μ,

$$I_\mu = \bigoplus_{\nu \not\geq \mu} A_\nu. \tag{1.11}$$

From this it is immediate that (1.8) holds with

$$R_{\lambda\nu}(q) = \sum_d \mathrm{mult}(\chi^\lambda, (A_\nu)_d) q^d. \tag{1.12}$$

Lascoux [11, Thm 6.5] anticipated our Theorem 6.2 by stating, without proof, a similar algebraic version of the atomic decomposition, in the setting of the cohomology ring of the flag variety.

Our proof departs from the purely elementary spirit of Garsia and Procesi, relying fairly heavily upon algebraic geometry. For us the relevant geometric interpretation is that of Kraft, de Concini and Procesi, wherein $\mathrm{Spec}\, R_\mu$ is recognized as the *scheme-theoretic intersection* of the diagonal matrices with a *nilpotent orbit variety* (the Zariski closure of a conjugacy class of nilpotent matrices). We describe this setting in detail in sections 4 and 5, while section 6 outlines the proof of our main results. Using a Frobenius-splitting result of Mehta and van der Kallen we deduce (Theorem 6.1) that there is a decomposition analogous to (1.10)–(1.11) for the nilpotent orbit varieties $\overline{C_\mu}$ in the nullcone, $\mathcal{N}$:

$$k[\mathcal{N}] = \bigoplus_{\nu \vdash n} \hat{A}_\nu \qquad \text{with } \mathcal{I}(C_\mu) = \bigoplus_{\nu \not\geq \mu} \hat{A}_\nu \qquad \forall\ \mu \vdash n. \tag{1.13}$$

This decomposition holds in prime characteristics and hence also in characteristic zero. Then, a recent result of Broer (intended by him for quite different purposes) enables us to carry the decomposition down to the diagonal matrices and complete the proof of Theorem 6.2.

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2 The q-Kostka polynomials

Our notation for partitions, tableaux, symmetric functions, and related objects will generally be compatible with Macdonald [16]. In particular, we write "λ is a partition of n" as $\lambda \vdash n$, and put our partitions in decreasing order, $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$. The only order on P_n, the set of partitions of n, that we will use is the "natural" or *dominance* partial order [16, I.1], defined by $\mu \leq \lambda$ if and only if

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i \qquad 1 \leq i \leq n. \tag{2.1}$$

Thus the smallest partition of n in dominance is $(1, 1, \ldots, 1) = (1^n)$ while the largest is (n). In fact P_n is a lattice under the dominance order [16, p. 11].

Tableaux always have entries drawn from $\{1, \ldots, n\}$ and are *column-strict*, i.e. weakly increasing in the rows, strictly in the columns. The *weight* of a tableau is the vector giving the number of occurrences of each integer in the tableau; a *standard* tableau is one of weight $(1,1,\ldots,1)$. The set of column-strict tableaux of shape λ and weight μ is written $\mathrm{CS}(\lambda,\mu)$, for $\lambda \vdash n$ and μ a composition of n. We will also need the set $\mathrm{CS}(\mu)$ of column-strict tableaux of weight μ and any shape, i.e. $\mathrm{CS}(\mu) = \bigcup_{\lambda \vdash n} \mathrm{CS}(\lambda,\mu)$. Combinatorially, the *Kostka number* $K_{\lambda\mu}$ is the cardinality of $\mathrm{CS}(\lambda,\mu)$; see [16, I.6].

The *Hall-Littlewood polynomials* $\{P_\mu(\mathbf{x};q)\}_{\mu\vdash n}$ are a remarkable basis for the symmetric polynomials in $\mathbf{x} = \{x_1,\ldots,x_n\}$, over $\mathbb{Q}(q)$; see [16, Chapter III] for their definition. The structure constants of this basis give the *Hall polynomials* [16, III.3], while, more important for our purposes, the *q-Kostka polynomials* $K_{\lambda\mu}(q)$ are the entries of the transition matrix from the Schur polynomials $\{s_\lambda(\mathbf{x})\}$ to this basis [16, III.6]:

$$s_\lambda(\mathbf{x}) = \sum_{\mu\vdash|\lambda|} K_{\lambda\mu}(q) P_\mu(\mathbf{x};q). \tag{2.2}$$

The specialization $q \to 1$ gives [16, III.2] $s_\lambda(\mathbf{x}) = \sum_{\mu\vdash n} K_{\lambda\mu}(1) m_\mu(\mathbf{x})$, the defining relation for the Kostka numbers, so $K_{\lambda\mu}(q)$ is indeed a q-analog of $K_{\lambda\mu}$.

From the definition above it follows that $K_{\lambda\mu}(q)$ is a polynomial with integer coefficients. Computing $K_{\lambda\mu}(q)$ for small n, one finds the coefficients are non-negative. Several different proofs of this fact appear in the literature. We will be especially concerned with those of Lascoux and Schützenberger [19, 13] and of Garsia and Procesi [7].

3 Cocharge and atoms

Lascoux and Schützenberger [19, 13, 12] obtain $K_{\lambda\mu}(q)$ combinatorially, by q-counting the column-strict tableaux of shape λ and content μ:

$$K_{\lambda\mu}(q) = \sum_{T\in\mathrm{CS}(\lambda,\mu)} q^{c(T)}. \tag{3.1}$$

The *charge* statistic $c(T)$ is computed directly from the tableau T by a somewhat complicated combinatorial process [16, p.242]. One sees that $c(T) \leq n(\mu)$ for $T \in \mathrm{CS}(\mu)$, where $n(\mu) = \sum_{i=1}^{n}(i-1)\mu_i$; and in many ways the *cocharge* $\hat{c}(T) = n(\mu) - c(T)$ is in fact more natural than charge. For instance, in [13], Lascoux and Schützenberger define a ranked poset on the column-strict tableaux of weight μ in which the rank of a tableau is its cocharge.

Using this poset and other tools, Lascoux and Schützenberger [12, 11] have explored the remarkable structure of the q-Kostka polynomials, or more precisely of their "cocharge" variants

$$\tilde{K}_{\lambda\mu}(q) = q^{n(\mu)} K_{\lambda\mu}(q^{-1}) = \sum_{T\in\mathrm{CS}(\lambda,\mu)} q^{\hat{c}(T)}. \tag{3.2}$$

First, $\tilde{K}_{\lambda\mu}(q)$ is a monotonically decreasing function of μ:

Theorem 3.1 (Lascoux and Schützenberger). *$\tilde{K}_{\lambda\mu}(q) - \tilde{K}_{\lambda\nu}(q)$ has non-negative integer coefficients for all partitions $\nu \geq \mu$.*

The proof is combinatorial, by construction of a shape- and cocharge-preserving injection $\theta^{\mu}_{\nu} : \mathrm{CS}(\nu) \hookrightarrow \mathrm{CS}(\mu)$ for every $\nu \geq \mu$. In particular, there is such a map into the standard tableaux for every $\nu \vdash n$. We will abbreviate this *standardization* map as $\theta_{\nu} : \mathrm{CS}(\nu) \hookrightarrow \mathrm{CS}((1^n))$. (Note that this map is not the same as the standardization map of Schensted [18].)

Lascoux and Schützenberger [13, 11] further strengthen the monotonicity result by studying the details of the standardization more closely. By construction the injections are compatible: if $\lambda \geq \mu \geq \nu$ then $\theta^{\nu}_{\mu} \circ \theta^{\mu}_{\lambda} = \theta^{\nu}_{\lambda}$. Thus if $\lambda \geq \mu$, then $\theta_{\lambda}\mathrm{CS}(\lambda) \subseteq \theta_{\mu}\mathrm{CS}(\mu)$. Furthermore,

$$(\theta_{\mu}\mathrm{CS}(\mu)) \cap (\theta_{\nu}\mathrm{CS}(\nu)) = \theta_{\mu\vee\nu}\mathrm{CS}(\mu \vee \nu). \tag{3.3}$$

So we may think of a "contribution from μ" which is disjoint from the contribution from any $\nu \neq \mu$. Formally, the *atom* associated with μ is

$$A(\mu) = \theta_{\mu}\mathrm{CS}(\mu) - \bigcup_{\nu>\mu} \theta_{\nu}\mathrm{CS}(\nu). \tag{3.4}$$

These results allow an "atomic decomposition" of $\mathrm{CS}((1^n))$ as a disjoint union of the atoms. The standardization maps are shape-preserving, so we may define

$$A(\lambda, \mu) = \theta_{\mu}\mathrm{CS}(\lambda, \mu) - \bigcup_{\nu>\mu} \theta_{\nu}\mathrm{CS}(\lambda, \nu). \tag{3.5}$$

Since the standardization maps also preserve cocharge, we now have:

Theorem 3.2 (Lascoux and Schützenberger).

$$\tilde{K}_{\lambda\mu}(q) = \sum_{\nu\geq\mu} R_{\lambda\mu}(q), \qquad \textit{where } R_{\lambda\mu}(q) = \sum_{T\in A(\lambda,\mu)} q^{\hat{c}(T)}. \tag{3.6}$$

Note $R_{\lambda\mu}(q)$ is also a polynomial with non-negative integer coefficients.

Example 3.3. The lattice of partitions P_n is a chain for $n \leq 5$, in which case the atomic decomposition reduces to the monotonicity result. Consider, therefore, partitions of 6; Table 1 gives $\tilde{K}_{\lambda\mu}(q)$ and $R_{\lambda\mu}(q)$ for the partitions $\lambda, \mu \geq (321)$. Note that the only incomparable pair in this set is $(411, 33)$.

4 Nilpotent orbit varieties

Our approach to the atomic decomposition of the q-Kostka polynomials is based on the nilpotent orbit varieties, one of several geometric constructions of these polynomials. Aspects of this construction are discussed by a number of authors, including Kostant [9], Steinberg [23], Spaltenstein [20], Springer [21, 22], Hotta and Springer [8], Kraft [10], de Concini and Procesi [5], Garsia and Procesi [7], Bergeron and Garsia [2], and Carrell [4].

$\tilde{K}_{\lambda\mu}(q)$	6	51	42	411	33	321
λ = 6	1	1	1	1	1	1
51		q	q	$q+q^2$	q	$q+q^2$
42			q^2	q^2	q^2	q^2+q^3
411				q^3	0	q^3
33					q^3	q^3
321						q^4

$R_{\lambda\mu}(q)$	6	51	42	411	33	321
λ = 6	1	0	0	0	0	0
51		q	0	q^2	0	0
42			q^2	0	0	q^3
411				q^3	0	0
33					q^3	0
321						q^4

Table 1: $\tilde{K}_{\lambda\mu}(q)$ and $R_{\lambda\mu}(q)$, for $\lambda, \mu \geq (321)$.

Let k be an algebraically closed field. We consider the variety of n by n matrices $\mathfrak{gl}_n(k)$ and its associated ring of polynomial functions $k[\mathfrak{gl}_n] = k[x_{11}, x_{12}, \ldots, x_{nn}]$. Write $\mathbf{X}$ for the n by n matrix of indeterminates x_{ij}. Denote by $\mathcal{N}$ the variety of nilpotent matrices in $\mathfrak{gl}_n(k)$, i.e. those matrices with sole eigenvalue 0. The ideal of $\mathcal{N}$ in $\mathfrak{gl}_n(k)$ is generated [9] by the coefficients of the characteristic polynomial $\det(tI - \mathbf{X})$, i.e.

$$k[\mathcal{N}] = k[\mathfrak{gl}_n]/\mathcal{I}(\mathcal{N}) = k[\mathbf{X}]/(E_1(\mathbf{X}), E_2(\mathbf{X}), \ldots, E_n(\mathbf{X})). \tag{4.1}$$

$GL_n(k)$ acts on $\mathfrak{gl}_n(k)$ by conjugation (the *adjoint action*), and $\mathcal{N}$ is stable under this action. Thus $\mathcal{N}$ is a union of GL_n-conjugacy classes. Since the only eigenvalue appearing is 0, each conjugacy class is determined by its Jordan block structure: Let $C_\mu \subset \mathcal{N}$ be the set of matrices with Jordan block form $\mu'_1, \mu'_2, \ldots, \mu'_n$, where μ' is the *conjugate partition* to μ. Equivalently,

$$C_\mu = \{N \in \mathcal{N} : \operatorname{rank} N^i = \mu_{i+1} + \cdots + \mu_n\}. \tag{4.2}$$

We will be particularly concerned with the ideal of C_μ, denoted $\mathcal{I}(C_\mu) \subseteq k[\mathfrak{gl}_n]$. Since C_μ is a GL_n-conjugacy class, $\mathcal{I}(C_\mu)$ is GL_n-stable as well, under the natural extension of the action to $k[\mathfrak{gl}_n]$. The ideals $\mathcal{I}(C_\mu)$ were studied by Kraft [10], de Concini and Procesi [5], and Tanisaki [24]; Weyman [25] found an explicit generating set for $\mathcal{I}(C_\mu)$, and also confirmed the conjectured generators of [5] and [24].

It is a standard fact that the Zariski closure $\overline{C_\mu}$, the variety cut out by $\mathcal{I}(C_\mu)$, is exactly

$$\overline{C_\mu} = \bigcup_{\nu \geq \mu} C_\nu. \tag{4.3}$$

Thus we have

$$\nu \geq \mu \iff \overline{C_\nu} \subseteq \overline{C_\mu} \iff \mathcal{I}(C_\nu) \supseteq \mathcal{I}(C_\mu). \tag{4.4}$$

The smallest of these *nilpotent orbit varieties* is $C_{(n)}$, which is closed and contains only the zero matrix, while the largest is $\overline{C_{(1^n)}} = \bigcup_{\nu \vdash n} C_\nu = \mathcal{N}$. Since all the $\overline{C_\mu}$ are contained in $\mathcal{N}$, the ideals $\mathcal{I}(C_\mu)$ are essentially interchangeable with the ideals $J_\mu = \mathcal{I}(C_\mu)/\mathcal{I}(\mathcal{N}) \subset k[\mathcal{N}]$.

5 Intersecting with the diagonal

Set-theoretically, the intersection of the nilpotent orbit variety $\overline{C_\mu}$ with the diagonal matrices $\mathfrak{t} \subset \mathfrak{gl}_n$ contains only the zero matrix. However, the *scheme-theoretic intersection* of $\overline{C_\mu}$ with $\mathfrak{t}$ forms an interesting zero-dimensional scheme, $\operatorname{Spec} k[\overline{C_\mu} \cap \mathfrak{t}]$, where

$$k[\overline{C_\mu} \cap \mathfrak{t}] = k[\mathfrak{gl}_n]/(\mathcal{I}(C_\mu) + \mathcal{I}(\mathfrak{t})). \tag{5.1}$$

The coordinate ring $k[\overline{C_\mu} \cap \mathfrak{t}]$ has finite k-dimension, since $\operatorname{Spec} k[\overline{C_\mu} \cap \mathfrak{t}]$ has only one underlying point. Furthermore, it carries an action of the *Weyl group* of GL_n, the symmetric group S_n, as follows. We view $S_n \subset GL_n$ as the subgroup of permutation matrices. While the GL_n action by conjugation does not preserve $\mathfrak{t}$, the restriction of this action to S_n does. Likewise, $\mathcal{I}(\mathfrak{t})$ is S_n-stable, as are the GL_n-stable sets C_μ and $\mathcal{I}(C_\mu)$; so the ideal $\mathcal{I}(C_\mu)+\mathcal{I}(\mathfrak{t})$, the quotient ring $k[\overline{C_\mu}\cap\mathfrak{t}]$, and the scheme $\operatorname{Spec} k[\overline{C_\mu}\cap\mathfrak{t}]$ all carry S_n-actions.

Since each ideal $\mathcal{I}(C_\mu) + \mathcal{I}(\mathfrak{t})$ contains the ideal $\mathcal{I}(\mathfrak{t}) = (x_{ij} : i \neq j)$ and the ideal $\mathcal{I}(\mathcal{N}) = (E_1(\mathbf{X}), E_2(\mathbf{X}), \ldots, E_n(\mathbf{X}))$, we lose nothing by working in the quotient ring $k[\mathcal{N} \cap \mathfrak{t}] = k[\mathbf{X}]/(\mathcal{I}(\mathcal{N}) + \mathcal{I}(\mathfrak{t}))$. Of course, $k[\mathbf{X}]/\mathcal{I}(\mathfrak{t}) \cong k[x_{11}, x_{22}, \ldots, x_{nn}]$, and we may identify the surviving variables x_{ii} with x_i; as in section 2, we use $\mathbf{x}$ to represent the variables $\{x_1, \ldots, x_n\}$. In $k[\mathbf{x}]$, the polynomial $E_i(\mathbf{X})$ becomes the elementary symmetric function $e_i(\mathbf{x})$. Define the ring R by:

$$k[\mathbf{X}]/(\mathcal{I}(\mathcal{N}) + \mathcal{I}(\mathfrak{t})) \cong k[\mathbf{x}]/(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})) = R, \tag{5.2}$$

and let the ideal I_μ be the image of $\mathcal{I}(C_\mu) + \mathcal{I}(\mathfrak{t})$ in R. (Explicit sets of generators for I_μ appear in [5, 24, 7].) As described in (1.4), we define $R_\mu = R/I_\mu$; we will tend to consider the isomorphism $R_\mu \cong k[\overline{C_\mu} \cap \mathfrak{t}]$ an equality. Note also that $I_{(1^n)} = (0)$ and $R_{(1^n)} = R$.

At this point we should remark that all of the ideals $\mathcal{I}(C_\mu)$ and $\mathcal{I}(\mathfrak{t})$ are *homogeneous* ideals—generated by homogeneous polynomials—and thus the quotients $k[\overline{C_\mu}]$, $k[\mathfrak{t}]$, and R_μ are *graded* rings. (Equivalently, we are noting that the k^*-action on $\mathfrak{gl}_n$ stabilizes C_μ and $\mathfrak{t}$.) So we have, in particular, $R_\mu = \oplus_{d \geq 0} (R_\mu)_d$ for all $\mu \vdash n$, where $(R_\mu)_d$ denotes the homogeneous component of degree d.

Now de Concini and Procesi [5] proved that R_μ is isomorphic as an S_n module to $\operatorname{Ind}_{S_\mu}^{S_n} 1$, and by Young's rule the character of $\operatorname{Ind}_{S_\mu}^{S_n} 1$ is $\sum_{\lambda \vdash n} K_{\lambda\mu} \chi^\lambda$. So, recording the multiplicity of the irreducible χ^λ in each degree $(R_\mu)_d$ as the coefficient of q^d, one has

$$\sum_d \operatorname{mult}(\chi^\lambda, (R_\mu)_d) q^d = \tilde{K}_{\lambda\mu}(q), \tag{5.3}$$

wherein the $\tilde{K}_{\lambda\mu}(q)$ must be some q-analogs of the Kostka numbers $K_{\lambda\mu}$. In fact, the polynomials $\tilde{K}_{\lambda\mu}(q)$ are the same polynomials introduced in section 3: $\tilde{K}_{\lambda\mu}(q) = q^{n(\mu)} K_{\lambda\mu}(q^{-1})$. This identification follows from the work of Kraft, de Concini, Procesi, Hotta, and Springer [10, 5, 21, 22, 8].

Garsia and Procesi give an elementary proof of this connection between the ideals I_μ and the q-Kostka polynomials in [7], which makes completely explicit several combinatorial properties of the q-Kostka polynomials. In particular, their monotonicity is immediate [7, p. 103]: Let μ and ν be partitions with $\nu \geq \mu$. As noted in section 4, we have $\mathcal{I}(C_\nu) \supseteq \mathcal{I}(C_\mu)$, and hence $I_\nu \supseteq I_\mu$. (One can also compute directly from the Tanisaki generators for I_μ that $I_\nu \supseteq I_\mu$.) Thus $R_\mu/I_\nu \cong R_\nu$, and the surjection of S_n-modules gives $\tilde{K}_{\lambda\mu}(q) \geq \tilde{K}_{\lambda\nu}(q)$. This surjection is an algebraic analog of Lascoux's injection $\mathrm{CS}(\nu) \hookrightarrow \mathrm{CS}(\mu)$.

5.1 Atoms

One important property which is not seen immediately from the Garsia-Procesi point of view is the atomic decomposition $\tilde{K}_{\lambda\mu} = \sum_\nu R_{\lambda\nu}$ with $R_{\lambda\nu}$ a polynomial with non-negative integer coefficients. Since $\tilde{K}_{\lambda\mu}(q)$ is obtained from the graded character of the S_n-module R_μ, we seek an isomorphism of graded S_n-modules

$$R_\mu \cong \bigoplus_{\nu \geq \mu} A_\nu. \tag{5.4}$$

Where can we find these modules A_ν? They most naturally live in R. We must decompose R as

$$R = \bigoplus_{\nu \vdash n} A_\nu \qquad \text{so that } I_\mu = \bigoplus_{\nu \not\geq \mu} A_\nu \quad \text{for all } \mu \vdash n. \tag{5.5}$$

Then we shall have immediately from (5.4) and (3.6) that

$$\sum_d \mathrm{mult}(\chi^\lambda, (A_\nu)_d) q^d = R_{\lambda\nu}(q). \tag{5.6}$$

Example 5.1. Using Macaulay 2 [1], we found bases of the spaces $A_\nu \subset R$ whose characters are described by the atomic polynomials $R_{\lambda\nu}$ of example 3.3. These bases are too large to present here; instead in Table 2 we give one vector in each irreducible S_n-representation. Note that this example is somewhat trivial, since there are only two incomparable partitions, and thus the corresponding lattice is automatically distributive; in addition, the multiplicity of representations in the graded components of the atoms never exceeds 1. The example would have to be significantly larger ($n = 10$) to avoid these failings.

6 Main theorems

Theorem 6.1. *Let k be an algebraically closed field. There exists a direct-sum decomposition of the coordinate ring of $\mathfrak{gl}_n$,*

$$k[\mathfrak{gl}_n] = \bigoplus_{\nu \vdash n} \hat{A}_\nu \tag{6.1}$$

compatible with the ideals of the conjugacy classes C_μ:

$$\mathcal{I}(C_\mu) = \bigoplus_{\nu \not\geq \mu} \hat{A}_\nu \qquad \forall\ \mu \vdash n. \tag{6.2}$$

		$A_\nu(q)$					
		6	51	42	411	33	321
λ	6	1	0	0	0	0	0
	51		x_6	0	x_6^2	0	0
	42			x_5x_6	0	0	$e_3(x_3, x_4, x_5, x_6)$
	411				$x_5x_6^2$	0	0
	33					$x_4x_5x_6$	0
	321						$x_4x_5x_6^2$

Table 2: Generators of irreducibles of type λ in $A_\nu(q)$, for $\nu \geq (321)$.

Theorem 6.2 (Atomic decomposition). *There exists a direct-sum decomposition of $k[\mathcal{N} \cap \mathfrak{t}] \cong R$,*

$$R = \bigoplus_{\nu \vdash n} A_\nu \qquad \text{such that } I_\mu = \bigoplus_{\nu \not\geq \mu} A_\nu \quad \text{for all } \mu \vdash n. \tag{6.3}$$

Thus we obtain the "atom polynomial,"

$$R_{\lambda\nu}(q) = \sum_d \operatorname{mult}(\chi^\lambda, (A_\nu)_d) q^d. \tag{6.4}$$

We only sketch the proofs in this extended abstract. First we show that the decomposition exists if and only if the ideals $\{I_\mu : \mu \vdash n\}$ generate a distributive sublattice of the lattice of all ideals in R. Similarly, there is such a decomposition of $k[\mathcal{N}]$ if and only if the lattice generated by the ideals of orbit varieties $\{\mathcal{I}(C_\mu) : \mu \vdash n\}$ is distributive. Geometrically the latter property is natural: we expect that the lattice generated by the $\mathcal{I}(C_\mu)$ is the dual of the lattice generated by the orbit varieties $\overline{C_\mu}$ under the set operations.

We show that the lattice generated by $\{\mathcal{I}(C_\mu) : \mu \vdash n\}$ is distributive by showing that every element of the lattice is radical. This fact is a consequence of a recent geometric result:

Theorem 6.3 (Mehta and van der Kallen [17]). *Let k be an algebraically closed field of positive characteristic. There is a Frobenius splitting ϕ of $\mathfrak{gl}_n$ such that every nilpotent orbit variety $\overline{C_\mu}$ is compatibly Frobenius split.*

Since the distributivity is proved in positive characteristic, we must make a standard technical argument to extend the result to characteristic 0. This proves Theorem 6.1. Finally, we address the intersection with the diagonal. Essentially, we show that the lattice generated by the $\mathcal{I}(C_\mu)$ together with $\mathcal{I}(\mathfrak{t})$ is still distributive, even though its elements are no longer radical.

This argument relies on a result of Broer, who in [3] extends Chevalley's restriction theorem to modules of covariants. His theorem holds in great generality, but we will need it only for the Lie group GL_n and Lie algebra $\mathfrak{gl}_n$. (We view $\mathfrak{gl}_n$ as the algebraic variety of all n by n matrices over k.) Let $T \subset GL_n$ be the subgroup of invertible diagonal matrices, and $\mathfrak{t} \subset \mathfrak{gl}_n$ its Lie algebra. Any GL_n-module M has a representation of the Weyl group S_n on the fixed-point set M^T of the T-action. Broer's theorem applies to *small* GL_n-modules: those which do not have the T-weight 2ϕ, where ϕ is the highest root of GL_n.

Theorem 6.4 (Broer [3]). *Let k be an algebraically closed field of characteristic zero, and let M be a small GL_n-module. For any nilpotent orbit variety $\overline{C_\mu}$, the map*

$$\mathrm{Hom}_{GL_n}(M, k[\overline{C_\mu}]) \to \mathrm{Hom}_{S_n}(M^T, k[\overline{C_\mu} \cap \mathfrak{t}]) \tag{6.5}$$

induced from the restriction $k[\overline{C_\mu}] \to k[\overline{C_\mu} \cap \mathfrak{t}]$ is an isomorphism of graded vector spaces.

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