Osculating Lattice Paths and Alternating Sign Matrices

R. Brak *

Department of Mathematics, The University of Melbourne
Parkville, Victoria 3052, Australia

May 8, 1997

Abstract

Osculating lattice paths are sets of directed lattice paths which are not allowed to cross or have common edges, but *are* allowed common vertices. We derive a constant term formula for the number of such lattice paths. The formula is obtained by solving a set of simultaneous recurrence relations.

Alternating sign matrices are in simple bijection with a subset of osculating lattice paths. This leads to a constant term formula for the number of alternating sign matrices.

Résumé

Par "chemins de contact" on entend des ensembles de chemins orientés dans un réseau qui ne se traversent pas et qui n'ont pas d'arêtes communes, mais qui peuvent avoir des noeuds communs. Nous établissons une formule de type "terme constant" pour le nombre de tels chemins. On obtient cette formule en résolvant un ensemble de récurrences simultanées.

Les matrices à signes alternées sont en bijection simple avec un sous-ensemble de chemins de contact. On obtient ainsi une nouvelle formule de type "terme constant" pour le nombre de telles matrices.

*email: brak@maths.mu.oz.au

1 Introduction

Osculating paths (see next section for a definition) are intimately associated with the combinatorial problem of enumerating alternating sign matrices (ASMs). Alternating sign matrices are square $n \times n$ matrices whose entries are either 0,$\pm$1, such that the non-zero elements in each row and column alternate between 1 and -1 and begin and end with 1. It is readily shown the there is a bijection[1] between alternating sign matrices (ASMs) and n certain osculating paths. An example of this bijection is illustrated in figure 2. The expression of RRM[2, 3]

$$a_n = \prod_{i=1}^{n-1} \frac{(3i+1)!}{(n+i)!} \tag{1}$$

for the number of $n \times n$ ASMs was first impressively proved by Zeilberger [4] who related it to a particular class of plane partitions. These partitions had been enumerated by Andrews [5] based on a result by Stembridge. A shorter derivation was subsequently obtained by Kuperberg[6] using a result of Izergin and Korepin[7]. A pure combinatorial

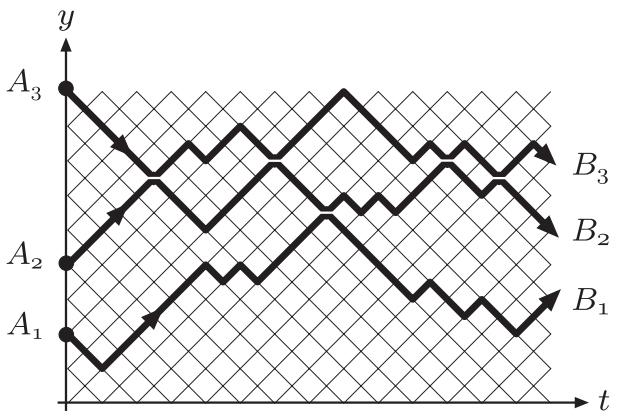

Figure 1: *An example of three osculating paths.*

proof remains an open problem which we hope this paper sheds a little light upon.

2 Principle Results

Before giving the principle theorems we need a few definitions.

Let S_n be the group of permutations of n objects with $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$. Let $\bar{\sigma}$ be the inverse of σ. Construct a bipartite graph representation from σ by starting with the usual two row representation of σ. For every number α in the top row draw a line starting at α and ending in the bottom row at σ_α, in such a way that only two lines intersect at any one point (call such a point an inversion vertex). If all the lines are draw such that the number of inversion vertices is minimal we will call this the *inversion* diagram of σ. If the lines are labeled by their starting numbers in the top row, then the inversion vertices can be labeled by the pair $\{\alpha, \beta\}$ with $\alpha < \beta$, where α and β are

the labels of the two lines. Figure 4 shows an example – drawn vertically rather than horizontally. Note, each inversion vertex represents an inversion of the permutation. Let C_σ be the set of inversion vertex labels (or set of inversion vertices for short), $\{\alpha, \beta\}$ of the permutation σ.

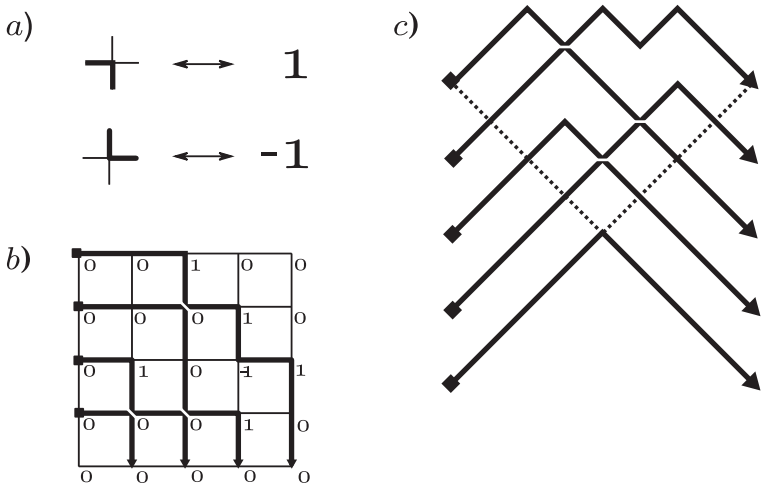

Figure 2: *An example of the bijection between osculating paths and a* 5×5 *alternating sign matrice. a) Bijection between* 1 *and* -1 *and the path steps, b) an example of the paths for the matrix shown and c) the resulting constrained osculating paths.*

Let $\mathcal{Q} = \{(u, v)|(u, v) \in \mathbb{Z} \times \mathbb{Z}\}$. Let $\Pi = \{(t, y)|(t, y) = (v - u, u + v), (u, v) \in \mathcal{Q}\}$ i.e. $\mathcal{Q}$ rotated 45° about the origin. A t-step path on Π is a sequence $w = (s_0, s_1, \ldots, s_t)$ of points in Π such that if $s_{t'} = (t', y)$ then $s_{t'+1}$ is either $s_{t'+1} = (t' + 1, y + 1)$ (a North East step) or $s_{t'+1} = (t' + 1, y - 1)$ (a South East step). Call t' the horizontal coordinate and y the height coordinate.

Definition 1 (Osculating). *Let* $\{y_\alpha\}_{\alpha=1\ldots n}$ *be a sequence of integers with* $y_1 \leq y_2 \leq \cdots \leq y_n$*, such that no three consecutive values are equal i.e. if* $y_\alpha = y_{\alpha+1}$ *then* $y_{\alpha-1} < y_a$ *and* $y_{\alpha+1} < y_{\alpha+2}$*. Such a sequence is called an* ***osculating sequence****. A pair for which* $y_\alpha = y_{\alpha+1}$ *is called an* ***osculation****. Let* $\{y_\alpha\}_{\alpha=1\ldots n}$ *and* $\{y'_\alpha\}_{\alpha=1\ldots n}$ *be osculating sequences. Let* $A_\alpha = (0, y'_\alpha)$ *and* $B_\alpha = (t, y_\alpha)$*,* $t \in \mathbb{N}$ *be the starting and ending points respectively of the* n*-tuple* $(w_1, \ldots, w_n)$ *of paths of* Π *such that*

1. *for each* α*,* w_α *is a* t*-step path from* A_α *to* B_α*.*
2. *If, for* $\alpha = 1 \ldots n$*,* $s_{t'} = (t', y''_\alpha) \in w_\alpha$*, then* $\{y''_\alpha\}_{\alpha=1\ldots n}$ *is an osculating sequence.*
3. *The paths* $(w_1, \ldots, w_n)$ *have no edges in common.*

Paths satisfying the above condition are called t*-step* ***osculating paths*** *starting at* $\{A_\alpha\}_n$ *and ending at* $\{B_\alpha\}_n$*. See figure 1 for an example.*

Let $\mathbb{Z}((x_1, x_2, \ldots, x_n, \omega))$ be a ring of formal Laurent series in the $n + 1$ variables $x_1, x_2, \ldots, x_n, \omega$ with coefficients in $\mathbb{Z}$. For convenience we will use the notation $\mathbb{Z}((x, \omega))$ for this ring. Later we will also use the quotient ring of $\mathbb{Z}((x, \omega))$, which we

denote $\mathbf{F}_{\mathbb{Z}((x,\omega))}$. Since the main results involve a constant term formula we define the operation as follows:

Definition 2 (Constant term). *Let $R \in \mathbb{Z}((x,\omega))$ then*

$$R = \sum_{r_1 \geq m_1} \sum_{r_2 \geq m_2} \cdots \sum_{r_n \geq m_n} \sum_{r_{n+1} \geq m_{n+1}} a_{r_1,r_2,\ldots,r_n,r_{n+1}} x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \omega^{r_{n+1}}$$

with $a_{r_1,r_2,\ldots,r_n,r_{n+1}} \in \mathbb{Z}$ and all $m_i \in \mathbb{Z}$. The constant term operation CT is defined as

$$CT\big[R\big] = \sum_{r \geq m_{n+1}} a_{r_1,r_2,\ldots,r_n,r} \omega^r. \tag{2}$$

i.e. we only take the constant term over the "x" variables.

Now that we have defined osculating paths we have the principle result of this paper.

Theorem 1. *Let $\lambda_\alpha = x_\alpha + 1/x_\alpha$ and $\Lambda_n = \prod_{\alpha=1}^n \lambda_\alpha$. The osculation generating function for t-step osculating paths starting at $\{A_\alpha\}_n$ and ending at $\{B_\alpha\}_n$ is given by*

$$\mathcal{R}_t(\omega) = CT\left[\Lambda_n^t \sum_{\sigma \in S_n} \epsilon_\sigma \prod_{\alpha=1}^n x_\alpha^{y_{\sigma_\alpha} - y'_\alpha} \prod_{\{i,j\} \in C_\sigma} \left\{ \frac{\lambda_i\lambda_j - \omega x_j/x_i}{\lambda_i\lambda_j - \omega x_i/x_j} \right\}\right] \tag{3}$$

where ϵ_σ is the signature of the permutation σ and C_σ is the set of inversion vertices of σ. The variable ω counts the number of osculations.

Note, $\mathcal{R}_t(0)$ is the number of *non*-intersecting paths and in this case (3) is readily shown to reduce to the Gessel-Viennot determinant.

To count the number of $n \times n$ alternating sign matrices we need n osculating paths with w_α, $\alpha = 1, \ldots, n$, starting at $A_\alpha = (0, 2\alpha)$ and ending at $B_\alpha = (2n-2, 2\alpha)$ with the initial $n - \alpha$ steps of each w_α in the NE direction and the final $n - \alpha$ steps of w_α in the SE direction, as illustrated in figure 2. This then leads to the following corollary.

Corollary 1. *The number of alternating sign matrices, $\mathcal{A}_1(n)$, of size $n \times n$ is*

$$CT\left[\Lambda_n^{2n-2} \sum_{\sigma \in S_n} \epsilon_\sigma \prod_{\alpha=1}^n \left\{ x_\alpha^{2(\sigma_\alpha - \alpha)} \left(\frac{x_\alpha^{-1} x_{\bar{\sigma}_\alpha}}{\lambda_\alpha \lambda_{\bar{\sigma}_\alpha}} \right)^{n-\alpha} \right\} \prod_{\{i,j\} \in C_\sigma} \left\{ \frac{\lambda_i\lambda_j - x_j/x_i}{\lambda_i\lambda_j - x_i/x_j} \right\}\right]. \tag{4}$$

Similar bijections[1] exist for alternating sign matrices which are vertically and horizontally symmetric (see Robbins[3] for definitions) of sizes $4n+1$ and $4n+3$. A very similar corollary is obtained[10].

[1] I would like to thank Mireille Bousquet-Melou for drawing my attention to these bijections

3 Proofs

The proof of theorem one rests on solving a recurrence relation with a set of boundary conditions, which enforce the osculation constraint, and an initial condition. If there are n paths then the number of boundary conditions equals the Fibbonacci number, F_n. The solution of the recurrence relation is trivial, however, in order to satisfy the boundary conditions one has to introduce a Bethe Ansatz[8]. This introduces $n!$ arbitrary constants, all but one are fixed by the boundary conditions. The final constant is fixed by the initial condition.

We begin by constructing a matching from an osculating sequence. Let Z_n be a linear graph of n vertices. Let $\mathcal{P}$ be the set of all matchings of Z_n. Let $\{y_\alpha\}_{\alpha=1\ldots n}$ be an osculating sequence. Label the α^{th} vertex of Z_n by the corresponding variable y_α. For all pairs for which $y_\alpha = y_{\alpha+1}$ colour the edge between α and $\alpha+1$. This defines a unique matching $M \in \mathcal{P}$. Let $\mathcal{I}_M$ be the set of isolated points in M and $\mathcal{E}_M$ the set of coloured edges in M.

For a given configuration of osculating paths and a given horizontal coordinate we get a set of height coordinates $(y_1, \ldots, y_n)$. This set naturally defines a corresponding osculating sequence $\{y_\alpha\}_{\alpha=1\ldots n}$. An example of this correspondence and the corresponding matching is shown in figure 3.

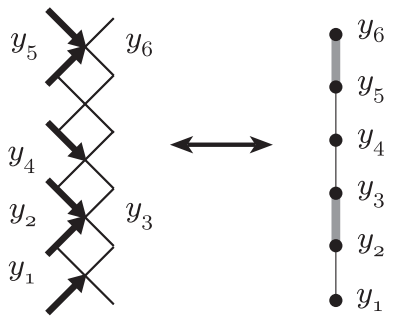

Figure 3: *An example of the matching obtained from a set of osculating vertices for a given configuration of osculating paths with a given horizontal coordinate, here* $y_2 = y_3$, $y_5 = y_6$, $\mathcal{I}_M = \{y_1, y_4\}$ *and* $\mathcal{E}_M = \{[y_2, y_3], [y_5, y_6]\}$.

Let $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{e} = (e_1, \ldots, e_n)$ and let $r(\mathbf{y}; t)$ be the osculation generating function for the number of t-step osculating paths from $\{A_\alpha\}_n$ to $\{B_\alpha\}_n$. If $y_1 < y_2 < \ldots < y_n$ then

$$r(\mathbf{y}; t+1) = \sum_{e_1=\pm 1} \cdots \sum_{e_n=\pm 1} r(\mathbf{y} + \mathbf{e}; t). \tag{5}$$

If $\mathbf{y}$ contains any osculations a different recurrence is necessary. Since $\mathbf{y}$ defines an osculating sequence $\{y_\alpha\}_{\alpha=1\ldots n}$, and hence a matching, we have that for each $M \in \mathcal{P}$,

$$r(\mathbf{y}; t+1) = \omega^{|\mathcal{E}_M|} \sum_{e_1 \in v_1^M} \cdots \sum_{e_n \in v_n^M} r(\mathbf{y} + \mathbf{e}; t) \tag{6}$$

where

$$v_i^M = \begin{cases} \{+1,-1\} & \text{if } y_i \in \mathcal{I}_M, \\ \{-1\} & \text{if } y_i \text{ is a lower vertex of some edge} \in \mathcal{E}_M \ , \\ \{+1\} & \text{if } y_i \text{ is an upper vertex of some edge} \in \mathcal{E}_M. \end{cases} \tag{7}$$

(an "upper" vertex of an edge in $\mathcal{E}_M$ is the one with the greater "α" label and conversely for a "lower" vertex). Since $|\mathcal{P}| = F_n$, there are F_n such equations. The initial condition is

$$r(\mathbf{y};0) = \omega^{|\mathcal{E}_{M'}|} \prod_{\alpha=1}^{n} \delta_{y_\alpha, y'_\alpha}. \tag{8}$$

where $\delta_{i,j}$ is the Kronecker delta and M' is the matching corresponding to the initial set of starting heights $(y'_1, \ldots, y'_n)$.

Equation (5) can be interpreted as the recurrence for a *single* random walker on $\mathbb{Z}^n$ with boundary conditions (6). For this reason we refer to equations (6) as the boundary conditions on equation (5).

Let $\mathbf{x} = \prod_{\alpha=1}^n x_\alpha$ and $\mathbf{x}^{\mathbf{y}} = \prod_{\alpha=1}^n x_\alpha^{y_\alpha}$, then $\mathbf{x}, \mathbf{x}^{\mathbf{y}} \in \mathbb{Z}((x,\omega))$. If $r(\mathbf{y};t) = \Lambda_n^t \mathbf{x}^{\mathbf{y}}$, then (5) is satisfied provided

$$\Lambda_n(x_1, \ldots, x_n) = \prod_{\alpha=1}^{n} \big(x_\alpha + x_\alpha^{-1}\big). \tag{9}$$

Since $\Lambda_n(x_1, \ldots, x_n)$ is invariant under any permutation of its arguments i.e. is a symmetric function, we have that $r(\mathbf{y};t) = \Lambda_n^t \mathbf{x}_\sigma^{\mathbf{y}}$, where $\mathbf{x}_\sigma^{\mathbf{y}} = \prod_{\alpha=1}^n x_{\sigma_\alpha}^{y_\alpha}$ for any $\sigma \in S_n$ is also a solution of (5). In general this gives us $n!$ solutions.

To satisfy the boundary conditions we try the following Ansatz

$$r(\mathbf{y};t) = A_0(\mathbf{x}) \Lambda_n^t \sum_{\sigma \in S_n} A_\sigma(\mathbf{x}) \mathbf{x}_\sigma^{\mathbf{y}} \tag{10}$$

where the non-zero elements $A_\sigma(\mathbf{x}), A_0(\mathbf{x}) \in \mathbf{F}_{\mathbb{Z}((x,\omega))}$ and *independent* of $\mathbf{y}$. In statistical mechanics such an Ansatz is known as a Bethe Ansatz. We will see that is $A_0(\mathbf{x})$ is conveniently written as a ratio of elements of $\mathbb{Z}((x,\omega))$ i.e. elements of $\mathbf{F}_{\mathbb{Z}((x,\omega))}$, but in fact it will turn out that they are actually elements of the subring of $\mathbf{F}_{\mathbb{Z}((x,\omega))}$ isomorphic to $\mathbb{Z}((x,\omega))$. This is important, since the constant term operation, which we will need, has only been defined on $\mathbb{Z}((x,\omega))$.

If $r(\mathbf{y};t)$is given by (10), then (6) is certainly satisfied if

$$\sum_{\sigma \in S_n} A_\sigma \left(\Lambda_n - \omega^{|\mathcal{E}_M|} \sum_{e_1 \in v_1^M} \cdots \sum_{e_n \in v_n^M} \mathbf{x}_\sigma^{\mathbf{e}} \right) \mathbf{x}_\sigma^{\mathbf{y}} = 0 \tag{11}$$

for each $M \in \mathcal{P}$. The choice of elements A_σ which ensure this is given by the following lemma.

Lemma 1. *If*

$$A_\sigma = \epsilon_\sigma \prod_{1\le\alpha<\beta\le n} s_{\sigma_\alpha\sigma_\beta} \tag{12}$$

where

$$s_{\alpha\beta} = \lambda_\alpha\lambda_\beta - \omega x_\alpha/x_\beta \tag{13}$$

then, for arbitrary $\mathbf{y}$*,*

$$\sum_{\sigma\in S_n} A_\sigma \Big(\Lambda_n - \omega^{|\mathcal{E}_M|} \sum_{e_1\in v_1^M} \cdots \sum_{e_n\in v_n^M} \mathbf{x}_\sigma^{\mathbf{e}}\Big)\mathbf{x}_\sigma^{\mathbf{y}} = 0 \tag{14}$$

is certainly true, for all $M \in \mathcal{P}$*.*

The lemma is readily, but tediously, proved by induction on $|\mathcal{E}_M|$. The details are omitted, but will appear in the full version of this paper[10].

Equation (13) is a standard result in the Bethe Ansatz solution of the six-vertex model[9] where similar path configurations are considered. There is, however, a fundamental difference between the paths considered here and those in the six vertex model. In six vertex model the partition function is the most important object, and to compute it the initial and final positions of the paths must be summmed over (within some range corresponding to the size of the physical system), unlike the situation here, where they are fixed. Furthermore the six-vertex calculations are performed using a strip of finite width, a constraint that complicates the computation in our case.

Finally, we have to satisfy the initial condition (8).

Lemma 2. *If*

$$r(\mathbf{y};t) = CT\Big[A_0(\mathbf{x})\Lambda_n^t \sum_{\sigma\in S_n} A_\sigma(\mathbf{x})\mathbf{x}_\sigma^{\mathbf{y}}\Big] \tag{15}$$

with

$$A_0(\mathbf{x}) = \frac{\omega^{|\mathcal{E}_{M'}|}}{n!} \sum_{\sigma'\in S_n} \frac{1}{A_{\sigma'}(\mathbf{x})}\mathbf{x}_{\sigma'}^{-\mathbf{y}'} \tag{16}$$

then the initial condition (8) is satisfied.

Proof. To prove the lemma we are reduced to showing that

$$CT\left[\frac{1}{n!}\sum_{\sigma\in S_n}\sum_{\sigma'\in S_n} \frac{A_\sigma(\mathbf{x})}{A_{\sigma'}(\mathbf{x})} \prod_{\alpha,\beta=1}^{n} x_{\sigma_\alpha}^{y_\alpha} x_{\sigma'_\beta}^{-y'_\beta}\right] = \prod_{\alpha=1}^{n} \delta_{y_\alpha,y'_\alpha}. \tag{17}$$

Clearly, if $\sigma = \sigma'$ then $\mathrm{CT}[\sum_{\sigma\in S_n}\prod_{\alpha=1}^n x_{\sigma_\alpha}^{y_\alpha - y'_\alpha}] = n!\prod_{\alpha=1}^n \delta_{y_\alpha,y'_\alpha}$. Thus we need to show that for $\sigma \neq \sigma'$ we have $\mathrm{CT}[(A_\sigma(\mathbf{x})/A_{\sigma'}(\mathbf{x})\prod_{\alpha=1}^n x_{\sigma_\alpha}^{y_{\sigma_\alpha} - y'_{\sigma'_\alpha}}] = 0$. This is readily done. □

Note, it can be shown[10] that $A_\sigma(\mathbf{x})/A_{\sigma'}(\mathbf{x}) \in \mathbb{Z}((x,\omega))$ and hence the CT operation is well defined.

The final step in the proof of theorem one is to show that

$$CT\left[\frac{1}{n!}\Lambda_n^t \sum_{\sigma\in S_n}\sum_{\sigma'\in S_n}\frac{A_\sigma(\mathbf{x})}{A_{\sigma'}(\mathbf{x})}\prod_{\alpha,\beta=1}^{n} x_{\sigma_\alpha}^{y_\alpha}x_{\sigma'_\beta}^{-y'_\beta}\right] \tag{18}$$

reduces to (3). Let $\sigma'_I = 123\cdots n$ be the identity permutation and $\bar{\sigma}'$ the inverse permutation of σ'. We now use the property that the constant term is invariant under a permutation of the variables in any term. We thus map any term $\mathbf{x}_\sigma^{\mathbf{y}}\mathbf{x}_{\sigma'}^{-\mathbf{y}'}$, $\sigma' \neq \sigma'_I$ into the term $\mathbf{x}_{\sigma''}^{\mathbf{y}}\mathbf{x}_{\sigma'_I}^{-\mathbf{y}'}$, where $\sigma'' = \bar{\sigma}' \circ \sigma$ (here $\sigma'' \circ \bar{\sigma}'$ denotes the composition of the two permutations σ'' and $\bar{\sigma}'$ considered as elements of S_n) i.e.

$$\frac{A_\sigma(\mathbf{x})}{A_{\sigma'}(\mathbf{x})}\prod_{\alpha,\beta=1}^{n} x_{\sigma_\alpha}^{y_\alpha}x_{\sigma'_\beta}^{-y'_\beta} \qquad \xrightarrow{\bar{\sigma}'} \qquad \frac{A_{\sigma''}(\mathbf{x})}{A_{\sigma'_I}(\mathbf{x})}\prod_{\alpha,\beta=1}^{n} x_{\sigma''_\alpha}^{y_\alpha}x_{\beta}^{-y'_\beta} \tag{19a}$$

$$= \qquad \frac{A_{\sigma''}(\mathbf{x})}{A_{\sigma'_I}(\mathbf{x})}\prod_{\alpha=1}^{n} x_{\alpha}^{y_{\bar{\sigma}''_\alpha}-y'_\alpha} \tag{19b}$$

Each term will now be repeated $n!$ times, thus (18) becomes

$$CT\left[\Lambda_n^t \sum_{\sigma\in S_n}\frac{A_{\bar{\sigma}}(\mathbf{x})}{A_{\sigma_I}(\mathbf{x})}\prod_{\alpha=1}^{n} x_{\alpha}^{y_{\sigma_\alpha}-y'_\alpha}\right]. \tag{20}$$

If (12) is substituted for $A_{\bar{\sigma}}$ and A_{σ_I} then in the ratio $A_{\bar{\sigma}}/A_{\sigma_I}$, in general, many of the $s_{\alpha\beta}$ factors cancel. The only parts that do not cancel are factors of the form $s_{\beta\alpha}/s_{\alpha\beta}$, but since the $s_{\alpha\beta}$ factor always arises from the identity permutation the factor $s_{\beta\alpha}$ corresponds to an inversion in the permutation σ (not $\bar{\sigma}$). As the set of inversions of σ, is the same as the vertex set, C_σ, we have that

$$\frac{A_{\bar{\sigma}}(\mathbf{x})}{A_{\sigma_I}(\mathbf{x})} = \prod_{\{i,j\}\in C_\sigma}\left\{\frac{\lambda_i\lambda_j - \omega x_j/x_i}{\lambda_i\lambda_j - \omega x_i/x_j}\right\}. \tag{21}$$

3.1 Combinatorial Interpretation

We now provide a (partial) combinatorial interpretation of the terms of expression for the number of osculating paths. The combinatorial interpretation will lead to the proof of corollary one.

The interpretation is based on the following simple observation: If there are n t-step paths then

$$\mathrm{CT}\left[\prod_{\alpha=1}^{n}(x_\alpha + \frac{1}{x_\alpha})^t x_\alpha^{y_\alpha - y'_\alpha}\right] = \prod_{\alpha=1}^{n}\binom{t}{t - \frac{y_\alpha - y'_\alpha}{2}} \tag{22}$$

This is just the number of n t-step paths from $(0, y'_\alpha)$ to (t, y_α) with common vertices an edges *allowed*. For each path the $(x_\alpha + 1/x_\alpha)^t$ generates *all* possible t-step paths

regardless of the starting and ending points. The x_α corresponding to a NE step and the $1/x_\alpha$ factor corresponding to a SE step (or visa versa). The $x_\alpha^{y_\alpha - y'_\alpha}$ factor in conjunction with the constant term operator $\mathrm{CT}[\,\cdot\,]$ ensures that of all possible paths only those that start and end at the correct points are picked out i.e. have the correct 'vertical displacement', $y_\alpha - y'_\alpha$.

Thus $\mathrm{CT}[\prod_{\alpha=1}^n \lambda_\alpha^t x_\alpha^{y_{\sigma_\alpha} - y'_\alpha}]$ is the number of n t-step unconstrained paths starting at $(0, y'_\alpha)$ and ending at (t, y_{σ_α}) respectively. This can be represented schematically by the set of inversion diagrams. Each term of (3) also carries a $A_{\bar{\sigma}}/A_{\sigma'_I}$ factor. Each inversion vertex between a pair of paths, starting at $(0, y'_\alpha)$ and $(0, y'_\beta)$, will give a "vertex operator" contribution $s_{\beta\alpha}/s_{\alpha\beta}$. This correspondence is illustrated in figure 4.

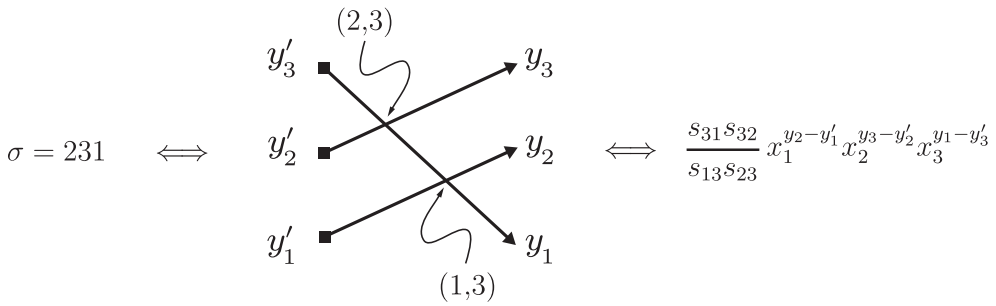

Figure 4: *Inversion diagram correspondence of one of the terms in equation (3) for the case of three paths. The lines of the inversion diagram can be interpreted as unconstained lattice paths between the respective heights.*

The action of each vertex operator then ensures that the osculation condition is met (the combinatorial manner in which this creates an involution is the subject of a future publication).

Note, if we put $\omega = 0$ them all the vertex operators are unity and (3) reduces to the determinant formula for non-intersecting paths. In a sense the Bethe Ansatz has "automated" the construction of the Gessel-Viennot involution.

To obtain (4) of corollary one we have to ensure that the first $n - \alpha$ steps of the path *starting* at $(0, y'_\alpha)$ are in the NE direction and the last $n - \beta$ steps of the path *ending* at $(0, y_\beta)$ are in the SE direction. The former constraint is achieved by replacing $n - \alpha$ factors of λ_α with $n - \alpha$ factors of x_α and the latter constraint is achieved by replacing $n - \beta$ factors of λ_β with $n - \beta$ factors of $1/x_\beta$. According to the above combinatorial interpretation the starting and ending points of the paths *inside* the CT operator depend on σ, thus, for each term the two constraints are enforced by inserting

$$\prod_{\alpha=1}^{n} \left(\frac{x_\alpha^{-1} x_{\bar{\sigma}_\alpha}}{\lambda_\alpha \lambda_{\bar{\sigma}_\alpha}} \right)^{n-\alpha} \tag{23}$$

for each permutation. This gives (4) and hence gives the corollary.

A more explicit, but very messy, derivation of the same result can be obtained by constraining the sums in the recurrence relations to enforce the above constraints.

Acknowledgments

I would like to thank the Australian Research Council for financial support for this project. I would also like to thank Mireille Bousquet-Melou for introducing me to the alternating sign matrix problem and John Essam for drawing my attention to the relationship between osculating paths and the six-vertex model.

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