P-recursiveness for pattern avoidance

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Abstract

Proving a first nontrivial instance of a conjecture of Noonan and Zeilberger we show that the number $S_r(n)$ of permutations of length n containing exactly r subsequences of type 132 is a P-recursive function of n. We also show the stronger statement that the ordinary generating function $G_r(x)$ of $S_r(n)$ is algebraic, in fact, it is rational in the variables x and $\sqrt{1-4x}$. We use this information to show that the degree of the polynomial recursion satisfied by $S_r(n)$ is r, for all $r \geq 1$.

Résumé

En prouvant la première spécialisation nontriviale d'une conjecture de Noonan et Zeilberger nous démontrons que le nombre $S_r(n)$ des permutations de $[n]$ contenant le motif 132 exactement r fois est une fonction polynômiellement récursive de n. De plus, la série génératrice $G_r(x)$ de $S_r(n)$ est algébrique, en effet, elle est rationelle en x et $\sqrt{1-4x}$. Nous utilisons cette information pour prouver que le degré de la récursion polynômielle satisfaite par $S_r(n)$ est r, pour tout $r \geq 1$.

1 Introduction and Background

Let $q = (q_1, q_2, \ldots, q_k) \in S_k$ be a permutation, and let $k \leq n$. We say that the permutation $p = (p_1, p_2, \ldots, p_n) \in S_n$ contains a subsequence (or pattern) of type q if there is a set of indices $1 \leq i_{q_1} < i_{q_2} < \ldots < i_{q_k} \leq n$ such that $p(i_1) < p(i_2) < \ldots < p(i_k)$. Otherwise we say that p is q-avoiding.

For example, a permutation is 132-avoiding if it doesn't contain three (not necessarily consecutive) elements among which the leftmost is the smallest and the middle one is the largest.

Recently, attention has been paid to the problem of counting n-permutations containing a *given number* r (as opposed to 0) of subsequences of a certain type q. In [4] Noonan has proved that the number of permutations in S_n containing exactly one 123-subsequence is $\frac{3}{n}\binom{2n}{n-3}$. The present author [1] has found a formula for the number of those containing exactly one 132-subsequence, namely $\binom{2n-3}{n-3}$.

The big question, however, is to describe this function for *any given* r, not just for $r = 0$ or 1. In [7] Noonan and Zeilberger conjectured that for *any* given subsequence q and for *any* given r, the number of n-permutations containing exactly r subsequences of type q is a P-recursive function of n. In the following three paragraphs we give a very brief summary of P-recursive and algebraic functions. The reader familiar with them can skip these paragraphs.

Recall that a function $f : \mathbf{N} \rightarrow \mathbf{C}$ is called P-recursive if there exist polynomials $P_0, P_1, \cdots, P_k \in \mathbf{C}[\mathbf{n}]$, with $P_k \neq 0$ so that

$$P_k(n)f(n+k) + P_{k-1}(n)f(n+k-1) + \cdots + P_0(n)f(n) = 0 \tag{1}$$

for all $n \in \mathbf{N}$. The continuous analogue of this notion is D-finiteness. Let $u(x) \in \mathbf{C}[[x]]$ be a power series. (We will be working over $\mathbf{C}$, the field of complex numbers, but a more general treatment is possible). If there exist polynomials $p_0(x), p_1(x), \cdots p_d(x)$ so that $p_d \neq 0$ and

$$p_d(x)u^{(d)}(x) + p_{d-1}(x)u^{(d-1)}(x) + \cdots + p_1(x)u'(x) + p_0(x)u(x) = 0, \tag{2}$$

then we say that u is D-finite. (Here $u^{(j)} = \frac{d^j u}{dx^j}$).

Some basic properties of these functions are listed here, for their proof see [5].

1. The function $f(n)$ is P-recursive if and only if its ordinary generating function $u(x) = F(x) = \sum_{n\geq 0} f(n)x^n$ is D-finite.

2. If $f(n)$ and $g(n)$ are P-recursive, then so are their sum and convolution.

3. If $f(n)$ is a P-recursive function and $g(n)$ is a function which disagrees with f on a finite number of n's only, then $g(n)$ is P-recursive as well.