p-elliptic hypermaps and the Klein map.

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April 1997

Abstract

We show that, on a p-elliptic hypermap, an automorphism ψ is of maximal order $p(1 + 2g/(p-1))$ if $o(\psi) = pm$ where p and m are coprime, and that the bound becomes $2pg/(p-1)$ otherwise. We classify the automorphism groups of p-elliptic hypermaps and give an exhaustive list when p is $g+1$ or $2g+1$. Finally, the Klein map is shown to be the only regular map admitting an automorphism of prime order $2g+1$ that is not $2g+1$-elliptic

Résumé

Nous montrons que sur une hypercarte p-elliptique un automorphisme est d'ordre au plus $p(1 + 2g/(p-1))$ si $o(\psi) = pm$ où p et m sont premiers entre eux, et que cette borne devient $2pg/(p-1)$ sinon. Nous classifions les groupes d'automorphismes des hypercartes p-elliptiques et en donnons une liste exhaustive quand p vaut $g+1$ ou $2g+1$. Finalement, la carte de Klein est la seule carte régulière qui admet un automorphisme d'ordre premier $2g+1$ et qui ne soit pas $2g+1$-elliptique.

1 Introduction

The combinatorial study of Riemann surfaces topology is a research field where either geometers such as G. Jones and D. Singerman (see [JoSi]) or group and graph theorists such as R.Cori and A. Machì (see [CoMa]) have developped to meet at the joint point of the *theory of maps*. This study is made effective by the tessellation of the surface into polygons that naturally leads to the consideration of a *map*: a pair of permutations (α, σ), one corresponding to the edges and the other to the vertices of these polygons.Then, topological propreties have combinatorial translations; for instance, connexity on the surface is ensured by the transitivity of $< \alpha, \sigma >$ (see below for a detailed explanation).

Some properties of the automorphisms of the surface correspond then to the properties of $Aut(\alpha, \sigma)$, centralizer of both α and σ in the symmetric group. An example is given by a classical result known as the Riemann-Hurwitz formula which relates the genus of the surface, the genus of it quotient surface with respect to an automorphism group G and the number of points fixed by all elements in G on the surface. Defining the *genus* of the map in function of the number of cycles of α, σ and $\alpha\sigma$, Machì showed that the formula holds for $Aut(\alpha, \sigma)$ (see [Ma]).

Maps theory has been generalized in a natural way to the one of hypermaps by Cori and can be related to hypergraph theory (see [Co] and below). Searching for links between a hypermap and its quotient hypermap with respect to an automorphism group, we have proposed in [Be2] the concept of *induced automorphism* which can be understood as follow. Let (α, σ) be a hypermap, $\psi \in Aut(\alpha, \sigma)$, G a subgroup of $Aut(\alpha, \sigma)$ and let $(\overline{\alpha}, \overline{\sigma})$ be the quotient hypermap of (α, σ) with respect to G. When ψ belongs to the normalizer of G in $Aut(\alpha, \sigma)$, the automorphism $\overline{\psi}$ induced by ψ on $(\overline{\alpha}, \overline{\sigma})$ is well defined.

This paper deals with the so called p-elliptic hypermaps introduced in [Be2]. The idea is to study hypermaps on which all automorphisms are induced on the sphere by operating a quotient with respect to a prime order group. These can be viewed as p-sheeted coverings of the sphere where p is a prime (see below).

For p-elliptic hypermaps we classify their automorphism groups. $Aut(\alpha, \sigma)$ is either C_{pn} where n is a divisor of $1 + 2g/(p-1)$, C_{pn} or D_{pn} where n is a divisor of $2g/(p-1)$; a semi-direct product of either C_n or a lifting of D_n by

C_p, where n is a divisor of $2 + 2g/(p-1)$; or $Aut(\alpha, \sigma)$ is of order $12p$, $24p$ or $60p$.

This gives sharp bounds for the order of the automorphism group and of the elements. Namely, on a p-elliptic hypermap, an automorphism ψ is of maximal order $p(1 + 2g/(p-1))$ if $o(\psi) = pm$ where p and m are coprime, and that the bound becomes $2pg/(p-1)$ otherwise. Fixing p, these bounds are reached for infinitely many g. Moreover, we show that for $g > 1$, if there exists an automorphism ϕ of prime order $2g+1$, then the hypermap except for one case is $(2g+1)$-elliptic. The exception happens for $g = 3$ and $Aut(\alpha, \sigma) = PSL_2(\mathbf{Z}_7)$, The *Klein map* we describe is a canonical example of it.

2 Hypermaps and automorphisms

Let us recall a few definitions that appear in the Cori-Machì's survey on hypermaps ([CoMa]).

Definition 2.1 *A* hypermap *is a pair of permutations* (α, σ) *on* B*, the set of* darts *(or* brins*), such that the group they generate is transitive on* B*. When* α *is a fixed-dart-free involution,* (α, σ) *is a* map*. The cycles of* α*,*σ *and* $\alpha^{-1}\sigma$ *are called* edges, vertices *and* faces*, respectively.*

Note that considering the permutation $\alpha^{-1}\sigma$ means doing first α^{-1} and then σ so that permutations are multiplied from left to right.

Euler's formula gives the relationship between the numbers of cycles of these three permutations:

$$z(\alpha) + z(\sigma) + z(\alpha^{-1}\sigma) = n + 2 - 2g$$

where $n = card(B)$, g is a non-negative integer, called the *genus* of (α, σ), and where for any permutation $\theta, z(\theta)$ denotes the number of its cycles (see [CoMa]).

Definition 2.2 *An* automorphism ϕ *of a hypermap* (α, σ) *is a permutation commuting with both* α *and* σ*:*

$$\alpha\phi = \phi\alpha \qquad \textit{and} \qquad \sigma\phi = \phi\sigma \quad .$$

Thus, the full automorphism group of (α, σ), denoted by $Aut(\alpha, \sigma)$, is the centralizer in $Sym(n)$ of the group generated by α and σ. A subgroup G of $Aut(\alpha, \sigma)$ is an *automorphism group* of (α, σ); the transitivity of $< \alpha, \sigma >$ implies that $Aut(\alpha, \sigma)$ is semi-regular (see [CoMa]). Recall here that a semi-regular group is defined by the fact that all its orbits are of the same length namely $\mid G \mid$.

We denote by $\chi_\theta(\phi)$ the number of cycles of a permutation θ fixed by an automorphism ϕ and by $\chi(\phi)$ the total number of cycles of α, σ, and $\alpha^{-1}\sigma$ fixed by ϕ; since a cycle of either α, σ or $\alpha^{-1}\sigma$ will be called a *point*, $\chi(\phi)$ represent the fixed points of ϕ on the hypermap. $o(\phi)$ will be the order of ϕ.

If (α, σ) is *planar* that is $g = 0$ then $\chi(\phi) = 2$ for all automorphisms ϕ. Moreover, $Aut(\alpha, \sigma)$ is one of C_n (cyclic), D_n (dihedral), A_4, S_4 and A_5 (see [CoMaPeVa]). We shall need this result later.

We now define an equivalence relation R on the set B.

Definition 2.3 *Let G be an automorphism group of the hypermap (α, σ). Two brins b_1 and b_2 are* equivalent, *$b_1 R b_2$, if they belong to the same orbit of G.*

This leads to the following definition.

Definition 2.4 *The* quotient hypermap *$(\overline{\alpha}, \overline{\sigma})$ of (α, σ) with respect to an automorphism group G, is a pair of permutations $(\overline{\alpha}, \overline{\sigma})$ acting on the set $\overline{B}$, where $\overline{B} = B/R$ and $\overline{\alpha}$, $\overline{\sigma}$ are the permutations induced by α and σ on $\overline{B}$.*

The following *Riemann-Hurwitz formula* relates the genus γ of $(\overline{\alpha}, \overline{\sigma})$ to the genus g of (α, σ):

$$(RH1) \qquad 2g - 2 = card(G)\big(2\gamma - 2\big) + \sum_{\phi \epsilon G - \{id\}} \chi(\phi) \quad .$$

It follows that $\gamma \leq g$. In case G is a cyclic group, $G =< \phi >$, $(RH1)$ becomes

$$(RH2) \qquad 2g - 2 = card(G)\big(2\gamma - 2\big) + \sum_{i=1}^{o(\phi)-1} \chi(\phi^i) \quad .$$

As a consequence, if $g \geq 2$ $\mid Aut(\alpha,\sigma) \mid \leq 84(g-1)$. This bound demonstrated by Hurwitz has been combinatoraly proved by Machì in [Ma].

If ϕ is an automorphism of order m, then, for all integers i, $\chi(\phi) \leq \chi(\phi^i)$, and when m and i are coprime $\chi(\phi) = \chi(\phi^i)$.

Let (α,σ) be a hypermap, G an automorphism group of (α,σ) and let $(\overline{\alpha},\overline{\sigma})$ be the quotient hypermap of (α,σ) with respect to G. Then for any element ψ of $Aut(\alpha,\sigma)$ such that the group G is normal in $< G,\psi >$, the permutation $\bar{\psi}$, defined as $\bar{\psi} = \psi/G$, is an automorphism of $(\overline{\alpha},\overline{\sigma})$.The two following operations on (α,σ) are equivalent:

(i) taking the quotient of (α,σ) first w.r.t. G and then to $< \bar{\psi} >$

(ii) taking the quotient of (α,σ) w.r.t. $< G,\psi >$.

Definition 2.5 *The permutation $\bar{\psi}$ is called the* induced automorphism *of ψ on $(\overline{\alpha},\overline{\sigma})$. We also say ψ* induces *$\bar{\psi}$ on $(\overline{\alpha},\overline{\sigma})$.*

We consider now the case in which an automorphism ϕ of prime order p is normal in $< \psi,\phi >$, where ψ is any element of $Aut(\alpha,\sigma)$.

Proposition 2.6 *Let ψ commute with ϕ.*

If ψ is of order m where p and m are coprime, then

$$\chi(\overline{\psi})p = \chi(\psi) + (p-1)\chi(\phi\psi).$$

If ψ is of order pm, p and m coprime, and ϕ belongs to $< \psi >$, then

$$\chi(\overline{\psi})p = \chi(\psi^p) + (p-1)\chi(\psi).$$

If ψ is of order $p^m n$, $m > 1$, p and n coprime, and ϕ belongs to $< \psi >$, then

$$\chi(\psi) = \chi(\overline{\psi}).$$

If ψ is of order pm, m being any integer, and ϕ does not belongs to $< \psi >$, then

$$\chi(\overline{\psi})p = \sum_{i=0}^{p-1} \chi(\psi\phi^i)$$

and

$$\chi(\psi\phi^i) \equiv 0 (mod \quad p).$$

Proposition 2.7 *If ψ does not commute with ϕ Then*

$$\chi(\psi) = \chi(\overline{\psi}).$$

Definition 2.8 *A hypermap (α, σ) of genus $g > 1$ is said to be p-*elliptic *if it admits an automorphism ϕ of prime order p such that:*

(1) the quotient hypermap $(\overline{\alpha}, \overline{\sigma})$ with respect to ϕ is planar, that is of genus 0,

(2) $< \phi >$ is normal in $Aut(\alpha, \sigma)$.

Definition 2.9 *On a hypermap (α, σ) of genus $g > 1$ an automophism ϕ of order p is said to be p-*elliptic *if it is normal in $Aut(\alpha, \sigma)$ and fixes $2 + 2g/(p-1)$ points.*

Note that Definition 2.9 is equivalent to Definition 2.8 since by (RH) if the quotient hypermap $(\overline{\alpha}, \overline{\sigma})$ with respect to ϕ is of genus 0 then ϕ fixes $2 + 2g/(p-1)$ points and vice versa.

Remark 2.10 *A 2-elliptic hypermap is a hyperelliptic hypermap and Property (1) implies Property (2) when $g > 1$ ([CoMa]).*

Since an automorphism on the sphere fixes exactly 2 points, it has been proved in [Be2] that:

Proposition 2.11 *an automorphism ψ on a p-elliptic hypermap of genus g has one of the following behaviors:*

i) it is a p-elliptic automorphism and therefore fixes $2+2g/(p-1)$ points;

ii) it fixes 2 points in common with a p-elliptic automorphism and therefore fixes only these 2 points;

iii) it fixes 1 point in common with a p-elliptic automorphism and therefore fixes either 1 or $p+1$ points;

iv) it fixes no point in common with a p-elliptic automorphism and therefore fixes either 0 or $2p$ points.

Note that a "non p-elliptic" automorphism fixes at most 2 points in common with a p-elliptic automorphism.

We shall also need the following result proved in [BiCo] and [Be1]: an automorphism of prime power order can not fix only one point.

3 A bound for the order of an automorphism

Let (α, σ) be a p-elliptic hypermap and ϕ be the p-elliptic automorphism. We know that ϕ fixes $2 + 2g/(p-1)$ points. Since $< \phi >$ is normal in $Aut(\alpha, \sigma)$ any automorphism permutes these fixed points among themselves, and we can define a homomorphism h from $Aut(\alpha, \sigma)$ to $S_{2+2g/(p-1)}$.

Lemma 3.1 *Let (α, σ) be a p-elliptic hypermap. An automorphism ψ induces a regular permutation on $2 + 2g/(p-1)$ points if ϕ and ψ have no common fixed point, $1 + 2g/(p-1)$ points if ϕ and ψ have one common fixed point and $2g/(p-1)$ points if ϕ and ψ have two common fixed points.*

Theorem 3.2 *Let (α, σ) be a p-elliptic hypermap and let ψ be an automorphism of (α, σ). Then either $o(\psi) = p\big(1 + 2g/(p-1)\big)$, where p and $(1 + 2g/(p-1)$ are coprime, or $o(\psi) \leq 2pg/(p-1)$.*

All these bounds are sharp as the two following examples show.

Example 3.3 *Let $g \geq 2$, p a prime such that $p-1$ divides $2g$, and suppose that $1+2g/(p-1)$ is coprime with p. consider the following hypermap (α, σ) on $n = p(1 + 2g/(p-1))$ brins: $\alpha^{-1}\sigma$ is the cycle $(1, 2, \cdots, n)$, α and σ are powers of $\alpha^{-1}\sigma$ of orders p and $1 + 2g/(p-1)$. Euler formula shows that g is the genus of (α, σ). $Aut(\alpha, \sigma) =< \alpha^{-1}\sigma >$ and the quotient hypermap with respect to α is planar. Thus (α, σ) is a p-elliptic hypermap with an automorphism of maximal order.*

Example 3.4 *Let p and g be such that $2g = p^k(p-1)$ where p is a prime and k is any positive integer. Consider (α, σ) on $n = 2pg/(p-1) = p^{k+1}$ brins: σ is the cycle $(1, 2, \cdots, n)$, α is a power of σ of order p; $\alpha^{-1}\sigma$ is thus a cycle of order n. Euler formula shows that g is the genus of (α, σ). $Aut(\alpha, \sigma) =< \sigma >$ and the quotient hypermap with respect to α is planar. Thus (α, σ) is a p-elliptic hypermap with an automorphism of maximal order.*

Corollary 3.5 *The order of an automorphism of a p-elliptic hypermap of genus g is either $4g + 2$, or at most $4g$.*

4 Automorphism groups of p-elliptic hypermaps

We indicate by $C_l \times_k C_p$ th group such that there exists ϕ, generator of C_p, and ψ C_l, generateur of C_p, and $\psi\phi = \phi^k\psi$. In the sequel, we will need a square, cubique and fourth root of unity in C_p, respectively denoted by -1, j et i; the corresponding groups being $G \times_{-1} C_p$, $G \times_j C_p$ and $G \times_i C_p$.

Theorem 4.1 *Let (α, σ) be a p-elliptic hypermap. Then $Aut(\alpha, \sigma)$ is either C_{pn} (cyclic) where n is a divisor of $1 + 2g/(p-1)$; C_{pn} or D_{pn} (dihedral) where n is a divisor of $2g/(p-1)$; a semi-direct product of either C_n or a lifting of D_n by C_p, where n is a divisor of $2 + 2g/(p-1)$; or $Aut(\alpha, \sigma)$ is of order $12p$, $24p$ or $60p$ (liftings of A_4, S_4, A_5 respectively).*

Corollary 4.2 *Let (α, σ) be a hyperelliptic hypermap. Then $Aut(\alpha, \sigma)$ is either C_{2n} where n is a divisor of $2g + 1$, C_{2n} or D_{2n} where n is a divisor of $2g$; $C_n \times C_2$ or a lifting of D_n by C_2, where n is a divisor of $2g + 2$; or $Aut(\alpha, \sigma)$ is of order 24, 48 or 120 (liftings of A_4, S_4, A_5 respectively).*

Corollary 4.3 *Let (α, σ) be a $(2g + 1)$-elliptic hypermap, then its automorphism group is either C_{2g+1} or C_{4g+2}; and if 3 divides g we also have $C_3 \times_j C_{2g+1}$.*

Remark 4.4 *The case C_{4g+2} happens for all $g > 1$ as we have seen in Example 1. The case C_{2g+1} happens for all $g > 1$ by taking $\alpha = \sigma^{-1} = (1, 2, \cdots, 2g + 1)$. The case $C_3 \times_j C_{2g+1}$ happens for instance on the hypermap of genus 3 described in [CoMa].*

Corollary 4.5 *Let (α, σ) be a $(g + 1)$-elliptic hypermap, then its automorphism group is one of the following: C_{g+1}, C_{2g+2}, C_{3g+3}, D_{g+1}, D_{2g+2}, $C_2 \times C_{2g+2}$ and if 4 divides g we also have $C_4 \times_i C_{g+1}$.*

Lemma 4.6 *Let (α, σ) be a hypermap of genus $g > 1$ and p a prime dividing the order of $Aut(\alpha, \sigma)$. If p equals $g+1$ or $2g+1$, then the p-Sylow subgroups are of order p.*

Proposition 4.7 *Let (α, σ) be a hypermap of genus $g > 1$ such that there exists an automorphism ϕ of prime order $2g+1$. Then, except for one case when $g = 3$ and $Aut(\alpha, \sigma) = PSL_2(\mathbf{Z}_7)$ (the simple group of order 168), (α, σ) is a $(2g+1)$-elliptic hypermap.*

Such a hypermap does exist: the "Klein map", a regular map on 168 brins vertices of length 3 and faces of length 7 for which $Aut(\alpha, \sigma)$ is $PSL_2(\mathbf{Z}_7)$. It is given in the last section.

Proposition 4.8 *Let (α, σ) be a hypermap of genus $g > 30$ such that there exist an automorphism ϕ of prime order $g+1$. Then (α, σ) is a $(g+1)$-elliptic hypermap.*

Corollary 4.9 *Let (α, σ) be a hypermap of genus $g = 2$. Then $Aut(\alpha, \sigma)$ is either $1, C_2$ or C_4, or else (α, σ) is 5-elliptic, 3-elliptic or hyperelliptic.*

Acknowledgements

I wish to thank Professor Antonio Machì for his encouragement and advice in directing this work.

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A The Klein map

We give here a combinatorial map (α, σ) which describes a regular triangulation of the Klein surface. This surface of genus 3 is a famous example where the Hurwitz bound is reached. Thus, this map admits $PSL_2(\mathbf{Z}_7)$, the simple group of order 168, as automorphism group. (α, σ) has 56 vertices, 84 edges, and 24 faces on 168 brins; it is represented on Figure 1 where we identify the 14 sides of the polygon in the following way: A-F, B-K, C-H, D-M, E-J, G-L, I-N. We recall that the Klein map represents an exceptionnal phenomenon for $2g+1$ prime order automorphisms where g is the genus, since we have seen by Proposition 4.7 that any hypermap admitting a $2g+1$ prime order automorphism is $2g+1$-elliptic and has an automorphism group equal C_{4g+2}, C_{2g+1} ou $C_3 \times_j C_{2g+1}$ except the Klein map.

α= (1, 21) (2, 23) (3, 4) (5, 27) (6, 7) (8, 30) (9, 10) (11, 31) (12, 13) (14, 34) (15, 16) (17, 37) (18, 19) (20, 40) (22, 48) (24, 44) (25, 51) (26, 53) (28, 58) (29, 62) (32, 67) (33, 70) (35, 78) (36, 79) (38, 87)

(39, 88) (41, 96) (42, 97) (43, 129) (45, 102) (46, 151) (47, 106) (49, 108) (50, 149) (52, 56) (54, 139) (55, 59) (57, 112) (60, 137) (61, 65) (63, 127) (64, 68) (66, 115) (69, 165) (71, 150) (72, 73) (74, 118) (75, 76) (77, 159) (80, 138) (81, 82) (83, 121) (84, 85) (86, 153) (89, 163) (90, 91) (92, 124) (93, 94) (95, 141) (98, 157) (99, 100) (101, 103) (104, 120) (105, 116) (107, 109) (110, 123) (111, 119) (113, 126) (114, 1 22) (117, 125) (128, 130) (131, 133) (132, 155) (134, 136) (135, 162) (140, 142) (143, 1 45) (144, 161) (146, 148) (147, 168) (152, 154) (156, 167) (158, 160) (164, 166)

σ = (1, 2, 3) (4, 5, 6) (7, 8, 9) (10, 11, 12) (13, 14, 15) (16, 17, 18) (19, 20, 21) (22, 23, 24) (25, 26, 27) (28, 29, 30) (31, 32, 33) (34, 35, 36) (37, 38, 39) (40, 41, 42) (43, 44, 45) (46, 47, 48) (49, 50, 51) (52, 53, 54) (55, 56, 57) (58, 59, 60) (61, 62, 63) (64, 65, 66) (67, 68, 69) (70, 71, 72) (73, 74, 75) (76, 77, 78) (79, 80, 81) (82, 83, 84) (85, 86, 87) (88, 89, 90) (91, 92, 93) (94, 95, 96) (97, 98, 99) (100, 101, 102) (103, 104, 105) (106, 107, 108) (109, 110, 111) (112, 113, 114) (115, 116, 117) (118, 119, 120) (121, 122, 123) (124, 125, 126) (127, 128, 129) (130, 131, 132) (133, 134, 135) (136, 137, 138) (139, 140, 141) (142, 143, 144) (145, 146, 147) (148, 149, 150) (151, 152, 153) (154, 155, 156) (157, 158, 159) (160, 161, 162) (163, 164, 165) (166, 167, 168)

$\alpha\sigma$= (1, 19, 16, 13, 10, 7, 4) (2, 24, 45, 100, 97, 40, 21) (3, 5, 25, 49, 106, 48, 23) (6, 8, 28, 59, 56, 53, 27) (9, 11, 32, 68, 65, 62, 30) (12, 14, 35, 76, 7 3, 70, 31) (15, 17, 38, 85, 82, 79, 34) (18, 20, 41, 94, 91, 88, 37) (22, 46, 152, 155, 130, 129, 44) (26, 54, 140, 143, 146, 149, 51) (29, 63, 128, 131, 134, 137, 58) (33, 71, 14 8, 147, 166, 165, 67) (36, 80, 136, 135, 160, 159, 78) (39, 89, 164, 167, 154, 153, 87) (42, 98, 15 8, 161, 142, 141, 96) (43, 127, 61, 66, 116, 103, 102) (47, 107, 110, 121, 84, 86, 151) (50, 150, 7 2, 74, 119, 109, 108) (52, 57, 113, 124, 93, 95, 139) (55, 60, 138, 81, 83, 122, 112) (64, 69, 163, 90, 92, 125, 115) (75, 77, 157, 99, 101, 104, 118) (105, 117, 126, 114, 123, 111, 120) (132, 15 6, 168, 145, 144, 162, 133)

Figure 1: Klein map