Geometric Representations of Graphs and the Four–Color Theorem

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In the last years there have been some major advances in our understanding of the topological and geometric structure of graphs such as the minor theorem of Robertson and Seymour and the spectral invariant $\mu(G)$ discovered by Colin de Verdière. The purpose of this talk is to give an introduction to these concepts and the main results and open problems.

As a start, recall Kuratowski's Theorem in the version using minors. We write $H \leq G$ if H is a minor of G.

$$G \text{ is planar} \iff K_5 \not\leq G \text{ and } K_{3,3} \not\leq G.$$

There are two well–known analoga:

$$G \text{ is outer–planar} \iff K_4 \not\leq G \text{ and } K_{2,3} \not\leq G.$$

$$G \text{ is a linear forest} \iff K_3 \not\leq G \text{ and } K_{1,3} \not\leq G.$$

What's more the chromatic number χ relates to these classes as follows:

$$\begin{aligned} G \text{ linear forest} &\implies \chi(G) \leq 2 \quad \text{(trivial)} \\ G \text{ outer–planar} &\implies \chi(G) \leq 3 \quad \text{(easy)} \\ G \text{ planar} &\implies \chi(G) \leq 4 \quad \text{(4–Color Theorem).} \end{aligned}$$

Two natural questions arise:

A. Why is χ connected to these classes in this way?

B. What is the next class or, in general, what is the hierarchy of classes that are generated in this way?

In a discussion of these questions we survey some of the recent work relating topological properties, minors, geometric representations and the chromatic number of graphs.